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A SHEAF THEORETIC REPRESENTATION OF RINGS WITH BOOLEAN ORTHOGONALITIES

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It is shown that certain associative rings with Boolean orthogonality are isomorphic to rings of global sections.

Let A be a ring and \perp a relation on A . For each subset S of A define

$$S^\perp = \{x \in A \mid x \perp s \text{ for all } s \in S\} \text{ and } S^{\perp\perp} = (S^\perp)^\perp.$$

When $S = \{s\}$ we write s^\perp and $s^{\perp\perp}$ instead of $\{s\}^\perp$ and $\{s\}^{\perp\perp}$. Subsets of A of the form S^\perp are *polars*. The relation \perp is a *Boolean orthogonality* if all polars are two-sided ideals and if, for all $x, y \in A$,

1. $x \perp y \rightarrow y \perp x$, 2. $x \perp x \rightarrow x = 0$, and
3. $x^{\perp\perp} \cap y^{\perp\perp} = (0) \rightarrow x \perp y$.

The set of polars is a Boolean algebra (see [3]) with meet and join defined by

$$B \wedge C = B \cap C \quad \text{and} \quad B \vee C = (B^\perp \wedge C^\perp)^\perp.$$

Boolean orthogonality have been studied by Davis [3], Cornish [1] and by Cornish and Stewart [2].

Throughout this paper we shall assume that A is an associative ring with an identity and with a Boolean orthogonality \perp . We shall also assume that the following *finiteness condition* is satisfied:

for any two elements $x, y \in A$ there is a finite set $F \subseteq A$ such that $x^{\perp\perp} \wedge y^{\perp\perp} = F^{\perp\perp}$.

Notice that if $F = \{f_1, \dots, f_n\}$, then $F^\perp = f_1^\perp \wedge \dots \wedge f_n^\perp$ and $F^{\perp\perp} = f_1^{\perp\perp} \vee \dots \vee f_n^{\perp\perp}$.

An ideal I of A is a \perp -ideal if $F^{\perp\perp} \subseteq I$ for every finite set $F \subseteq I$, and I is \perp -prime if $I \neq A$ and whenever the intersection of two polars B and C is contained in I , either $B \subseteq I$ or $C \subseteq I$.

LEMMA. *Assume that P is either a \perp -prime ideal or $P = A$, that I is a \perp -ideal and that $x \in A \setminus I$ is such that $x^{\perp\perp} \wedge x^{\perp\perp} \subseteq I$ implies that $a \in P$. Then there is a \perp -prime \perp -ideal Q such that $I \subseteq Q \subseteq P$ and $x \notin Q$.*

Proof. Using Zorn's Lemma select a \perp -ideal $Q \supseteq I$ maximal with respect to the property " $x \notin Q$ and $x^{\perp\perp} \wedge x^{\perp\perp} \subseteq Q$ implies that $a \in P$ ". Clearly $I \subseteq Q \subseteq P$.

Suppose that B and C are polars neither of which is contained in Q . Choose $b \in B \setminus Q$ and $c \in C \setminus Q$. Then

$$B' = \cup \{F^{\perp\perp} \mid F \text{ is a finite subset of } \{b\} \cup Q\}$$

and

$$C' = \cup \{G^{\perp\perp} \mid G \text{ is a finite subset of } \{c\} \cup Q\}$$

are \perp -ideals which properly contain Q . By the maximality of Q either $x^{\perp\perp} \subseteq B'$ or $x^{\perp\perp} \wedge b_1^{\perp\perp} \subseteq B'$ for some $b_1 \in A \setminus P$, and $x^{\perp\perp} \subseteq C'$ or $x^{\perp\perp} \wedge c_1^{\perp\perp} \subseteq C'$ for some $c_1 \in A \setminus P$. Thus we obtain finite sets $\{b, f_1, \dots, f_n\} \subseteq \{b\} \cup Q$ and $\{c, g_1, \dots, g_m\} \subseteq \{c\} \cup Q$ such that one of $x^{\perp\perp}$, $x^{\perp\perp} \wedge b_1^{\perp\perp}$, $x^{\perp\perp} \wedge c_1^{\perp\perp}$ or $x^{\perp\perp} \wedge b_1^{\perp\perp} \wedge c_1^{\perp\perp}$ is contained in

