SECOND ORDER DIFFERENTIAL OPERATORS WITH SELF-ADJOINT EXTENSIONS

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Let \( \mathcal{H} \) denote the Hilbert space of square summable analytic functions on the unit disk, and consider those formal differential operators

\[
L = p_2 \frac{d^2}{dz^2} + p_1 \frac{d}{dz} + p_0
\]

which give rise to symmetric operators in \( \mathcal{H} \). Examples have been given where the symmetric operators associated with these formal operators have defect indices \((0, 0)\) and \((2, 2)\) and hence are either self-adjoint or have self-adjoint extensions in \( \mathcal{H} \). In this note a class of symmetric operators with defect indices \((1, 1)\) is given.

Let \( \mathcal{A} \) denote the space of functions analytic on the unit disk and \( \mathcal{H} \) the subspace of square summable functions in \( \mathcal{A} \) with inner product

\[
(f, g) = \int \int_{|z| < 1} f(z) \overline{g(z)} \, dx \, dy.
\]

A complete orthonormal set for \( \mathcal{H} \) is obtained by normalizing the powers of \( z \). From this it follows that \( \mathcal{H} \) is identical with the space of power series \( \sum_{n=0}^{\infty} a_n z^n \) which satisfy

\[
(1.1) \quad \sum_{n=0}^{\infty} |a_n|^2/(n+1) < \infty.
\]

Let \( L \) be such that it maps polynomials into \( \mathcal{H} \) and has the property \((Lz^n, z^m) = (z^n, Lz^m)\), \( n, m = 0, 1, 2, \cdots \). Let \( \mathcal{D}_0 \) be the subspace of polynomials and set \( T_0 f = Lf \) for \( f \) in \( \mathcal{D}_0 \). Then \( T_0 \) is symmetric and the defect indices \( m^+ \) and \( m^- \) of its closure, \( S \), are just the number of linearly independent solutions of \( Lu = iu \) and \( Lu = -iu \) respectively which are in \( \mathcal{H} \). See [2]. In [2] and [3] examples of such symmetric operators \( S \) with defect indices \((0, 0)\) and \((2, 2)\) are provided. We now give a class of operators with defect indices \((1, 1)\).
2. Consider the operator \( L \),

\[
L = (c_1 z^3 + \bar{c}_1 z) \frac{d^2}{dz^2} + ((c_2 + 3c_1)z^2 + \bar{c}_2) \frac{d}{dz} + 2c_2 z.
\]

In [3] it is shown that \( L \) gives rise to symmetric \( T_0 \). Concerning the defect indices of its closure \( S \), we have the following.

**Theorem 2.1.** Let \( L \) be the operator of (2.1) then \( S \) has defect indices \( m^+ = m^- = 1 \).

**Proof.** The idea of the proof is to show that the equation \( L \phi = \pm i \phi \) has precisely one power series solution \( \phi(z) = \sum_{j=0}^{\infty} a_j z^j \) and that there exists a \( K > 0 \) and a positive integer \( p \) such that \( |a_j| \leq Kj^{-1/p} \) for \( j \) sufficiently large. Consequently the series \( \sum_{j=0}^{\infty} |a_j|^2/(j + 1) \) converges and \( \phi \) belongs to \( \mathcal{H} \), and \( m^+ = m^- = 1 \).

Dividing \( L \phi = \pm i \phi \) by \( c_1 \) we have the differential equation

\[
(\bar{c}_1 z + \omega) \phi'' + [(3 + \alpha)z^2 + \beta] \phi' + 2\alpha z \phi = \lambda \phi,
\]

where \( \omega = \bar{c}_1/c_1, \alpha = c_2/c_1, \beta = \bar{c}_2/c_1, \) and \( \lambda = \pm i/c_1 \).

Substituting \( \sum_{j=0}^{\infty} a_j z^j \) into (2.2) we obtain

\[
\beta a_1 + \sum_{j=1}^{\infty} \left[ (j + 1)(\omega j + \beta) a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1} \right] z^j = \lambda a_0 + \sum_{j=1}^{\infty} \lambda a_j z^j \quad \lambda \neq 0.
\]

If \( \beta = 0 \) we have \( a_0 = 0 \) and (2.3) can be solved recursively for \( a_2, a_3, \ldots \), in terms of \( a_1 \) since \( \omega j + \beta \) never vanishes. Thus we have but one analytic solution

\[
\phi(z) = z(1 + a_2 z^2 + \cdots).
\]

If \( \beta \neq 0 \), we have \( a_1 = \lambda a_0 / \beta \) and (2.3) can be solved recursively for \( a_2, a_3, \ldots \), provided that \( (\omega j + \beta) \) never vanishes for \( j = 1, 2, \ldots \). Thus we are able to obtain the single formal power series solution \( \phi(z) = 1 + a_1 z + a_2 z^2 + \cdots \). The case when \( (\omega j + \beta) \) vanishes for some positive integer \( j \) presents some complications and will be considered later in the proof. Solving (2.3) for \( a_{j+1} \) we have

\[
a_{j+1} = \frac{1}{\omega} \left\{ \frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + \left( 1 + \frac{\beta}{\omega} \right) j + \frac{\beta}{\omega}} \right\}.
\]
But $\beta/\omega = \bar{c}_z/\bar{c}_1 = \bar{\alpha}$, hence (2.4) becomes

$$a_{j+1} = \frac{1}{\omega} \left\{ \frac{[j^2 + j\alpha + (\alpha - 1)]a_j + \lambda a_j}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}} \right\}. \tag{2.4}$$

