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**COVERING THE VERTICES OF A GRAPH BY  
VERTEX-DISJOINT PATHS**

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## COVERING THE VERTICES OF A GRAPH BY VERTEX-DISJOINT PATHS

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Define the path-covering number  $\mu(G)$  of a finite graph  $G$  to be the minimum number of vertex-disjoint paths required to cover the vertices of  $G$ . Let  $g(n, k)$  be the minimum integer so that every graph,  $G$ , with  $n$  vertices and at least  $g(n, k)$  edges has  $\mu(G) \leq k$ . A relationship between  $\mu(G)$  and the degree sequence for a graph  $G$  is found; this is used to show that

$$\frac{1}{2}(n - k)(n - k - 1) + 1 \leq g(n, k) \leq \frac{1}{2}(n - 1)(n - k - 1) + 1$$

A further extremal problem is solved.

**1. Introduction.** A graph  $G$  is a finite collection  $\mathcal{V}(G)$  of vertices (or points) some pairs of which are joined by a single edge; the collection of edges is denoted by  $\mathcal{E}(G)$ .  $H$  is a *subgraph* of  $G$  if  $\mathcal{V}(H) \subseteq \mathcal{V}(G)$  and  $\mathcal{E}(H) \subseteq \mathcal{E}(G)$ . If  $H$  and  $K$  are two vertex-disjoint graphs,  $H \cup K$  is the graph with  $\mathcal{V}(H \cup K) = \mathcal{V}(H) \cup \mathcal{V}(K)$  and  $\mathcal{E}(H \cup K) = \mathcal{E}(H) \cup \mathcal{E}(K)$ ;  $H + K$  is  $H \cup K$  together with all  $|\mathcal{V}(H)| \cdot |\mathcal{V}(K)|$  possible choices of edges joining a vertex of  $H$  to a vertex of  $K$ .  $\bar{G}$  denotes the complement of  $G$ ;  $\Gamma_n$  denotes the complete graph with  $n$  vertices and  $\Gamma_{m,n}$  denotes the complete bipartite graph,  $\bar{\Gamma}_m + \bar{\Gamma}_n$ .

Let  $G$  be a graph. A path of length  $n$  in  $G$  is an ordered sequence  $P = \langle a_1, a_2, \dots, a_n \rangle$  of distinct points, where if  $n \geq 2$ ,  $a_i$  is adjacent to  $a_{i+1}$  for  $1 \leq i \leq n - 1$ .  $\langle a_1, a_2, \dots, a_n \rangle$  is the same path as  $\langle a_n, a_{n-1}, \dots, a_1 \rangle$ . If  $P$  and  $Q$  are paths, by  $P * Q$  we shall mean that one end-point,  $a$  of  $P$ , is adjacent to one end-point,  $b$  of  $Q$ , and that  $P * Q$  is formed by joining  $a$  to  $b$ . More specifically we may write  $Pa * bQ$  or  $P * bQ$  or  $Pa * Q$  to specify, in varying degrees, which end-point of  $P$  is joined to which end-point of  $Q$ . Also,  $\langle a_1, a_2, \dots, a_n \rangle * \langle b_1, b_2, \dots, b_m \rangle = \langle a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \rangle$  where  $a_n$  must be adjacent to  $b_1$ . A *Hamilton-path* is a path of length  $|\mathcal{V}(G)|$ . A *path-cover* of  $G$  is a collection,  $\mathcal{S}$ , of vertex-disjoint paths such that every vertex of  $G$  lies on some path in  $\mathcal{S}$ . The *path-covering number*, denoted by  $\mu(G)$ , of  $G$  is defined by:

$$\mu(G) = \text{Min} \{ |\mathcal{S}| : \mathcal{S} \text{ is a path-cover of } G \}.$$

A *minimal path-cover* (M.P.C.) of  $\bar{G}$  is a path-cover,  $\mathcal{S}$  of  $G$ , with  $|\mathcal{S}| = \mu(G)$ .

We note that  $\mu(G)$  is an invariant of  $G$  and remark that a graph,  $G$ , has a Hamilton-path if and only if  $\mu(G) = 1$ . It has been shown by Nash-Williams [1] and others that the problem of classifying all Hamiltonian graphs is equivalent to that of classifying all graphs which have a Hamilton-path. Thus a classification of all graphs with  $\mu(G) = k$  ( $k = 1, 2, 3, \dots$ ) would also solve the Hamiltonian problem as a special case.

