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A GENERALIZED JENSEN'S INEQUALITY

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A generalized Jensen's inequality for conditional expectations of Bochner-integrable functions which extends the results of Dubins and Scalora is proved using a different method.

1. Introduction. Let (Ω, \mathbf{F}, P) be a probability space, $(\mathbf{U}, \|\cdot\|)$ a complex (or real) Banach space and $(\mathbf{V}, \|\cdot\|, \cong_v)$ an ordered Banach space over the complex (or real) field such that the positive cone $\{v \in \mathbf{V} : v \cong_v \theta\}$ is closed. Let x be a Bochner-integrable function on (Ω, \mathbf{F}, P) to \mathbf{U} . Let \mathbf{G} be a sub- σ -field of the σ -field \mathbf{F} and let f be a function on $\Omega \times \mathbf{U}$ to \mathbf{V} such that for each $u \in \mathbf{U}$ the function $f(\cdot, u)$ is strongly measurable with respect to \mathbf{G} and such that for each $\omega \in \Omega$ the function $f(\omega, \cdot)$ is continuous and convex in the sense that $tf(\omega, u_1) + (1-t)f(\omega, u_2) \cong_v f(\omega, tu_1 + (1-t)u_2)$ whenever $u_1, u_2 \in \mathbf{U}$ and $0 \leq t \leq 1$. For any Bochner-integrable function z on (Ω, \mathbf{F}, P) to any Banach space \mathbf{W} , we define $E[z | \mathbf{G}]$ "a conditional expectation of z relative to \mathbf{G} " as a Bochner-integrable function on (Ω, \mathbf{F}, P) to \mathbf{W} such that $E(z | \mathbf{G})$ is strongly measurable with respect to \mathbf{G} and that

$$\int_A E[z | \mathbf{G}](\omega) dP = \int_A z(\omega) dP, \quad A \in \mathbf{G},$$

where the integrals are Bochner-integrals.

The purpose of this note is to prove the following generalized Jensen's inequality:

THEOREM. *If $f(\cdot, x(\cdot))$ is Bochner-integrable, then*

$$(J) \quad E[f(\cdot, x(\cdot)) | \mathbf{G}](\omega) \cong_v f(\omega, E[x | \mathbf{G}](\omega)) \quad \text{a.e.}$$

The above theorem extends the results of Dubins [2] (cf. Mayer [5, p. 79]) and Scalora [6, p. 360, Theorem 2.3]. It is proved in [2] that the theorem is true for the case in which the spaces \mathbf{U} and \mathbf{V} are both the real numbers \mathbf{R} , while in [6] Scalora uses the methods of Hille-Phillips [4] to prove the theorem when the function $f(\omega, u)$ is replaced by a continuous, subadditive positive-homogeneous function $g(u)$ on \mathbf{U} to \mathbf{V} . It should be noted that the method of the proof used here is different than those used previously, the previous methods appear to be ineffective for a proof of the extension.

2. Preliminaries. We refer to [4] and [6] for the definitions and basic properties of the concepts of Bochner-integrals and the conditional expectation of a Bochner-integrable function. Our proof of the theorem is based on the following lemmas. Unless otherwise specified, functions in Lemma 1–5 are defined on (Ω, \mathbf{F}, P) to U .

LEMMA 1. ([4, p. 73, Corollary 1]). *A function is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countably-valued functions.*

LEMMA 2. (Egoroff's theorem, [4, p. 72] or [3, p. 149]). *A sequence $\{z_j\}_{j=1}^{\infty}$ of strongly measurable functions is almost uniformly convergent to a function z if and only if*

$$\|z_i(\omega) - z(\omega)\| \rightarrow 0 \text{ a.e. as } i \rightarrow \infty.$$

The following lemma is an immediate consequence of Lemma 1 and Lemma 2.

LEMMA 3. *If z is a strongly measurable function, then for any positive number M there exists a sequence $\{z_i\}_{i=1}^{\infty}$ of simple functions such that $\|z_i(\omega)\| \leq \|z(\omega)\| + M$ a.e., $i = 1, 2, \dots$, and $\|z_i(\omega) - z(\omega)\| \rightarrow 0$ a.e. as $i \rightarrow \infty$.*

LEMMA 4. ([6, p. 356, Theorem 2.2]).

