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A GENERALIZED JENSEN'S INEQUALITY

TING-ON TO AND KAI WING YIP

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A generalized Jensen's inequality for conditional expectations of Bochner-integrable functions which extends the results of Dubins and Scalora is proved using a different method.

1. Introduction. Let $(\Omega, \mathbf{F}, P)$ be a probability space, $(\mathbf{U}, \|\cdot\|)$ a complex (or real) Banach space and $(\mathbf{V}, \|\cdot\|, \geqq_v)$ an ordered Banach space over the complex (or real) field such that the positive cone $\{v \in \mathbf{V} : v \geqq_v \theta\}$ is closed. Let x be a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$. Let $\mathbf{G}$ be a sub-σ-field of the σ-field $\mathbf{F}$ and let f be a function on $\Omega \times \mathbf{U}$ to $\mathbf{V}$ such that for each $u \in \mathbf{U}$ the function $f(\cdot, u)$ is strongly measurable with respect to $\mathbf{G}$ and such that for each $\omega \in \Omega$ the function $f(\omega, \cdot)$ is continuous and convex in the sense that $tf(\omega, u_1) + (1-t)\ f(\omega, u_2) \geqq_v f(\omega, tu_1 + (1-t)u_2)$ whenever $u_1,\ u_2 \in \mathbf{U}$ and $0 \leqq t \leqq 1$. For any Bochner-integrable function z on $(\Omega, \mathbf{F}, P)$ to any Banach space $\mathbf{W}$, we define $E[z \mid \mathbf{G}]$ "a conditional expectation of z relative to $\mathbf{G}$" as a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{W}$ such that $E(z \mid \mathbf{G}]$ is strongly measurable with respect to $\mathbf{G}$ and that

$$\int_A E[z \mid \mathbf{G}](\omega) dP = \int_A z(\omega) dP, \qquad A \in \mathbf{G},$$

where the integrals are Bochner-integrals.

The purpose of this note is to prove the following generalized Jensen's inequality:

THEOREM. *If $f(\cdot, x(\cdot))$ is Bochner-integrable, then*

$$\text{(J)} \qquad E[f(\cdot, x(\cdot)) \mid \mathbf{G}](\omega) \geqq_v f(\omega, E[x \mid \mathbf{G}](\omega)) \qquad \text{a.e.}$$

The above theorem extends the results of Dubins **[2]** (cf. Mayer **[5**, p. 79]) and Scalora **[6**, p. 360, Theorem 2.3]. It is proved in **[2]** that the theorem is true for the case in which the spaces $\mathbf{U}$ and $\mathbf{V}$ are both the real numbers $\mathbf{R}$, while in **[6]** Scalora uses the methods of Hille-Phillips **[4]** to prove the theorem when the function $f(\omega, u)$ is replaced by a continuous, subadditive positive-homogeneous function $g(u)$ on $\mathbf{U}$ to $\mathbf{V}$. It should be noted that the method of the proof used here is different than those used previously, the previous methods appear to be ineffective for a proof of the extension.

2. Preliminaries. We refer to [4] and [6] for the definitions and basic properties of the concepts of Bochner-integrals and the conditional expectation of a Bochner-integrable function. Our proof of the theorem is based on the following lemmas. Unless otherwise specified, functions in Lemma 1–5 are defined on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$.

LEMMA 1. ([4, p. 73, Corollary 1]). *A function is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countably-valued functions.*

LEMMA 2. (Egoroff's theorem, [4, p. 72] or [3, p. 149]). *A sequence $\{z_i\}_{i=1}^{\infty}$ of strongly measurable functions is almost uniformly convergent to a function z if and only if*

$$\|z_i(\omega) - z(\omega)\| \to 0 \text{ a.e. } as\, i \to \infty.$$

The following lemma is an immediate consequence of Lemma 1 and Lemma 2.

