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**SECOND ORDER DIFFERENTIAL OPERATORS WITH  
SELF-ADJOINT EXTENSIONS**

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## SECOND ORDER DIFFERENTIAL OPERATORS WITH SELF-ADJOINT EXTENSIONS

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Let  $\mathcal{H}$  denote the Hilbert space of square summable analytic functions on the unit disk, and consider those formal differential operators

$$L = p_2 \frac{d^2}{dz^2} + p_1 \frac{d}{dz} + p_0$$

which give rise to symmetric operators in  $\mathcal{H}$ . Examples have been given where the symmetric operators associated with these formal operators have defect indices  $(0, 0)$  and  $(2, 2)$  and hence are either self-adjoint or have self-adjoint extensions in  $\mathcal{H}$ . In this note a class of symmetric operators with defect indices  $(1, 1)$  is given.

Let  $\mathcal{A}$  denote the space of functions analytic on the unit disk and  $\mathcal{H}$  the subspace of square summable functions in  $\mathcal{A}$  with inner product

$$(f, g) = \iint_{|z| < 1} f(z) \overline{g(z)} \, dx dy.$$

A complete orthonormal set for  $\mathcal{H}$  is obtained by normalizing the powers of  $z$ . From this it follows that  $\mathcal{H}$  is identical with the space of power series  $\sum_{n=0}^{\infty} a_n z^n$  which satisfy

$$(1.1) \quad \sum_{n=0}^{\infty} |a_n|^2 / (n+1) < \infty.$$

Let  $L$  be such that it maps polynomials into  $\mathcal{H}$  and has the property  $(Lz^n, z^m) = (z^n, Lz^m)$ ,  $n, m = 0, 1, 2, \dots$ . Let  $\mathcal{D}_0$  be the subspace of polynomials and set  $T_0 f = Lf$  for  $f$  in  $\mathcal{D}_0$ . Then  $T_0$  is symmetric and the defect indices  $m^+$  and  $m^-$  of its closure,  $S$ , are just the number of linearly independent solutions of  $Lu = iu$  and  $Lu = -iu$  respectively which are in  $\mathcal{H}$ . See [2]. In [2] and [3] examples of such symmetric operators  $S$  with defect indices  $(0, 0)$  and  $(2, 2)$  are provided. We now give a class of operators with defect indices  $(1, 1)$ .

## 2. Consider the operator $L$ ,

$$(2.1) \quad L = (c_1 z^3 + \bar{c}_1 z) \frac{d^2}{dz^2} + ((c_2 + 3c_1)z^2 + \bar{c}_2) \frac{d}{dz} + 2c_2 z.$$

In [3] it is shown that  $L$  gives rise to symmetric  $T_0$ . Concerning the defect indices of its closure  $S$ , we have the following.

**THEOREM 2.1.** *Let  $L$  be the operator of (2.1) then  $S$  has defect indices  $m^+ = m^- = 1$ .*

*Proof.* The idea of the proof is to show that the equation  $L\phi = \pm i\phi$  has precisely one power series solution  $\phi(z) = \sum_{j=0}^{\infty} a_j z^j$  and that there exists a  $K > 0$  and a positive integer  $p$  such that  $|a_j| \leq K j^{-1/p}$  for  $j$  sufficiently large. Consequently the series  $\sum_{j=0}^{\infty} |a_j|^2 / (j+1)$  converges and  $\phi$  belongs to  $\mathcal{H}$ , and  $m^+ = m^- = 1$ .

Dividing  $L\phi = \pm i\phi$  by  $c_1$  we have the differential equation

$$(2.2) \quad (z^3 + \omega z)\phi'' + [(3 + \alpha)z^2 + \beta]\phi' + 2\alpha z\phi = \lambda\phi,$$

where  $\omega = \bar{c}_1/c_1$ ,  $\alpha = c_2/c_1$ ,  $\beta = \bar{c}_2/c_1$ , and  $\lambda = \pm i/c_1$ .

Substituting  $\sum_{j=0}^{\infty} a_j z^j$  into (2.2) we obtain

$$(2.3) \quad \beta a_1 + \sum_{j=1}^{\infty} [(j+1)(\omega j + \beta)a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1}]z^j \\ = \lambda a_0 + \sum_{j=1}^{\infty} \lambda a_j z^j \quad \lambda \neq 0.$$

If  $\beta = 0$  we have  $a_0 = 0$  and (2.3) can be solved recursively for  $a_2, a_3, \dots$ , in terms of  $a_1$  since  $\omega j + \beta$  never vanishes. Thus we have but one analytic solution

$$\phi(z) = z(1 + a_2 z^2 + \dots).$$

If  $\beta \neq 0$ , we have  $a_1 = \lambda a_0 / \beta$  and (2.3) can be solved recursively for  $a_2, a_3$ , etc., provided that  $(\omega j + \beta)$  never vanishes for  $j = 1, 2, \dots$ . Thus we are able to obtain the single formal power series solution  $\phi(z) = 1 + a_1 z + a_2 z^2 + \dots$ . The case when  $(\omega j + \beta)$  vanishes for some positive integer  $j$  presents some complications and will be considered later in the proof. Solving (2.3) for  $a_{j+1}$  we have

$$(2.4) \quad a_{j+1} = \frac{1}{\omega} \left\{ \frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + \left(1 + \frac{\beta}{\omega}\right)j + \frac{\beta}{\omega}} \right\}.$$