(a) *If $z(\omega) = u$ on Ω then $E[z | \mathbf{G}](\omega) = u$ a.e.*

(b) *If z and $z_i, i = 1, 2, \dots$, are Bochner-integrable functions such that $z(\omega) = \sum_{i=1}^n t_i z_i(\omega)$ a.e. where t_i are scalars then $E[z | \mathbf{G}](\omega) = \sum_{i=1}^n t_i E[z_i | \mathbf{G}](\omega)$ a.e.*

(c) $\|E[z | \mathbf{G}](\omega)\| \leq E[\|z\| | \mathbf{G}](\omega)$ a.e., *for any Bochner-integrable function z .*

(d) *If z is a Bochner-integrable function and $z_i, i = 1, 2, \dots$, are strongly measurable functions such that $\|z_i(\omega) - z(\omega)\| \rightarrow 0$ a.e. as $i \rightarrow \infty$, and if there is a real-valued integrable function $y(\omega) \geq 0$ such that $\|z_i(\omega)\| \leq y(\omega)$ a.e., $i = 1, 2, \dots$, then z_i 's are Bochner-integrable and $\|E[z_i | \mathbf{G}](\omega) - E[z | \mathbf{G}](\omega)\| \rightarrow 0$ a.e. as $i \rightarrow \infty$.*

LEMMA 5. *If z is a Bochner-integrable function and $z_i, i = 1, 2, \dots$, are strongly measurable functions such that $\|z_i(\omega) - z(\omega)\| \rightarrow 0$ uniformly a.e. as $i \rightarrow \infty$, then there exists an integer N such that $z_i, i = N, N + 1, \dots$, are Bochner-integrable functions, and*

$$\|E[z_i | \mathbf{G}](\omega) - E[z | \mathbf{G}](\omega)\| \rightarrow 0 \text{ uniformly}$$

a.e. as $i \rightarrow \infty$.

Proof. An immediate consequence of Lemma 4 and the fact that $E[\cdot | \mathbf{G}]$ is a positive operator on the space of all real-valued integrable functions.

LEMMA 6. *If z is a strongly measure function on (Ω, \mathbf{G}, P) to a Banach space \mathbf{W} , and if on (Ω, \mathbf{F}, P) , y is a numerically-valued integrable function such that zy is a Bochner-integrable function with values in \mathbf{W} , then*

$$E[zy | \mathbf{G}](\omega) = zE[y | \mathbf{G}](\omega) \text{ a.e..}$$

Proof. By using Lemma 3 and Lemma 4, the proof when \mathbf{W} is the real numbers \mathbf{R} as given by Billingsley [1, p. 110, Theorem 10.1] can be applied to obtain the general result.

LEMMA 7. *Let g be a convex function on \mathbf{U} to \mathbf{V} . If $u_i \in \mathbf{U}$ and $t_i \in \mathbf{R}$, $t_i \geq 0$, $i = 1, 2, \dots, n$, such that*

$$\sum_{i=1}^n t_i = 1, \text{ then } \sum_{i=1}^n t_i g(u_i) \geq_v g\left(\sum_{i=1}^n t_i u_i\right).$$

Proof. By induction.

3. Proof of the theorem. We first note that if $F \in \mathbf{F}$ with $P(F) > 0$ and z is a simple function on (Ω, \mathbf{F}, P) to \mathbf{U} such that $\chi_F f(\cdot, z(\cdot))$ is Bochner-integrable, then

$$(1) \quad E[\chi_F f(\cdot, z(\cdot)) | \mathbf{G}](\omega) \geq_v E[\chi_F | \mathbf{G}](\omega) f(\omega, \frac{E[\chi_F z | \mathbf{G}](\omega)}{E[\chi_F | \mathbf{G}](\omega)}) \text{ a.e. on } F.$$

To see this, let $z = \sum_{i=1}^n u_i \chi_{A_i}$, where $u_i \in \mathbf{U}$ and A_i 's are disjoint sets of \mathbf{F} such that $\sum_{i=1}^n \chi_{A_i} = 1$. It is clear that $F \subset \{\omega : E[\chi_F | \mathbf{G}](\omega) > 0\}$ a.e.. Since $f(\cdot, u_i)$ is strongly measurable with respect to \mathbf{G} and $f(\omega, \cdot)$ is convex, by using Lemma 4, (b), Lemma 6 and Lemma 7, we then have

$$\begin{aligned} & \frac{1}{E[\chi_F | \mathbf{G}](\omega)} E[\chi_F f(\cdot, z(\cdot)) | \mathbf{G}](\omega) \\ &= \frac{1}{E[\chi_F | \mathbf{G}](\omega)} \sum_{i=1}^n f(\omega, u_i) E[\chi_F \chi_{A_i} | \mathbf{G}](\omega) \text{ a.e. on } F. \end{aligned}$$

$$\begin{aligned} &\cong_v f\left(\omega, \frac{1}{E[\chi_F | \mathbf{G}](\omega)} \sum_{i=1}^n u_i E[\chi_F \chi_{A_i} | \mathbf{G}](\omega)\right) \text{ a.e. on } F \\ &= f\left(\omega, \frac{E[\chi_{FZ} | \mathbf{G}](\omega)}{E[\chi_F | \mathbf{G}](\omega)}\right) \text{ a.e. on } F. \end{aligned}$$

Nextly, since x is assumed to be a Bochner-integrable function on (Ω, \mathbf{F}, P) to \mathbf{U} , x is strongly measurable, and hence by the definition of strong measurability (or by Lemma 3) there exists a sequence $\{x_i\}_{i=1}^\infty$ of simple functions on (Ω, \mathbf{F}, p) to \mathbf{U} such that $\|x_i(\omega) - x(\omega)\| \rightarrow 0$ a.e.. Furthermore, since $f(\omega, \cdot)$ is continuous on \mathbf{U} it follows that $\|f(\omega, x_i(\omega)) - f(\omega, x(\omega))\| \rightarrow 0$ a.e..