LEMMA 3. *If z is a strongly measurable function, then for any positive number M there exists a sequence $\{z_i\}_{i=1}^{\infty}$ of simple functions such that $\|z_i(\omega)\| \leqq \|z(\omega)\| + M$ a.e., i = $1, 2, \cdots$, and $\|z_i(\omega) - z(\omega)\| \to 0$* a.e. *as $i \to \infty$.*

LEMMA 4. ([6, p. 356, Theorem 2.2]).

(a) *If $z(\omega) = u$ on Ω then $E[z \mid \mathbf{G}](\omega) = u$* a.e.

(b) *If z and z_i, $i = 1, 2, \cdots$, are Bochner-integrable functions such that $z(\omega) = \sum_{i=1}^{n} t_i z_i(\omega)$* a.e. *where t_i are scalars then $E[z \mid \mathbf{G}](\omega) = \sum_{i=1}^{n} t_i E[z_i \mid \mathbf{G}](\omega)$* a.e.

(c) $\|E[z \mid \mathbf{G}](\omega)\| \leqq E[\|z\| \mid \mathbf{G}](\omega)$ a.e., *for any Boxhner-integrable function z.*

(d) *If z is a Bochner-integrable function and $z_i, i = 1, 2, \cdots$, are strongly measurable functions such that $\|z_i(\omega) - z(\omega)\| \to 0$* a.e. *as $i \to \infty$, and if there is a real-valued integrable function $y(\omega) \geqq 0$ such that $\|z_i(\omega)\| \leqq y(\omega)$* a.e., *$i = 1, 2, \cdots$, then z_i's are Bochner-integrable and $\|E[z_i \mid \mathbf{G}](\omega) - E[z \mid \mathbf{G}](\omega)\| \to 0$* a.e. *as $i \to \infty$.*

LEMMA 5. *If z is a Bochner-integrable function and $z_i, i = 1, 2, \cdots$, are strongly measurable functions such that $\|z_i(\omega) - z(\omega)\| \to 0$ uniformly* a.e. *as $i \to \infty$, then there exists an integer N such that $z_i, i = N, N+1, \cdots$, are Bochner-integrable functions, and*

$$\|E[z_i \mid \mathbf{G}](\omega) - E[z \mid \mathbf{G}](\omega)\| \to 0 \textit{ uniformly}$$

a.e. *as $i \to \infty$.*

Proof. An immediate consequence of Lemma 4 and the fact that $E[\cdot|\mathbf{G}]$ is a positive operator on the space of all real-valued integrable functions.

LEMMA 6. *If z is a strongly measure function on $(\Omega,\mathbf{G},P)$ to a Banach space $\mathbf{W}$, and if on $(\Omega,\mathbf{F},P)$, y is a numerically-valued integrable function such that zy is a Bochner-integrable function with values in $\mathbf{W}$, then*

$$E[zy\,|\,\mathbf{G}](\omega)=zE[y\,|\,\mathbf{G}](\omega)\ \text{a.e.}.$$

Proof. By using Lemma 3 and Lemma 4, the proof when $\mathbf{W}$ is the real numbers $\mathbf{R}$ as given by Billingsley [**1**, p. 110, Theorem 10.1] can be applied to obtain the general result.

LEMMA 7. *Let g be a convex function on $\mathbf{U}$ to $\mathbf{V}$. If $u_i\in\mathbf{U}$ and $t_i\in\mathbf{R}$, $t_i\geqq 0$, $i=1,2,\cdots n$, such that*

$$\sum_{i=1}^{n} t_i=1,\ \textit{then}\sum_{i=1}^{n} t_i g(u_i)\geqq_v g\left(\sum_{i=1}^{n} t_i u_i\right).$$

Proof. By induction.

