SECOND ORDER DIFFERENTIAL OPERATORS WITH
SELF-ADJOINT EXTENSIONS

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Let $H$ denote the Hilbert space of square summable analytic functions on the unit disk, and consider those formal differential operators

$$L = p_2 \frac{d^2}{dz^2} + p_1 \frac{d}{dz} + p_0$$

which give rise to symmetric operators in $H$. Examples have been given where the symmetric operators associated with these formal operators have defect indices $(0, 0)$ and $(2, 2)$ and hence are either self-adjoint or have self-adjoint extensions in $H$. In this note a class of symmetric operators with defect indices $(1, 1)$ is given.

Let $A$ denote the space of functions analytic on the unit disk and $\mathcal{H}$ the subspace of square summable functions in $A$ with inner product

$$(f, g) = \int \int_{|z|<1} f(z) \overline{g(z)} \, dx dy.$$

A complete orthonormal set for $\mathcal{H}$ is obtained by normalizing the powers of $z$. From this it follows that $\mathcal{H}$ is identical with the space of power series $\Sigma_{n=0}^\infty a_n z^n$ which satisfy

$$(1.1) \quad \sum_{n=0}^\infty |a_n|^2/(n + 1) < \infty.$$ 

Let $L$ be such that it maps polynomials into $\mathcal{H}$ and has the property $(Lz^n, z^m) = (z^n, Lz^m)$, $n, m = 0, 1, 2, \cdots$. Let $D_0$ be the subspace of polynomials and set $T_0f = Lf$ for $f$ in $D_0$. Then $T_0$ is symmetric and the defect indices $m^+$ and $m^-$ of its closure, $S$, are just the number of linearly independent solutions of $Lu = iu$ and $Lu = -iu$ respectively which are in $\mathcal{H}$. See [2]. In [2] and [3] examples of such symmetric operators $S$ with defect indices $(0, 0)$ and $(2, 2)$ are provided. We now give a class of operators with defect indices $(1, 1)$. 

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2. Consider the operator $L$

(2.1) \[ L = (c_1 z^3 + \bar{c}_1 z) \frac{d^2}{dz^2} + ((c_2 + 3c_1)z^2 + \bar{c}_2) \frac{d}{dz} + 2c_2 z. \]

In [3] it is shown that $L$ gives rise to symmetric $T_0$. Concerning the defect indices of its closure $S$, we have the following.

**Theorem 2.1.** Let $L$ be the operator of (2.1) then $S$ has defect indices $m^* = m^- = 1$.

**Proof.** The idea of the proof is to show that the equation $L \phi = \pm i \phi$ has precisely one power series solution $\phi(z) = \sum_{j=0}^\infty a_j z^j$ and that there exists a $K > 0$ and a positive integer $p$ such that $|a_j| \leq K j^{-1/p}$ for $j$ sufficiently large. Consequently the series $\sum_{j=0}^\infty |a_j|^2/(j+1)$ converges and $\phi$ belongs to $\mathcal{H}$, and $m^* = m^- = 1$.

Dividing $L \phi = \pm i \phi$ by $c_1$ we have the differential equation

(2.2) \[ (z^3 + \omega z) \phi'' + [(3 + \alpha)z^2 + \beta] \phi' + 2\alpha z \phi = \lambda \phi, \]

where $\omega = \bar{c}_1/c_1$, $\alpha = c_2/c_1$, $\beta = \bar{c}_2/c_1$, and $\lambda = \pm i/c_1$.

Substituting $\sum_{j=0}^\infty a_j z^j$ into (2.2) we obtain

(2.3) \[ \beta a_1 + \sum_{j=1}^\infty [(j+1)(\omega j + \beta)a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1}]z^j = \lambda a_0 + \sum_{j=1}^\infty \lambda a_j z^j \quad \lambda \neq 0. \]

If $\beta = 0$ we have $a_0 = 0$ and (2.3) can be solved recursively for $a_2, a_3, \cdots$, in terms of $a_1$ since $\omega j + \beta$ never vanishes. Thus we have but one analytic solution

\[ \phi(z) = z(1 + a_2 z^2 + \cdots). \]

If $\beta \neq 0$, we have $a_1 = \lambda a_0/\beta$ and (2.3) can be solved recursively for $a_2, a_3, \cdots$, etc., provided that $(\omega j + \beta)$ never vanishes for $j = 1, 2, \cdots$. Thus we are able to obtain the single formal power series solution $\phi(z) = 1 + a_1 z + a_2 z^2 + \cdots$. The case when $(\omega j + \beta)$ vanishes for some positive integer $j$ presents some complications and will be considered later in the proof. Solving (2.3) for $a_{j+1}$ we have

(2.4) \[ a_{j+1} = \frac{1}{\omega} \left\{ \frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + \left(1 + \frac{\beta}{\omega}\right) j + \frac{\beta}{\omega}} \right\}. \]
But $\beta/\omega = \tilde{c}_2/\tilde{c}_1 = \tilde{\alpha}$, hence (2.4) becomes

$$a_{j+1} = \frac{1}{\omega} \left\{ -\frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1 + \tilde{\alpha})j + \tilde{\alpha}} a_{j-1} + \lambda a_j \right\}. \tag{2.4}$$