Therefore, by Lemma 2 we can find an increasing sequence, $\Omega_1 \subset \Omega_2 \subset \dots$, of sets of \mathbf{F} with $P(\Omega - \Omega_k) < 1/k$, $k = 1, 2, \dots$, such that

(2) $\|\chi_{\Omega_k}(\omega)x_i(\omega) - \chi_{\Omega_k}(\omega)x(\omega)\| \rightarrow 0$ uniformly a.e. and

(3) $\|\chi_{\Omega_k}(\omega)f(\omega, x_i(\omega)) - \chi_{\Omega_k}(\omega)f(\omega, x(\omega))\| \rightarrow 0$ uniformly a.e., as $i \rightarrow \infty$, for each $k = 1, 2, \dots$.

According to Lemma 5, (2) implies

(2') $\|E[\chi_{\Omega_k} x_i | \mathbf{G}](\omega) - E[\chi_{\Omega_k} x | \mathbf{G}](\omega)\| \rightarrow 0$ uniformly a.e. as $i \rightarrow \infty$, for each $k = 1, 2, \dots$, and (3) implies

(3') $\|E[\chi_{\Omega_k} f(\cdot, x_i(\cdot)) | \mathbf{G}](\omega) - E[\chi_{\Omega_k} f(\cdot, x(\cdot)) | \mathbf{G}](\omega)\| \rightarrow 0$ uniformly a.e. as $i \rightarrow \infty$, for each $k = 1, 2, \dots$.

Now by using the continuity of $f(\omega, \cdot)$ again, it follows from (2') that

$$(4) \quad \left\| f\left(\omega, \frac{E[\chi_{\Omega_k} x_i | \mathbf{G}](\omega)}{E[\chi_{\Omega_k} | \mathbf{G}](\omega)}\right) - f\left(\omega, \frac{E[\chi_{\Omega_k} x | \mathbf{G}](\omega)}{E[\chi_{\Omega_k} | \mathbf{G}](\omega)}\right) \right\| \rightarrow 0$$

a.e. on Ω_k as $i \rightarrow \infty$.

On the other hand, from (1) we obtain

$$(1') \quad E[\chi_{\Omega_k} f(\cdot, x_i(\cdot)) | \mathbf{G}](\omega) \cong_v E[\chi_{\Omega_k} | \mathbf{G}](\omega) f\left(\omega, \frac{E[\chi_{\Omega_k} x_i | \mathbf{G}](\omega)}{E[\chi_{\Omega_k} | \mathbf{G}](\omega)}\right)$$

a.e. on Ω_k , for each $k = 1, 2, \dots$, and each $i = 1, 2, 3, \dots$.

Letting $i \rightarrow \infty$ in (1') and using (3') and (4), we obtain

$$(1'') \quad E[\chi_{\Omega_k} f(\cdot, x(\cdot)) | \mathbf{G}](\omega) \cong_v E[\chi_{\Omega_k} | \mathbf{G}](\omega) f\left(\omega, \frac{E[\chi_{\Omega_k} x | \mathbf{G}](\omega)}{E[\chi_{\Omega_k} | \mathbf{G}](\omega)}\right),$$

a.e. on Ω_k , since the positive cone of $(\mathbf{V}; \cong_v)$ is closed.

Finally, since $|\chi_{\Omega_k}(\omega)| \leq 1$ and $\chi_{\Omega_k}(\omega) \rightarrow 1$ a.e., by using Lemma 4, (a) and (d), and the continuity of $f(\omega, \cdot)$, when $k \rightarrow \infty$ we have

$$(J) \quad E[f(\cdot, x(\cdot)) | \mathbf{G}](\omega) \cong_v f(\omega, E[x | \mathbf{G}](\omega)) \quad \text{a.e.}$$

4. Remark. In particular, when \mathbf{G} is the trivial sub- σ -field $\mathbf{Z} = \{\Omega, \phi\}$, inequality (J) reduces to

$$(J') \quad \int_{\Omega} f(\omega, x(\omega)) dP \cong_v f\left(\omega, \int_{\Omega} x(\omega) dP\right).$$

When the function $f(\omega, u)$ is replaced by a continuous and convex function g on \mathbf{U} to \mathbf{V} , inequalities (J) and (J') become

$$(K) \quad E[g(x(\cdot)) | \mathbf{G}](\omega) \cong_v g(E[x | \mathbf{G}](\omega)) \quad \text{a.e. and}$$

$$(K') \quad \int_{\Omega} g(x(\omega)) dP \cong_v g\left(\int_{\Omega} x(\omega) dP\right).$$

As we have mentioned in the introduction, this result extends a theorem of Scalora [6] in which the stronger condition that g is subadditive and positive-homogeneous is assumed.

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