$$\begin{aligned} & \{b, f_1, \dots, f_n\}^{\perp\perp} \wedge \{c, g_1, \dots, g_m\}^{\perp\perp} \\ &= (b^{\perp\perp} \vee f_1^{\perp\perp} \vee \dots \vee f_n^{\perp\perp}) \wedge (c^{\perp\perp} \vee g_1^{\perp\perp} \vee \dots \vee g_m^{\perp\perp}) \\ &= (b^{\perp\perp} \wedge c^{\perp\perp}) \vee H^{\perp\perp} \end{aligned}$$

where H is a finite subset of Q (we have used the distributivity of the Boolean algebra of polars and also the finiteness condition). If $b^{\perp\perp} \wedge c^{\perp\perp} \subseteq Q$, then $x^{\perp\perp} \subseteq Q$ or $x^{\perp\perp} \wedge d^{\perp\perp} \subseteq Q$ for some $d \in A \setminus P$ both of which contradict the choice of Q . Thus $B \cap C \not\subseteq Q$ and we conclude that Q is \perp -prime.

For the remainder of this paper \bar{X} will be fixed set of \perp -prime ideals which contains all \perp -prime \perp -ideals and which is *full* (that is, if I is a sum of polars and $I \neq A$, then $I \subseteq P$ for some $P \in \bar{X}$).

PROPOSITION 2. (Cornish [1]). *For each $P \in \bar{X}$,*

$$\{x \in A \mid x^{\perp} \not\subseteq P\} = \cap \{R \in \bar{X} \mid R \subseteq P\} = \cap \{Q \in \bar{X} \mid Q \subseteq P$$

and Q is a \perp -prime \perp -ideal\}.

Proof. Suppose that $x^{\perp} \not\subseteq P$ and R is a \perp -prime ideal contained in P . Then $x^{\perp\perp} \wedge x^{\perp} = (0) \subseteq R$ and so $x \in R$.

If $x^{\perp} \subseteq P$, then by Lemma 1 (take $I = x^{\perp}$) there is a \perp -prime \perp -ideal $Q \subseteq P$ such that $x \notin Q$. This establishes the result.

The set described in the proposition will be denoted by O_p . We note that $O_p = P$ if and only if P is minimal in \bar{X} .

Let $P \in \bar{X}$. The set O_p , being an intersection of \perp -ideals, is itself a \perp -ideal. Define a relation (also denoted by \perp) on A/O_p by

$$(x + O_p) \perp (y + O_p) \leftrightarrow x^{\perp\perp} \wedge y^{\perp\perp} \subseteq O_p.$$

This relation is well-defined because if $x_1 = x + a$ and $y_1 = y + b$ where $a, b \in O_p$, then

$$x_1^{\perp\perp} \wedge y_1^{\perp\perp} = (x + a)^{\perp\perp} \wedge (y + b)^{\perp\perp} \subseteq (x^{\perp\perp} \vee a^{\perp\perp}) \wedge (y^{\perp\perp} \vee b^{\perp\perp})$$

and so $x_1^{\perp\perp} \wedge y_1^{\perp\perp} \subseteq (x^{\perp\perp} \wedge y^{\perp\perp}) \vee F^{\perp\perp}$ where F is a finite subset of O_p . It is routine to check that

$$x^{\perp} + O_p \subseteq (x + O_p)^{\perp} \quad \text{and} \quad x^{\perp\perp} + O_p \subseteq (x + O_p)^{\perp\perp}$$

for each $x \in A$, and that the relation \perp is a Boolean orthogonality on A/O_p .