Historically, O. Ore [3] first introduced the graphical invariant  $\mu$ . In [2] some elementary properties of  $\mu$  are derived. In §2 we generalize a result of O. Ore (Theorem 2.1 in [3]) and in §3 we consider two extremal problems involving  $\mu$ .

**2. Valency considerations.** In this section we derive a connection between the path-covering number and the degree sequence of a graph. We begin with some definitions:

**DEFINITION 2.1.** Let  $A$  be a finite set and  $f$  a real-valued function defined on the collection of subsets of  $A$ . For  $B \subseteq A$  and for any integer  $i$  with  $1 \leq i \leq |B|$ , define the function  $S_i$  by:

$$S_i(f, B) = \sum_{\substack{C \subseteq B \\ |C|=i}} f(C).$$

**DEFINITION 2.2.** If  $G$  is a graph,  $B \subseteq \mathcal{V}(G)$ , and either  $H \subseteq \mathcal{V}(G)$  or  $H$  is a subgraph of  $G$ , then define the generalized valence function,  $\rho$ , by

$$\rho_H(B) = \text{the number of vertices of } H \text{ which are adjacent to every member of } B.$$

If  $x$  is a vertex of  $G$ , then we write  $\rho(x)$  for  $\rho_G(\{x\})$ .

**DEFINITION 2.3.** Let  $G$  be a graph and  $X \subseteq \mathcal{V}(G)$  with  $|X| = k \geq 2$ . Define:

$$D(G, X) = \frac{1}{k} S_1(\rho_G, X) - \sum_{i=1}^k (-1)^i \binom{k-i}{k} S_i(\rho_G, X).$$

The following lemma is easily verified:

LEMMA 2.4. If  $X = \{x_1, x_2, \dots, x_k\}$ , and  $1 \leq m \leq k - 1$ , then

$$\sum_{i=1}^k S_m(f, X - \{x_i\}) = (k - m)S_m(f, X).$$

We now state the main result of this section:

THEOREM 2.5. Let  $G$  be a graph with  $\mu = \mu(G) \geq 2$ ,  $|\mathcal{V}(G)| = n$  and  $k$  an integer with  $2 \leq k \leq \mu$ , then there exists a set  $X$  consisting of  $k$  mutually non-adjacent vertices of  $G$ , satisfying:

$$(2.6) \quad \mu \leq n - D(G, X).$$

Note that the case  $k = 2$  reduces to the result of Ore (Theorem 2.1 in [3]):

$$\mu \leq n - \rho(x_1) - \rho(x_2).$$

*Proof.* Let  $\mathcal{S} = \{P_1, P_2, \dots, P_\mu\}$  be a M.P.C. for  $G$ . For each  $1 \leq i \leq k$ , let  $x_i$  be an end-vertex of  $P_i$ . Since  $\mathcal{S}$  is a M.P.C.,  $x_i$  is not adjacent to  $x_j$  for  $i \neq j$ .

Let  $X = \{x_1, x_2, \dots, x_k\}$ . We first show that for  $1 \leq i \leq k$  and  $1 \leq j \leq \mu$ , the inequality:

$$(2.7) \quad \rho_{P_j}(\{x_i\}) \leq |P_j| - \left(1 - \sum_{l=1}^{k-1} (-1)^l S_l(\rho_{P_j}, X - \{x_i\})\right)$$

holds. Let  $P_j$  be the path  $\langle a_1, a_2, \dots, a_t \rangle$ , let  $1 \leq m \leq k$ ,  $m \neq i$ , and consider the following cases:

- (i)  $i = j$ . In this case assume that  $x_i = a_1$ .
- (ii)  $m = j$ . In this case assume that  $x_m = a_t$ .
- (iii)  $m \neq j$  and  $i \neq j$ .