3. Proof of the theorem. We first note that if $F\in\mathbf{F}$ with $P(F)>0$ and z is a simple function on $(\Omega,\mathbf{F},P)$ to $\mathbf{U}$ such that $\chi_F f(\cdot,z(\cdot))$ is Bochner-integrable, then

$$\text{(1)}\quad E[\chi_F f\Big(\cdot,z(\cdot)|\mathbf{G}](\omega)\geqq_v E[\chi_F\,|\,\mathbf{G}](\omega)f(\omega,\frac{E[\chi_F z\,|\,\mathbf{G}](\omega)}{E[\chi_F\,|\mathbf{G}](\omega)}\Big)\ \text{a.e. on } F.$$

To see this, let $z=\Sigma_{i=1}^n u_i\chi_{A_i}$, where $u_i\in\mathbf{U}$ and A_i's are disjoint sets of $\mathbf{F}$ such that $\Sigma_{i=1}^n\chi_{A_i}=1$. It is clear that $F\subset\{\omega: E[\chi_F|\mathbf{G}](\omega)>0\}$ a.e.. Since $f(\cdot,u_i)$ is strongly measurable with respect to $\mathbf{G}$ and $f(\omega,\cdot)$ is convex, by using Lemma 4, (b), Lemma 6 and Lemma 7, we then have

$$\frac{1}{E[\chi_F\,|\,\mathbf{G}](\omega)}E[\chi_F f(\cdot,z(\cdot))\,|\,\mathbf{G}](\omega)$$

$$=\frac{1}{E[\chi_F\,|\,\mathbf{G}](\omega)}\sum_{i=1}^{n} f(\omega,u_i)\,E[\chi_F\chi_{A_i}\,|\,\mathbf{G}](\omega)\ \text{a.e. on } F.$$

$$\geqq_v f\left(\omega, \frac{1}{E[\chi_F|\mathbf{G}](\omega)} \sum_{i=1}^{n} u_i E[\chi_F \chi_{A_i}|\mathbf{G}](\omega)\right) \text{a.e. on } F$$

$$= f\left(\omega, \frac{E[\chi_F z|\mathbf{G}](\omega)}{E[\chi_F|\mathbf{G}](\omega)}\right) \text{ a.e. on } F.$$

Nextly, since x is assumed to be a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$, x is strongly measurable, and hence by the definition of strong measurability (or by Lemma 3) there exists a sequence $\{x_i\}_{i=1}^{\infty}$ of simple functions on $(\Omega, \mathbf{F}, p)$ to $\mathbf{U}$ such that $\|x_i(\omega) - x(\omega)\| \to 0$ a.e.. Furthermore, since $f(\omega, \cdot)$ is continuous on $\mathbf{U}$ it follows that $\|f(\omega, x_i(\omega)) - f(\omega, x(\omega))\| \to 0$ a.e..

Therefore, by Lemma 2 we can find an increasing sequence, $\Omega_1 \subset \Omega_2 \subset \cdots$, of sets of $\mathbf{F}$ with $P(\Omega - \Omega_k) < 1/k$, $k = 1, 2, \cdots$, such that

(2) $\|\chi_{\Omega_{k_0}}(\omega) x_i(\omega) - \chi_{\Omega_{k_0}}(\omega) x(\omega)\| \to 0$ uniformly a.e. and

(3) $\|\chi_{\Omega_k}(\omega) f(\omega, x_i(\omega)) - \chi_{\Omega_k}(\omega) f(\omega, x(\omega))\| \to 0$ uniformly a.e., as $i \to \infty$, for each $k = 1, 2, \cdots$.

According to Lemma 5, (2) implies

(2′) $\|E[\chi_{\Omega_k} x_i|\mathbf{G}](\omega) - E[\chi_{\Omega_k} x|\mathbf{G}](\omega)\| \to 0$ uniformly a.e. as $i \to \infty$, for each $k = 1, 2, \cdots$, and (3) implies

(3′) $\|E[\chi_{\Omega_k} f(\cdot, x_i(\cdot))|\mathbf{G}](\omega) - E[\chi_{\Omega_k} f(\cdot, x(\cdot))|\mathbf{G}](\omega)\| \to 0$ uniformly a.e. as $i \to \infty$, for each $k = 1, 2, \cdots$.