PROPOSITION 3. *For each $P \in \bar{X}$, $\bar{P} = P/O_p$ is a \perp -prime ideal of A/O_p which contains all proper polars of A/O_p .*

Proof. Let \bar{B} and \bar{C} be polars in A/O_p such that $\bar{B} \cap \bar{C} \subseteq \bar{P}$. Suppose that $\bar{B} \not\subseteq \bar{P}$. Then there is an element $b \in A$ such that $b + O_p \in \bar{B} \setminus \bar{P}$. Let $c + O_p \in \bar{C}$. Then

$$(b^{\perp\perp} + O_p) \cap (c^{\perp\perp} + O_p) \subseteq (b + O_p)^{\perp\perp} \cap (c + O_p)^{\perp\perp} \subseteq \bar{B} \cap \bar{C} \subseteq \bar{P}$$

and so $b^{\perp\perp} \cap c^{\perp\perp} \subseteq P$. Since $b \notin P$ we conclude that $c \in P$ and so $\bar{C} \subseteq \bar{P}$. Thus \bar{P} is \perp -prime.

Suppose that $a^{\perp\perp} \wedge b^{\perp\perp} \subseteq O_p$. Then there is a finite set $\{f_1, \dots, f_n\} \subseteq O_p$ such that

$$a^{\perp\perp} \wedge b^{\perp\perp} = \{f_1, \dots, f_n\}^{\perp\perp} = f_1^{\perp\perp} \vee \dots \vee f_n^{\perp\perp}.$$

For each $i = 1, \dots, n$, $f_i \in O_p$ and so $f_i^{\perp} \not\subseteq P$. Thus $f_1^{\perp} \wedge \dots \wedge f_n^{\perp} \not\subseteq P$. Also, $b^{\perp\perp} \wedge f_1^{\perp} \wedge \dots \wedge f_n^{\perp} \subseteq a^{\perp}$ because $a^{\perp\perp} \wedge b^{\perp\perp} \wedge f_1^{\perp} \wedge \dots \wedge f_n^{\perp} = (0)$. If $a \notin O_p$, then $a^{\perp} \subseteq P$ and so, since $f_1^{\perp} \wedge \dots \wedge f_n^{\perp} \not\subseteq P$, $b^{\perp\perp} \subseteq P$. Thus \bar{P} contains $(a + O_p)^{\perp}$ for all $a \notin O_p$. It follows that \bar{P} contains all proper polars of A/O_p .

Let S be the disjoint union of the factor rings A/O_P . The relation (also denoted by \perp) on the product

$$\Pi\{A/O_P \mid P \in \bar{X}\}$$

$$= \{f: \bar{X} \rightarrow S \mid f(P) \in A/O_P \text{ for all } P \in \bar{X}\}$$

defined by

$$f \perp g \leftrightarrow f(P) \perp g(P) \text{ in } A/O_P \text{ for all } P \in \bar{X}$$

is a Boolean orthogonality. Each $a \in A$ determines a function $\hat{a} \in \Pi\{A/O_P \mid P \in \bar{X}\}$ defined by $\hat{a}(P) = a + O_P$. It follows from Lemma 1 that $\cap\{P \mid P \text{ is a } \perp\text{-prime } \perp\text{-ideal}\} = (0)$ and so $\cap\{O_P \mid P \in \bar{X}\} = (0)$. Thus we obtain the usual embedding

$$A \xrightarrow{\sim} \hat{A} \subseteq \Pi\{A/O_P \mid P \in \bar{X}\}.$$

This embedding respects orthogonalities; that is, $a \perp b$ in A if and only if $\hat{a} \perp \hat{b}$ in the product.