Let

$$A = \{r : a_r \text{ is adjacent to } x_i\},$$

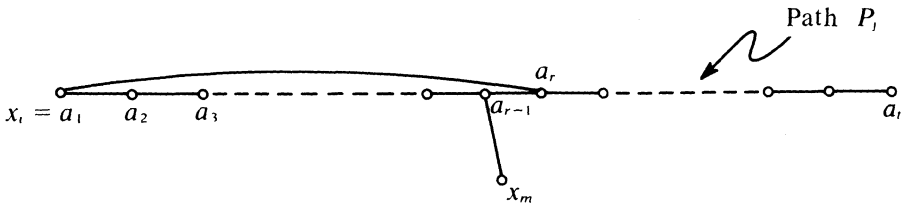
$$B_m = \{r : a_{r-1} \text{ is adjacent to } x_m\}$$

and

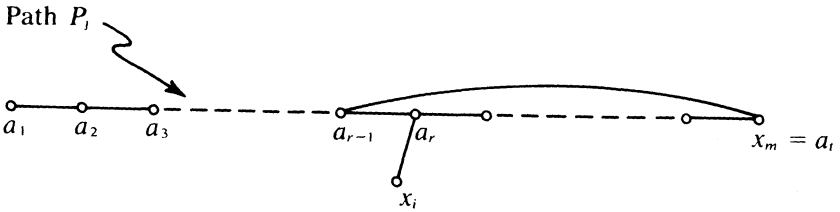
$$B = \bigcup_{\substack{1 \leq m \leq k \\ m \neq i}} B_m.$$

We claim that  $A \cap B_m = \emptyset$ , for if  $r \in A \cap B_m$ , then in each case we can

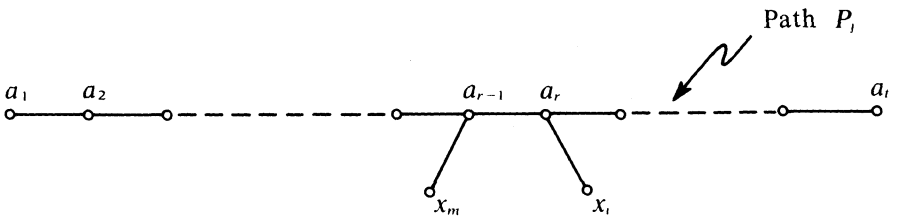
construct a path-cover,  $\mathcal{T}$  for  $G$ , as follows (see Figure 2.8):



Case (i)



Case (ii)



Case (iii)

FIGURE 2.8

In case (i), let:

$$\mathcal{T} = \mathcal{S} \cup \{ \langle a_t, a_{t-1}, \dots, a_r, x_i, a_2, a_3, \dots, a_{r-1} \rangle * x_m P_m \} - \{ P_i, P_m \}.$$

In case (ii), let:

$$\mathcal{T} = \mathcal{S} \cup \{ \langle a_1, a_2, \dots, a_{r-1}, x_m, a_{r-1}, a_{r-2}, \dots, a_r \rangle * x_i P_i \} - \{ P_i, P_m \}.$$

In case (iii), let:

$$\mathcal{T} = \mathcal{S} \cup \{ \langle a_1, \dots, a_{r-1} \rangle * x_m P_m, \langle a_t, a_{r-1}, \dots, a_r \rangle * x_i P_i \} - \{ P_i, P_j, P_m \}.$$

In either case,  $|\mathcal{T}| = |\mathcal{S}| - 1 < |\mathcal{S}|$ , contradicting the minimality of  $\mathcal{S}$ . Hence  $A \cap B_m = \phi$ . Also, in each case  $a_1 \notin A$ ; so  $A \subseteq P_j - B \cup \{a_1\}$ . This gives  $|A| \leq |P_j| - |B \cup \{a_1\}|$ , since  $B \cup \{a_1\} \subseteq P_j$ . But then, since  $a_1 \notin B$ , we get:

$$(2.9) \quad |A| \leq |P_j| - (1 + |B|).$$

For  $1 \leq m \leq k$ , let:

$$C_m = \{r: a_r \text{ is adjacent to } x_m\}.$$

Then since  $x_m$  is not adjacent to  $a_1$ ,  $|C_m| = |B_m|$  and:

$$\begin{aligned} |B| &= \left| \bigcup_{\substack{1 \leq m \leq k \\ m \neq i}} B_m \right| = \left| \bigcup_{\substack{1 \leq m \leq k \\ m \neq i}} C_m \right| \\ &= \sum_{l=1}^{k-1} (-1)^{l+1} \sum_{\substack{1 \leq m_1 < m_2 < \dots < m_l \leq k \\ m_1, m_2, \dots, m_l \neq i}} |C_{m_1} \cap C_{m_2} \cap \dots \cap C_{m_l}| \\ (2.10) \quad &= - \sum_{l=1}^{k-1} (-1)^l S_l(\rho_P, X - \{x_i\}). \end{aligned}$$