Thus we obtain the estimate

$$|a_{j+1}| \leq \frac{1}{|\omega|} \left| \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1 + \tilde{\alpha})j + \tilde{\alpha}} \right| |a_{j-1}|$$

$$+ \frac{1}{|\omega|} \frac{1}{j^2 + (1 + \tilde{\alpha})j + \tilde{\alpha}} |a_j| \tag{2.5}$$

Since $|\omega| = 1$ we have

$$|a_{j+1}| \leq |u_1(j)| \cdot |a_{j-1}| + |u_2(j)| \cdot |a_j| \tag{2.6}$$

where

$$u_1(j) = \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1 + \tilde{\alpha})j + \tilde{\alpha}},$$

and

$$u_2(j) = \frac{\lambda}{j^2 + (1 + \tilde{\alpha})j + \tilde{\alpha}}.$$

We now estimate $|u_1(j)|$ and $|u_2(j)|$ for large $j$. Since $|u_2(j)|$ tends to zero as $j^{-2}$ it follows that there exists an $M > 0$ such that

$$|u_2(j)| \leq \frac{M}{j^2}, \quad \text{for } j \text{ sufficiently large.} \tag{2.7}$$

Concerning $|u_1(j)|$ we obtain, upon dividing,

$$u_1(j) = \left(1 - \frac{1}{j}\right) + \frac{\alpha}{j} \operatorname{Im}(\alpha) i + O(j^{-2}).$$

Thus $|u_1(j)|^2 = 1 - 2/j + O(j^{-2})$, and hence by a direct calculation,

$$|u_1(j)| = 1 - \frac{1}{j} + O(j^{-2}).$$

For $\xi > 0$, we note that $|u_1(j)| \leq 1 - \xi j^{-1}$ for $j$ sufficiently large if and only if $-1 < -\xi$, or $\xi < 1$. Hence we have
\begin{align*}
(2.8) \quad |u_i(j)| & \leq 1 - \frac{\xi}{j}, \quad \text{for } j \text{ sufficiently large} \\
\text{and } 0 < \xi < 1.
\end{align*}

Using (2.6), (2.7), and (2.8) we obtain, for \( j \) sufficiently large,

\begin{align*}
|a_{j+1}| & \leq (1 - \xi j^{-1}) |a_{j-1}| + Mj^{-2}\|a_j\| \\
& \leq (1 - \xi j^{-1} + Mj^{-2}) M(j), \quad 0 < \xi < 1,
\end{align*}

where \( M(j) = \max\{|a_{j-1}|, |a_j|\} \).

Thus, for sufficiently large \( j \), we have

\begin{align*}
(2.9) \quad |a_{j+1}| & \leq (1 - \gamma j^{-1}) M(j),
\end{align*}

where \( 0 < \gamma = \frac{\xi}{2} < \frac{1}{2} \).

Now consider the expression \((1 - \gamma j^{-1}) (j - 1)^{-1/p}\), where \( p \) is a positive integer. This is dominated by \((j + 1)^{-1/p}\) for \( j \) sufficiently large if and only if

\[ j^{p+1} + (-p\gamma + 1)j^p + \cdots \leq j^{p+1} - j^p. \]

Hence, if and only if \(-p\gamma + 1 < -1\) or \(-p\gamma < -2\). Since \( \gamma > 0 \), \( p > 2/\gamma \). Thus we have

\begin{align*}
(2.10) \quad (1 - \gamma j^{-1}) (j - 1)^{-1/p} & \leq (j + 1)^{-1/p}, \quad p > \frac{2}{\gamma}.
\end{align*}

We now show that there exists a positive constant \( K \) for which

\[ |a_j| \leq K j^{-1/p} \quad \text{for } j \geq 1. \]

Let \( j_i \) be such that (2.9) and (2.10) hold for \( j > j_i \). Let \( K = \max_{j > j_i} |a_j| j^{1/p} \) so that \( |a_j| \leq K j^{-1/p} \) for \( j \geq j_i \). Using (2.9) it follows that

\[ |a_{j_i+1}| \leq (1 - \gamma j_i^{-1}) M(j_i), \]

where

\begin{align*}
M(j_i) & = \text{Max} (K j_i^{-1/p}, K (j_i - 1)^{-1/p}) \\
& = K (j_i - 1)^{-1/p}.
\end{align*}

Hence,

\[ |a_{j_i+1}| \leq (1 - \gamma j_i^{-1}) K (j_i - 1)^{-1/p}, \]
and using (2.10) we have

\[ |a_{j+1}| \leq K(j_1 + 1)^{-1/p}. \]

We now proceed inductively to establish

\[ |a_{j+k}| \leq K(j_1 + k)^{-1/p}, \quad k = 2, 3, \ldots. \]