But  $\beta/\omega = \bar{c}_2/\bar{c}_1 = \bar{\alpha}$ , hence (2.4) becomes

$$(2.4) \quad a_{j+1} = \frac{1}{\omega} \left\{ \frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}} \right\}.$$

Thus we obtain the estimate

$$(2.5) \quad |a_{j+1}| \leq \frac{1}{|\omega|} \left| \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}} \right| |a_{j-1}| + \frac{|\lambda|}{|\omega|} \frac{1}{|j^2 + (1 + \bar{\alpha})j + \bar{\alpha}|} |a_j|.$$

Since  $|\omega| = 1$  we have

$$(2.6) \quad |a_{j+1}| \leq |u_1(j)| |a_{j-1}| + |u_2(j)| |a_j|,$$

where

$$u_1(j) = \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}},$$

and

$$u_2(j) = \frac{\lambda}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}}.$$

We now estimate  $|u_1(j)|$  and  $|u_2(j)|$  for large  $j$ . Since  $|u_2(j)|$  tends to zero as  $j^{-2}$  it follows that there exists an  $M > 0$  such that

$$(2.7) \quad |u_2(j)| \leq \frac{M}{j^2}, \quad \text{for } j \text{ sufficiently large.}$$

Concerning  $|u_1(j)|$  we obtain, upon dividing,

$$u_1(j) = \left(1 - \frac{1}{j}\right) + \frac{2}{j} \operatorname{Im}(\alpha)i + O(j^{-2}).$$

Thus  $|u_1(j)|^2 = 1 - 2/j + O(j^{-2})$ , and hence by a direct calculation,

$$|u_1(j)| = 1 - \frac{1}{j} + O(j^{-2}).$$

For  $\xi > 0$ , we note that  $|u_1(j)| \leq 1 - \xi j^{-1}$  for  $j$  sufficiently large if and only if  $-1 < -\xi$ , or  $\xi < 1$ . Hence we have

$$(2.8) \quad |u_1(j)| \leq 1 - \frac{\xi}{j}, \quad \text{for } j \text{ sufficiently large}$$

and  $0 < \xi < 1$ .

Using (2.6), (2.7), and (2.8) we obtain, for  $j$  sufficiently large,

$$\begin{aligned} |a_{j+1}| &\leq (1 - \xi j^{-1}) |a_{j-1}| + M j^{-2} |a_j| \\ &\leq (1 - \xi j^{-1} + M j^{-2}) M(j), \quad 0 < \xi < 1, \end{aligned}$$

where  $M(j) = \max\{|a_{j-1}|, |a_j|\}$ .

Thus, for sufficiently large  $j$ , we have

$$(2.9) \quad |a_{j+1}| \leq (1 - \gamma j^{-1}) M(j),$$

where  $0 < \gamma = \xi/2 < \frac{1}{2}$ .

Now consider the expression  $(1 - \gamma j^{-1})(j - 1)^{-1/p}$ , where  $p$  is a positive integer. This is dominated by  $(j + 1)^{-1/p}$  for  $j$  sufficiently large if and only if

$$j^{p+1} + (-p\gamma + 1)j^p + \dots \leq j^{p+1} - j^p.$$

Hence, if and only if  $-p\gamma + 1 < -1$  or  $-p\gamma < -2$ . Since  $\gamma > 0$ ,  $p > 2/\gamma$ . Thus we have

$$(2.10) \quad (1 - \gamma j^{-1})(j - 1)^{-1/p} \leq (j + 1)^{-1/p}, \quad p > \frac{2}{\gamma}.$$

We now show that there exists a positive constant  $K$  for which  $|a_j| \leq K j^{-1/p}$  for  $j \geq 1$ . Let  $j_1$  be such that (2.9) and (2.10) hold for  $j > j_1$ . Let  $K = \max_{j \leq j_1} |a_j| j^{1/p}$  so that  $|a_j| \leq K j^{-1/p}$  for  $j \leq j_1$ . Using (2.9) it follows that

$$|a_{j+1}| \leq (1 - \gamma j_1^{-1}) M(j_1),$$

where

$$\begin{aligned} M(j_1) &= \text{Max}(K j_1^{-1/p}, K(j_1 - 1)^{-1/p}) \\ &= K(j_1 - 1)^{-1/p}. \end{aligned}$$