So since  $|A| = \rho_P(\{x_i\})$ , (2.7) follows from (2.9) and (2.10). Summing (2.7) for  $1 \leq i \leq k$  and applying Lemma 2.4, we get:

$$(2.11) \quad S_1(\rho_P, X) \leq k|P_j| - \left( k - \sum_{l=1}^{k-1} (-1)^l (k-l) S_l(\rho_P, X) \right).$$

Summing (2.11) for  $1 \leq j \leq \mu$ , we get:

$$S_1(\rho_G, X) \leq kn - \left( k\mu - \sum_{l=1}^{k-1} (-1)^l (k-l) S_l(\rho_G, X) \right).$$

from which (2.6) follows.

### 3. Extremal problems.

DEFINITION 3.1. Let  $k$  and  $n$  be integers with  $1 \leq k \leq n$ . Define:

$$g(n, k) = \text{Min} \{ m : \text{every graph, } G, \text{ with } |\mathcal{V}(G)| = n \text{ and} \\ |\mathcal{E}(G)| \geq m \text{ has } \mu(G) \leq k \}.$$

In this section we determine bounds for  $g(n, k)$ . See [4] for techniques in proving the following:

LEMMA 3.2.

$$(3.3) \quad \sum_{i=1}^{k-1} (-1)^i \binom{k-i}{k} \binom{k}{i} = -1 \quad \text{if } k \geq 2,$$

$$(3.4) \quad \sum_{i=2}^k (-1)^i (k-i+1) \binom{k}{i-1} = k \quad \text{if } k \geq 2,$$

$$(3.5) \quad \sum_{i=2}^j (-1)^i (k-i+1) \binom{j-1}{i-1} = k \quad \text{if } 3 \leq j \leq k.$$

LEMMA 3.6. Let  $K$  be a graph with  $|\mathcal{V}(K)| = s \geq 1$ , and let  $k$  be an integer with  $k \geq 2$ , and suppose  $H = \bar{\Gamma}_k + K$ , then:

$$D(H, \mathcal{V}(\bar{\Gamma}_k)) = 2s.$$

*Proof.* For  $1 \leq i \leq k-1$  and  $B \subseteq \mathcal{V}(\bar{\Gamma}_k)$  with  $|B| = i$ , each member of  $B$  is adjacent to every member of  $\mathcal{V}(K)$ . There are  $\binom{k}{i}$  choices for  $B$  and  $|\mathcal{V}(K)| = s$ ; thus:

$$S_i(\rho_H, \mathcal{V}(\bar{\Gamma}_k)) = s \binom{k}{i}.$$

This gives:

$$\begin{aligned} D(H, \mathcal{V}(\bar{\Gamma}_k)) &= \frac{s}{k} \binom{k}{1} - \sum_{i=1}^{k-1} (-1)^i s \binom{k-i}{k} \binom{k}{i} \\ &= s \left[ 1 - \sum_{i=1}^{k-1} (-1)^i \binom{k-i}{k} \binom{k}{i} \right] \\ &= 2s, \quad \text{using (3.3).} \end{aligned}$$

**THEOREM 3.7.** For  $1 \leq k \leq n$ ,

$$(3.8) \quad g(n, k) \leq \frac{1}{2}(n-1)(n-k-1) + 1.$$

*Proof.* Let  $G$  be a graph with  $|\mathcal{V}(G)| = n$ , and  $|\mathcal{E}(G)| \geq \frac{1}{2}(n-1)(n-k-1) + 1$ . Suppose  $\mu(G) > k$  and  $X = \{x_1, x_2, \dots, x_k, x_{k+1}\}$  is a set of mutually nonadjacent vertices of  $G$ .

$G$  may be considered to have been obtained from the complete graph  $\Gamma_n$  through the elimination of at most:

$$\frac{1}{2}n(n-1) - \frac{1}{2}(n-1)(n-k-1) - 1 = \frac{1}{2}(n-1)(k+1) - 1$$

edges.  $\frac{1}{2}k(k+1)$  are removed in obtaining, from  $\Gamma_n$ , the graph  $H$  in which only members of  $X$  are nonadjacent. Thus, to obtain  $G$  from  $H$ , at most:

$$(3.9) \quad \frac{1}{2}(n-1)(k+1) - 1 - \frac{1}{2}k(k+1) = \frac{1}{2}(n-k-1)(k+1) - 1$$

edges are removed from  $H$ .