Now by using the continuity of $f(\omega, \cdot)$ again, it follows from (2′) that

$$(4) \qquad \left\| f\left(\omega, \frac{E[\chi_{\Omega_k} x_i|\mathbf{G}](\omega)}{E[\chi_{\Omega_k}|\mathbf{G}](\omega)}\right) - f\left(\omega, \frac{E[\chi_{\Omega_k} x|\mathbf{G}](\omega)}{E[\chi_{\Omega_k}|\mathbf{G}](\omega)}\right) \right\| \to 0$$

a.e. on Ω_k as $i \to \infty$.

On the other hand, from (1) we obtain

$$(1') \quad E(\chi_{\Omega_k} f(\cdot, x_i(\cdot))|\mathbf{G}](\omega) \geqq_v E[\chi_{\Omega_k}|\mathbf{G}](\omega) f\left(\omega, \frac{E[\chi_{\Omega_k} x_i|\mathbf{G}](\omega)}{E[\chi_{\Omega_k}|\mathbf{G}](\omega)}\right)$$

a.e. on Ω_k, for each $k = 1, 2, \cdots$, and each $i = 1, 2, 3 \cdots$.

Letting $i \to \infty$ in (1′) and using (3′) and (4), we obtain

$$(1'') \quad E[\chi_{\Omega_k} f(\cdot, x(\cdot))|\mathbf{G}](\omega) \geqq_v E[\chi_{\Omega_k}|\mathbf{G}](\omega) f\left(\omega, \frac{E[\chi_{\Omega_k} x|\mathbf{G}](\omega)}{E[\chi_{\Omega_k}|\mathbf{G}](\omega)}\right),$$

a.e. on Ω_k, since the positive cone of $(\mathbf{V}; \geqq_v)$ is closed.

Finally, since $|\chi_{\Omega_k}(\omega)| \leqq 1$ and $\chi_{\Omega_k}(\omega) \to 1$ a.e., by using Lemma 4, (a) and (d), and the continuity of $f(\omega, \cdot)$, when $k \to \infty$ we have

(J) $$E[f(\cdot, x(\cdot))|\mathbf{G}](\omega) \geqq_v f(\omega, E[x|\mathbf{G}](\omega)) \quad \text{a.e.}.$$

4. Remark. In particular, when $\mathbf{G}$ is the trivial sub-σ-field $\mathbf{Z} = \{\Omega, \phi\}$, inequality (J) reduces to

(J′) $$\int_\Omega f(\omega, x(\omega))dP \geqq_v f\left(\omega, \int_\Omega x(\omega)dP\right).$$

When the function $f(\omega, u)$ is replaced by a continuous and convex function g on $\mathbf{U}$ to $\mathbf{V}$, inequalilties (J) and (J′) become

(K) $$E[g(x(\cdot))|\mathbf{G}](\omega) \geqq_v g(E[x|\mathbf{G}](\omega)) \quad \text{a.e. and}$$

(K′) $$\int_\Omega g(x(\omega))dP \geqq_v g\left(\int_\Omega x(\omega)dP\right).$$

As we have mentioned in the introduction, this result extends a theorem of Scalora **[6]** in which the stronger condition that g is subadditive and positive-homogeneous is assumed.

REFERENCES

1. Patrick Billingsley, *Ergodic Theory and Information, New York, J. Wiley and Sons*, 1965.
2. *L. E. Dubins, Rises and upcrossings of nonnegative martingales*, Illinois J. Math., **6** (1962), 226–241.
3. N. Dunford and J. T. Schwartz, *Linear Operators. Part I* : General Theory, Interscience, New York, 1958.
4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publ., Vol. 31, Providence, 1957.
5. P. A. Meyer, *Probability and Potentials*, London, Blaisdell Publishing Col, 1966.
6. F. S. Scalora, *Abstract martingale convergence theorems*, Pacific J. Math., **11** (1961), 347–374.

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UNIVERSITY OF CANTERBURY, CHRISTCHURCH, NEW ZEALAND
AND
UNITED COLLEGE, CHINESE UNIVERSITY OF HONG KONG

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