Hence,

$$|a_{j+1}| \leq (1 - \gamma j_1^{-1}) K(j_1 - 1)^{-1/p},$$

and using (2.10) we have

$$(2.11) \quad |a_{j_1+1}| \leq K(j_1 + 1)^{-1/p}.$$

We now proceed inductively to establish

$$(2.12) \quad |a_{j_1+k}| \leq K(j_1 + k)^{-1/p}, \quad k = 2, 3, \dots$$

Let

$$\begin{aligned} K_1 &= \max_{j \leq j_1+1} |a_j| j^{1/p} \\ &= \max \{K, K(j_1 + 1)^{-1/p}\} \leq K, \end{aligned}$$

making use of (2.11). Using (2.9) we have

$$|a_{j_1+2}| \leq (1 - \gamma(j_1 + 1)^{-1})M(j_1 + 1),$$

where,

$$\begin{aligned} M(j_1 + 1) &= \text{Max} (|a_{j_1+1}|, |a_{j_1}|) \\ &= \text{Max} (K(j_1 + 1)^{-1/p}, K(j_1)^{-1/p}) \\ &= K(j_1)^{-1/p}. \end{aligned}$$

It follows from (2.10) that

$$\begin{aligned} |a_{j_1+2}| &\leq (1 - \gamma(j_1 + 1)^{-1})K(j_1)^{-1/p} \\ &\leq K(j_1 + 2)^{-1/p}. \end{aligned}$$

Continuing on in this manner we establish (2.12). Hence any solution  $\sum_{j=0}^{\infty} a_j z^j$  whose coefficients satisfy (2.4) is in  $\mathcal{H}$ . To complete the proof we have only to deal with the case where  $j\omega + \beta$  vanishes for some positive integer  $j$ .

We now consider the case when  $j\omega + \beta$  vanishes for some positive integer  $n$ . The analytic solution obtained from (2.3) by taking  $a_0 = a_1 = \dots = a_n = 0$ , and solving recursively for  $a_{n+2}, a_{n+3}, \dots$ , in terms of  $a_{n+1}$  is, as we have seen, in  $\mathcal{H}$ . If there were a second analytic solution corresponding to  $a_0 \neq 0$  it would be in  $\mathcal{H}$  as well, and  $m^+ (m^-)$  would be 2. We now show that this is not the case, i.e.,  $m^+ = m^- = 1$ . To do this we make use of the following result.

Let  $\mu$  be such that  $\text{Im}(\mu) > 0$  and let  $\mathcal{D}_\mu^+$  be the nullspace of the operator  $S^* - \mu$ . Then the dimension of  $\mathcal{D}_\mu^+$  is equal to  $m^+$ . Similarly,

let  $\text{Im}(\mu) < 0$  and let  $\mathcal{D}_\mu^-$  be the nullspace of the operator  $S^* - \mu$ , then the dimension of  $\mathcal{D}_\mu^-$  is equal to  $m^-$ , [1, p. 1232].

Using this we see that  $m^+$  is just the number of linearly independent solutions of  $L\phi = \mu\phi$  in  $\mathcal{H}$  for any  $\mu$  such that  $\text{Im}(\mu) > 0$ . Similarly,  $m^-$  is the number of linearly independent solutions of  $L\phi = \mu\phi$  in  $\mathcal{H}$  for any  $\mu$  such that  $\text{Im}(\mu) < 0$ . Hence, if we can show that there exist  $\mu$  such that  $\text{Im} \mu > 0$  ( $\text{Im} \mu < 0$ ) for which there is no analytic solution corresponding to  $a_0 \neq 0$  we will have shown that  $m^+ = m^- = 1$ .

Consider (2.3), where  $\lambda$  is now  $\mu/c_2$ , and suppose that  $\beta = -n\omega$ . Taking  $j = 1, 2, \dots, n$  we obtain the following set of  $n + 1$  linear equations in  $a_0$  thru  $a_n$ :

$$\begin{aligned} -n\omega a_1 &= \lambda a_0 \\ (j+1)(j-n)\omega a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1} &= \lambda a_j, \\ & \qquad \qquad \qquad j = 1, 2, \dots, n-1 \\ (n^2 + n\alpha + \alpha - 1)a_{n-1} &= \lambda a_n. \end{aligned}$$

Thus we are led to consider the homogeneous system

$$\begin{aligned} -\lambda a_0 - n\omega a_1 &= 0 \\ 2\alpha a_0 - \lambda a_1 + 2(2-n)\omega a_2 &= 0 \\ (n^2 + n\alpha - 2n)a_{n-2} - \lambda a_{n-1} - n\omega a_n &= 0 \\ (n^2 + n\alpha + \alpha - 1)a_{n-1} - \lambda a_n &= 0 \end{aligned}$$

Since the parameter  $\lambda = \mu/c_2$  appears only on the diagonal the system determinant  $D_n(\lambda)$  is a polynomial in  $\lambda$  of degree  $n + 1$ ,

$$D_n(\lambda) = (-1)^{n+1} \lambda^{n+1} + \dots.$$

Thus  $D_n(\lambda)$  vanishes at most  $n + 1$  points in the complex plane, and we can find  $\mu$  in the upper half-plane and lower half-plane for which  $D_n(\mu/c_2) \neq 0$ . Thus  $a_0 = a_1 = \dots = a_n = 0$  and there is only one analytic solution of  $L\phi = \mu\phi$ .

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