We wish to compute  $D(G, X)$ . By Lemma 3.6,

$$(3.10) \quad D(H, X) = 2(n-k-1).$$

Now suppose that at some stage in the transformation from  $H$  to  $G$ , we have obtained a graph  $K$  with  $\mathcal{E}(H) \supseteq \mathcal{E}(K) \supseteq \mathcal{E}(G)$  and  $\mathcal{V}(K) = \mathcal{V}(H) = \mathcal{V}(G)$ . Let  $L = K - e$  where  $e \in \mathcal{E}(K) - \mathcal{E}(G)$ . We wish to know the effect,  $f(e) = D(L, X) - D(K, X)$ , on  $D$ , of removing the edge  $e$ . Since  $e$  is an edge of  $H$ , it cannot join two points of  $X$ . If neither end-point of  $e$  is in  $X$ , then  $f(e) = 0$  since  $S_i(\rho_K, X) = S_i(\rho_L, X)$  for  $1 \leq i \leq k$ . Now suppose that one end-point,  $y_1$ , of  $e$  is in  $X$  and that the other end-point,  $v$ , is not in  $X$ . Let  $y_1, y_2, \dots, y_j$  be the points of  $X$  which are adjacent to  $v$  in the graph  $K$ . Note that  $1 \leq j \leq k+1$ .



If  $1 \leq i \leq j$  and  $B \subseteq \{y_2, y_3, \dots, y_j\}$  with  $|B| = i - 1$ , and  $C = B \cup \{y_1\}$ , then  $|C| = i$  and  $v$  is adjacent to every member of  $C$  in the graph  $K$  but not in the graph  $L$ . There are  $\binom{j-1}{i-1}$  choices for such a set  $C$ . Furthermore, for any other combination of a vertex,  $t$ , and a set  $A \subseteq X$  with  $|A| = i$ ,  $t$  is adjacent to every member of  $A$  in the graph  $L$ . Thus:

$$S_i(\rho_L, X) - S_i(\rho_K, X) = \begin{cases} -\binom{j-1}{i-1} & \text{for } i \leq i \leq j \\ 0 & \text{for } i > j. \end{cases}$$

This gives:

$$f_j = f(e) = D(L, X) - D(K, X)$$

$$= \begin{cases} -\left[ \frac{1}{k+1} - \sum_{i=1}^k (-1)^i \binom{k-i+1}{k+1} \binom{k}{i-1} \right] & \text{if } j = k+1 \\ -\left[ \frac{1}{k+1} - \sum_{i=1}^j (-1)^i \binom{k-i+1}{k+1} \binom{j-1}{i-1} \right] & \text{if } 1 \leq j \leq k \end{cases}$$

$$= \begin{cases} -\frac{1}{k+1} \left[ k+1 - \sum_{i=2}^k (-1)^i (k-i+1) \binom{k}{i-1} \right] & \text{if } j = k+1 \\ -\frac{1}{k+1} \left[ k+1 - \sum_{i=2}^k (-1)^i (k-i+1) \binom{j-1}{i-1} \right] & \text{if } 2 \leq j \leq k \\ -1 & \text{if } j = 1 \end{cases}$$

$$= \begin{cases} -\frac{1}{k+1} & \text{if } 3 \leq j \leq k+1 \\ -\frac{2}{k+1} & \text{if } j = 2 \\ -1 & \text{if } j = 1 \end{cases}$$

using (3.4) and (3.5).

Notice that  $f_1 \leq f_2 \leq \dots \leq f_k \leq f_{k+1} < 0$  and that in order to realize the effect  $f_j$ , edges with effects  $f_{k+1}, f_k, \dots, f_{j+1}$  must first be removed. Hence when  $(k + 1)$  edges are removed, the combined effect is at least:

$$\sum_{i=1}^{k+1} f_i = -2.$$

So if  $r$  edges are removed in obtaining  $G$  from  $H$ ,

$$(3.11) \quad D(G, X) - D(H, X) \cong -\frac{2r}{k+1}.$$

Using (3.9) and (3.10) in (3.11) now gives:

$$(3.12) \quad D(G, X) \cong [2(n-k-1) - (n-k-1) + 2/(k+1)] > n-k-1.$$

But Theorem 2.5 guarantees the existence of a set  $X$  as constructed above, and satisfying:

$$D(G, X) \leq n - \mu(G) \leq n - k - 1.$$

This contradicts (3.12) and completes the proof of the theorem.