Let

\[ K_1 = \max_{j \leq j_1 + 1} |a_j|^{1/p} \]

\[ = \max \{K, K(j_1 + 1)^{-1/p}\} \leq K, \]

making use of (2.11). Using (2.9) we have

\[ |a_{j+2}| \leq (1 - \gamma(j_1 + 1)^{-1}M(j_1 + 1), \]

where,

\[ M(j_1 + 1) = \text{Max} \left( |a_{j+1}|, |a_j| \right) \]

\[ = \text{Max} \left( K(j_1 + 1)^{-1/p}, K(j_1)^{-1/p} \right) \]

\[ = K(j_1)^{-1/p}. \]

It follows from (2.10) that

\[ |a_{j+2}| \leq (1 - \gamma(j_1 + 1)^{-1})K(j_1)^{-1/p} \leq K(j_1 + 2)^{-1/p}. \]

Continuing on in this manner we establish (2.12). Hence any solution \( \sum_{j=0}^{\infty} a_j z^j \) whose coefficients satisfy (2.4) is in \( \mathcal{K} \). To complete the proof we have only to deal with the case where \( j\omega + \beta \) vanishes for some positive integer \( j \).

We now consider the case when \( j\omega + \beta \) vanishes for some positive integer \( n \). The analytic solution obtained from (2.3) by taking \( a_0 = a_1 = \cdots = a_n = 0 \), and solving recursively for \( a_{n+2}, a_{n+3}, \ldots \), in terms of \( a_{n+1} \) is, as we have seen, in \( \mathcal{K} \). If there were a second analytic solution corresponding to \( a_0 \neq 0 \) it would be in \( \mathcal{K} \) as well, and \( m^+(m^-) \) would be 2. We now show that this is not the case, i.e., \( m^+ = m^- = 1 \). To do this we make use of the following result.

Let \( \mu \) be such that \( \text{Im}(\mu) > 0 \) and let \( D^+_{\mu} \) be the nullspace of the operator \( S^* - \mu \). Then the dimension of \( D^+_{\mu} \) is equal to \( m^+ \). Similarly,
let \( \text{Im}(\mu) < 0 \) and let \( D^{-}_{\mu} \) be the nullspace of the operator \( S^* - \mu \), then the dimension of \( D^{-}_{\mu} \) is equal to \( m^{-} \), [1, p. 1232].

Using this we see that \( m^{+} \) is just the number of linearly independent solutions of \( L\phi = \mu \phi \) in \( \mathcal{H} \) for any \( \mu \) such that \( \text{Im}(\mu) > 0 \). Similarly, \( m^{-} \) is the number of linearly independent solutions of \( L\phi = \mu \phi \) in \( \mathcal{H} \) for any \( \mu \) such that \( \text{Im}(\mu) < 0 \). Hence, if we can show that there exist \( \mu \) such that \( \text{Im} \mu > 0 \) (\( \text{Im} \mu < 0 \)) for which there is no analytic solution corresponding to \( a_{0} \neq 0 \) we will have shown that \( m^{+} = m^{-} = 1 \).

Consider (2.3), where \( \lambda \) is now \( \mu/c_{2} \), and suppose that \( \beta = -n\omega \). Taking \( j = 1, 2, \cdots, n \) we obtain the following set of \( n + 1 \) linear equations in \( a_{0} \) thru \( a_{n} \):

\[
\begin{align*}
-j \omega a_{1} &= \lambda a_{0} \\
(j + 1)(j - n)\omega a_{j+1} + (j^{2} + j\alpha + \alpha - 1)a_{j-1} &= \lambda a_{j}, \\
& \quad j = 1, 2, \cdots, n - 1 \\
(n^{2} + n\alpha + \alpha - 1)a_{n-1} &= \lambda a_{n}.
\end{align*}
\]

Thus we are led to consider the homogeneous system

\[
\begin{align*}
-\lambda a_{0} - n\omega a_{1} &= 0 \\
2\alpha a_{0} - \lambda a_{1} + 2(2 - n)\omega a_{2} &= 0 \\
(n^{2} + n\alpha - 2n)a_{n-2} - \lambda a_{n-1} - n\omega a_{n} &= 0 \\
(n^{2} + n\alpha + \alpha - 1)a_{n-1} - \lambda a_{n} &= 0
\end{align*}
\]

Since the parameter \( \lambda = \mu/c_{2} \) appears only on the diagonal the system determinant \( D_{n}(\lambda) \) is a polynomial in \( \lambda \) of degree \( n + 1 \),

\[
D_{n}(\lambda) = (-1)^{n+1} \lambda^{n+1} + \cdots
\]

Thus \( D_{n}(\lambda) \) vanishes at most \( n + 1 \) points in the complex plane, and we can find \( \mu \) in the upper half-plane and lower half-plane for which \( D_{n}(\mu/c_{2}) \neq 0 \). Thus \( a_{0} = a_{1} = \cdots = a_{n} = 0 \) and there is only one analytic solution of \( L\phi = \mu \phi \).

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