We define a topology on \bar{X} by declaring the basic open sets to be the subsets of the form

$$\bar{X}(a) = \{P \in \bar{X} \mid a^{\perp\perp} \not\subseteq P\}.$$

Notice that $\bar{X}(a) \cap \bar{X}(b) \supseteq \bar{X}(c)$ for all $c \in a^{\perp\perp} \wedge b^{\perp\perp}$ and so these sets do qualify as a topological base.

Suppose that $\{\bar{X}(a) \mid a \in C\}$ is a cover of \bar{X} consisting of basic open sets. Then $\Sigma\{a^{\perp\perp} \mid a \in C\} = A$ because \bar{X} is full. Since A has an identity there is a finite set $F \subseteq C$ such that $\Sigma\{a^{\perp\perp} \mid a \in F\} = A$. Thus $\{\bar{X}(a) \mid a \in F\}$ covers \bar{X} and so \bar{X} is quasi-compact.

Give S the topology generated by sets of the form $\hat{a}[U] = \{a + O_P \mid P \in U\}$ where U is open in \bar{X} and $a \in A$. We obtain a sheaf of rings (S, π, \bar{X}) where $\pi: S \rightarrow \bar{X}$ is the projection onto \bar{X} .

Let $\Gamma = \{f \mid f \in \Pi\{A/O_P \mid P \in \bar{X}\} \text{ is continuous}\}$ be the ring of global sections. The following observation shows that $\hat{A} \subseteq \Gamma$: for all $x, y \in A$, $\{P \in \bar{X} \mid x - y \in O_P\}$ is open in \bar{X} . To see this notice that if $x - y \in O_Q$, then $Q \in \bar{X}(u) \subseteq \{P \in \bar{X} \mid x - y \in O_P\}$ where u is any element in $(x - y)^{\perp} \setminus Q$.

THEOREM 4. $\hat{A} = \Gamma$.

Proof. Let $f \in \Gamma$. Since \bar{X} is quasi-compact there are finite sets $\{a_1, \dots, a_n\}$ and $\{v_1, \dots, v_n\}$ such that $\bar{X} = \bar{X}(a_1) \cup \dots \cup \bar{X}(a_n)$ and $f(P) = v_i + O_P$ for all $P \in \bar{X}(a_i)$.

Notice that $v_i - v_j \in \cap\{O_P \mid P \in \bar{X}(a_i) \cap \bar{X}(a_j)\}$, so $(v_i - v_j)^{\perp} \subseteq$

$Q \in \bar{X}$ implies that $a_i^{\perp\perp} \wedge a_j^{\perp\perp} \subseteq Q$. It follows from Lemma 1 (take $P = A$ and $I = (v_i - v_j)^\perp$ for each $x \notin (v_i - v_j)^\perp$) that

$$(v_i - v_j)^\perp = \cap \{Q \mid (v_i - v_j)^\perp \subseteq Q \in \bar{X}\}$$

and so $a_i^{\perp\perp} \wedge a_j^{\perp\perp} \subseteq (v_i - v_j)^\perp$. Thus $(v_i - v_j)^{\perp\perp} \wedge a_i^{\perp\perp} \subseteq a_j^\perp$.

Since $\bar{X} = \bar{X}(a_1) \cup \dots \cup \bar{X}(a_n)$ and \bar{X} is full, $a_1^{\perp\perp} + \dots + a_n^{\perp\perp} = A$. Choose $u_i \in a_i^{\perp\perp}$ such that $1 = u_1 + \dots + u_n$ and let $v = u_1 v_1 + \dots + u_n v_n$. Then

$$v - v_j = u_1(v_1 - v_j) + \dots + u_n(v_n - v_j) \in a_j^\perp \subseteq O_P$$

for all $P \in \bar{X}(a_i)$. Thus $f(P) = v_j + O_P = v + O_P$ for all $P \in \bar{X}(a_i)$ and so $f = \hat{v} \in \hat{A}$.

f-rings (Keimal [4]). Let A be an f -ring with identity. The relation defined by $x \perp y \leftrightarrow |x| \wedge |y| = 0$ is a Boolean orthogonality and $x^{\perp\perp} \wedge y^{\perp\perp} = (|x| \wedge |y|)^{\perp\perp}$. Let \bar{X} be the set of irreducible ℓ -ideals. Then \bar{X} is full because polars are ℓ -ideals and sums of ℓ -ideals are again ℓ -ideals. Also, all \perp -prime \perp -ideals are irreducible ℓ -ideals and so A is isomorphic to the f -ring of all global sections of the sheaf (S, π, \bar{X}) .