**COROLLARY 3.13.** For  $n \geq 4$ ,  $g(n, n-3) = n$ .

*Proof.* The bipartite graph  $\Gamma_{1, n-1}$  is a graph with  $n$  vertices,  $(n-1)$  edges and path-covering number  $(n-2)$ . Thus  $g(n, n-3) \geq n$ . The reverse inequality is given by Theorem 3.7.

To obtain a lower bound for  $g(n, k)$ , consider the graph  $G = \Gamma_{n-k} \cup \bar{\Gamma}_k$ ; then  $\mu(G) = k+1$ , while  $|\mathcal{V}(G)| = n$  and  $|\mathcal{E}(G)| = \frac{1}{2}(n-1)(n-k-1)$ . This gives:

**PROPOSITION 3.14.** For  $n > k \geq 1$

$$(3.15) \quad g(n, k) \geq \frac{1}{2}(n-k)(n-k-1) + 1.$$

The following proposition gives some results that are easily verified:

**PROPOSITION 3.15.**

- (i)  $g(n, n) = 0$ ,  $g(n+1, n) = 1$ ,  $g(n+2, n) = 2$  for  $n \geq 1$
- (ii)  $g(6, 2) = 7$
- (iii)  $g(n+1, k+1) \geq g(n, k)$  for  $n \geq k \geq 1$ .

Part (iii) can be seen by letting  $G = H \cup \{x\}$  where  $H$  is a graph with  $n$  vertices,  $g(n, k) - 1$  edges, and  $\mu(H) = k+1$ , and  $x$  is an isolated vertex with  $x \notin \mathcal{V}(H)$ . Then  $G$  has  $(n+1)$  vertices,  $g(n, k) - 1$  edges, and  $(G) = k+2$ .

In the case  $k = 1$ , the upper bound in (3.8) is seen to be the same as the lower bound in (3.15) and hence equality holds for  $g(n, k)$  in both inequalities. However, Corollary 3.13 shows that the upper bound in (3.8) and not the lower bound in (3.15) is achieved in the case  $k = n - 3$ . Part (ii) of Proposition 3.15 shows a case where the lower bound and not the upper bound is achieved. It is conjectured that for small values of  $k$ ,  $g(n, k)$  is close to the lower bound in (3.15), while for large values of  $k$ ,  $g(n, k)$  is closer to the upper bound in (3.8).

We now turn to another extremal problem. Let  $v$  and  $n$  be integers with  $0 \leq v \leq n$ . Define:

$$h(n, v) = \text{Min}\{k : \text{every graph, } G, \text{ with } |\mathcal{V}(G)| = n \text{ and } \rho(x) \geq v \\ \text{for every } x \in \mathcal{V}(G), \text{ has } \mu(G) \leq k\}.$$

THEOREM 3.16.

$$(n, v) = \begin{cases} 1 & \text{if } v \geq \frac{n}{2} \\ n - 2v & \text{if } v < \frac{n}{2}. \end{cases}$$

*Proof.* The case  $v \geq \frac{n}{2}$  and the upper bound  $h(n, v) \leq n - 2v$  if  $v < \frac{n}{2}$  follows from 0. Ore's result (the note to Theorem 2.5). If  $v < \frac{n}{2}$ , let  $K = \Gamma_{v, n-v}$ . Then clearly  $|\mathcal{V}(K)| = n$  and  $\rho(x) \geq v$  for every  $x \in \mathcal{V}(G)$ ; and in [2] (Theorem 2.2.10) we show that  $\mu(K) = n - 2v$ . Hence

$$h(n, v) \geq n - 2v$$

completing the proof of the theorem.