Reduced rings (Koh [5]). Let A be a ring with identity and no nonzero nilpotent elements. The relation defined by $x \perp y \leftrightarrow xy = 0$ is a Boolean orthogonality and $x^{\perp\perp} \wedge y^{\perp\perp} = (xy)^{\perp\perp}$. Let \bar{X} be the set of all prime ideals of A . Clearly \bar{X} is full. Also, all \perp -prime \perp -ideals are completely prime and so A is isomorphic to the ring of global sections of the sheaf (S, π, \bar{X}) . Each stalk A/O_P is reduced (Proposition 2) and the prime ideal P/O_P contains all zero divisors (Proposition 3).

Semiprime rings. Let A be a semiprime ring with identity. The relation defined by $x \perp y \leftrightarrow (x)(y) = (0)$ is a Boolean orthogonality. However, the finiteness condition may not be satisfied as the following example shows.

Let R be a semiprime ring with identity, R' the ring of 3×3 matrices with entries from R ,

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Define \bar{x} and \bar{y} in $P = \Pi\{R_n \mid R_n = R' \text{ for } n = 1, 2, \dots\}$ by

$$\bar{x}(n) = \begin{cases} x & \text{if } n \equiv 1 \pmod{2} \\ 0 & \text{if } n \not\equiv 1 \pmod{2} \end{cases}$$

$$\bar{y}(n) = \begin{cases} y & \text{if } n \equiv 1 \pmod{3} \\ 0 & \text{if } n \not\equiv 1 \pmod{3}. \end{cases}$$

Notice that $\bar{x}\bar{y} = \bar{y}\bar{x} = \bar{x}^2 = \bar{y}^2 = 0$. Let E be the subring of P which is generated by the identity of P , \bar{x} , \bar{y} and

$$\begin{aligned} \Sigma\{R_n \mid R_n = R' \text{ for } n = 1, 2, \dots\}. \text{ Then} \\ \bar{x}^{\perp\perp} = \{f \in E \mid f(n) = 0 \text{ for } n \not\equiv 1 \pmod{2}\}, \\ \bar{y}^{\perp\perp} = \{f \in E \mid f(n) = 0 \text{ for } n \not\equiv 1 \pmod{3}\}, \end{aligned}$$

and so

$$\bar{x}^{\perp\perp} \wedge \bar{y}^{\perp\perp} = \{f \in E \mid f(n) = 0 \text{ for } n \not\equiv 1 \pmod{6}\}.$$

If $\bar{x}^{\perp\perp} \wedge \bar{y}^{\perp\perp} = \{f_1, \dots, f_n\}^{\perp\perp}$, then at least one of the f_i must satisfy $f_i(n) \neq 0$ for infinitely many positive integers n . But then there are integers α, β and γ such that $f_i(n) = (\alpha + \beta\bar{x} + \gamma\bar{y})(n)$ for all but a finite number of positive integers n . This is incompatible with the requirement that $f_i(n) = 0$ for $n \not\equiv 1 \pmod{6}$.

When the finiteness condition is satisfied (for instance, when A satisfies the maximum condition on annihilators), A is isomorphic to the ring of all global sections of the sheaf (S, π, \bar{X}) where \bar{X} is the set of prime ideals of A . Each stalk A/O_P is semiprime (Proposition 1) and the prime ideal P/O_P contains all two-sided annihilator ideals of A/O_P (Proposition 2).

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