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## CONTENTS

Zvi Artstein and John A. Burns, <i>Integration of compact set-valued functions</i> .....	297
J. A. Beachy and W. D. Blair, <i>Rings whose faithful left ideals are cofaithful</i> .....	1
Mark Benard, <i>Characters and Schur indices of the unitary reflection group [321]<sup>3</sup></i> .....	309
H. L. Bentley and B. J. Taylor, <i>Wallman rings</i> .....	15
E. Berman, <i>Matrix rings over polynomial identity rings II</i> .....	37
Simeon M. Berman, <i>A new characterization of characteristic functions of absolutely continuous distributions</i> .....	323
Monte B. Boisen, Jr. and Philip B. Sheldon, <i>Pre-Prüfer rings</i> .....	331
A. K. Boyle and K. R. Goodearl, <i>Rings over which certain modules are injective</i> .....	43
J. L. Brenner, R. M. Crabwell and J. Riddell, <i>Covering theorems for finite nonabelian simple groups. V</i> .....	55
H. H. Brungs, <i>Three questions on duo rings</i> .....	345
Iracema M. Bund, <i>Birnbaum-Orlicz spaces of functions on groups</i> ....	351
John D. Elwin and Donald R. Short, <i>Branched immersions between 2-manifolds of higher topological type</i> .....	361
J. K. Finch, <i>The single valued extension property on a Banach space</i> .....	61
J. R. Fisher, <i>A Goldie theorem for differentially prime rings</i> .....	71
Eric M. Friedlander, <i>Extension functions for rank 2, torsion free abelian groups</i> .....	371
J. Froemke and R. Quackenbusch, <i>The spectrum of an equational class of groupoids</i> .....	381
B. J. Gardner, <i>Radicals of supplementary semilattice sums of associative rings</i> .....	387
Shmuel Glasner, <i>Relatively invariant measures</i> .....	393
G. R. Gordh, Jr. and Sibe Mardešić, <i>Characterizing local connectedness in inverse limits</i> .....	411
S. Graf, <i>On the existence of strong liftings in second countable topological spaces</i> .....	419
S. Gudder and D. Strawther, <i>Orthogonally additive and orthogonally increasing functions on vector spaces</i> .....	427
F. Hansen, <i>On one-sided prime ideals</i> .....	79
D. J. Hartfiel and C. J. Maxson, <i>A characterization of the maximal monoids and maximal groups in <math>\beta x</math></i> .....	437
Robert E. Hartwig and S. Brent Morris, <i>The universal flip matrix and the generalized faro-shuffle</i> .....	445

John Allen Beachy and William David Blair, <i>Rings whose faithful left ideals are cofaithful</i> .....	1
Herschel Lamar Bentley and Barbara June Taylor, <i>Wallman rings</i> .....	15
Elizabeth Berman, <i>Matrix rings over polynomial identity rings. II</i> .....	37
Ann K. Boyle and Kenneth R. Goodearl, <i>Rings over which certain modules are injective</i> .....	43
J. L. Brenner, Robert Myrl Cranwell and James Riddell, <i>Covering theorems for finite nonabelian simple groups. V</i> .....	55
James Kenneth Finch, <i>The single valued extension property on a Banach space</i> .....	61
John Robert Fisher, <i>A Goldie theorem for differentially prime rings</i> .....	71
Friedhelm Hansen, <i>On one-sided prime ideals</i> .....	79
Jon Craig Helton, <i>Product integrals and the solution of integral equations</i> .....	87
Barry E. Johnson and James Patrick Williams, <i>The range of a normal derivation</i> .....	105
Kurt Kreith, <i>A dynamical criterion for conjugate points</i> .....	123
Robert Allen McCoy, <i>Baire spaces and hyperspaces</i> .....	133
John McDonald, <i>Isometries of the disk algebra</i> .....	143
H. Minc, <i>Doubly stochastic matrices with minimal permanents</i> .....	155
Shahbaz Noorvash, <i>Covering the vertices of a graph by vertex-disjoint paths</i> .....	159
Theodore Windle Palmer, <i>Jordan *-homomorphisms between reduced Banach *-algebras</i> .....	169
Donald Steven Passman, <i>On the semisimplicity of group rings of some locally finite groups</i> .....	179
Mario Petrich, <i>Varieties of orthodox bands of groups</i> .....	209
Robert Horace Redfield, <i>The generalized interval topology on distributive lattices</i> .....	219
James Wilson Stepp, <i>Algebraic maximal semilattices</i> .....	243
Patrick Noble Stewart, <i>A sheaf theoretic representation of rings with Boolean orthogonalities</i> .....	249
Ting-On To and Kai Wing Yip, <i>A generalized Jensen's inequality</i> .....	255
Arnold Lewis Villone, <i>Second order differential operators with self-adjoint extensions</i> .....	261
Martin E. Walter, <i>On the structure of the Fourier-Stieltjes algebra</i> .....	267
John Wermer, <i>Subharmonicity and hulls</i> .....	283
Edythe Parker Woodruff, <i>A map of <math>E^3</math> onto <math>E^3</math> taking no disk onto a disk</i> .....	291