Thus we obtain the estimate

$$|a_{j+1}| \leq \frac{1}{|\omega|} \left| \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}} \right| |a_{j-1}|$$

$$+ \frac{|\lambda|}{|\omega|} \frac{1}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}} |a_j|. \tag{2.5}$$

Since $|\omega| = 1$ we have

$$|a_{j+1}| \leq |u_1(j)| |a_{j-1}| + |u_2(j)| |a_j|, \tag{2.6}$$

where

$$u_1(j) = \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}},$$

and

$$u_2(j) = \frac{\lambda}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}}.$$

We now estimate $|u_1(j)|$ and $|u_2(j)|$ for large $j$. Since $|u_2(j)|$ tends to zero as $j^{-2}$ it follows that there exists an $M > 0$ such that

$$|u_2(j)| \leq \frac{M}{j^2}, \text{ for } j \text{ sufficiently large.} \tag{2.7}$$

Concerning $|u_1(j)|$ we obtain, upon dividing,

$$u_1(j) = \left(1 - \frac{1}{j}\right) + \frac{2}{j} \text{Im}(\alpha)i + O(j^{-2}).$$

Thus $|u_1(j)|^2 = 1 - 2/j + O(j^{-2})$, and hence by a direct calculation,

$$|u_1(j)| = 1 - \frac{1}{j} + O(j^{-2}).$$

For $\xi > 0$, we note that $|u_1(j)| \leq 1 - \xi j^{-1}$ for $j$ sufficiently large if and only if $-1 < -\xi$, or $\xi < 1$. Hence we have
\[(2.8) \quad |u_1(j)| \leq 1 - \frac{\xi}{j}, \quad \text{for } j \text{ sufficiently large and } 0 < \xi < 1.\]

Using (2.6), (2.7), and (2.8) we obtain, for \(j\) sufficiently large,

\[
|a_{j+1}| \leq (1 - \xi j^{-1}) |a_{j-1}| + Mj^{-2} |a_j| \leq (1 - \xi j^{-1} + Mj^{-2}) M(j), \quad 0 < \xi < 1,
\]

where \(M(j) = \max\{|a_{j-1}|, |a_j|\}\).

Thus, for sufficiently large \(j\), we have

\[(2.9) \quad |a_{j+1}| \leq (1 - \gamma j^{-1}) M(j),\]

where \(0 < \gamma = \xi / 2 < \frac{1}{2}\).

Now consider the expression \((1 - \gamma j^{-1})(j - 1)^{-1/p}\), where \(p\) is a positive integer. This is dominated by \((j + 1)^{-1/p}\) for \(j\) sufficiently large if and only if

\[j^{p+1} + (-p\gamma + 1)j^p + \cdots \leq j^{p+1} - j^p.\]

Hence, if and only if \(-p\gamma + 1 < -1\) or \(-p\gamma < -2\). Since \(\gamma > 0\), \(p > 2/\gamma\). Thus we have

\[(2.10) \quad (1 - \gamma j^{-1})(j - 1)^{-1/p} \leq (j + 1)^{-1/p}, \quad p > 2/\gamma.\]

We now show that there exists a positive constant \(K\) for which \(|a_j| \leq K j^{-1/p}\) for \(j \geq 1\). Let \(j_1\) be such that (2.9) and (2.10) hold for \(j > j_1\). Let \(K = \max_{j \geq j_1} |a_j| j^{1/p}\) so that \(|a_j| \leq K j^{-1/p}\) for \(j \geq j_1\). Using (2.9) it follows that

\[|a_{j_1+1}| \leq (1 - \gamma j_1^{-1}) M(j_1),\]

where

\[M(j_1) = \max (K j_1^{-1/p}, K (j_1 - 1)^{-1/p}) = K (j_1 - 1)^{-1/p}.\]

Hence,

\[|a_{j_1+1}| \leq (1 - \gamma j_1^{-1}) K (j_1 - 1)^{-1/p},\]
and using (2.10) we have
\begin{equation}
|a_{j+1}| \leq K(j+1)^{-1/p}.
\end{equation}

We now proceed inductively to establish
\begin{equation}
|a_{j+k}| \leq K(j+k)^{-1/p}, \quad k = 2, 3, \ldots.
\end{equation}

Let
\[
K_1 = \max_{j \in \mathbb{Z}} |a_j| j^{1/p} = \max \{K, K(j+1)^{-1/p}\} \leq K,
\]

making use of (2.11). Using (2.9) we have
\[
|a_{j+2}| \leq (1 - \gamma(j+1)^{-1})M(j+1),
\]

where,
\[
M(j+1) = \max( |a_{j+1}|, |a_j| ) = \max(K(j+1)^{-1/p}, K(j)^{-1/p}) = K(j)^{-1/p}.
\]

It follows from (2.10) that
\[
|a_{j+2}| \leq (1 - \gamma(j+1)^{-1})K(j)^{-1/p} \leq K(j+2)^{-1/p}.
\]

Continuing on in this manner we establish (2.12). Hence any solution \(\sum_{j=0} a_jz^j\) whose coefficients satisfy (2.4) is in \(\mathcal{H}\). To complete the proof we have only to deal with the case where \(j\omega + \beta\) vanishes for some positive integer \(j\).

We now consider the case when \(j\omega + \beta\) vanishes for some positive integer \(n\). The analytic solution obtained from (2.3) by taking \(a_0 = a_1 = \cdots = a_n = 0\), and solving recursively for \(a_{n+2}, a_{n+3}, \ldots\), in terms of \(a_{n+1}\) is, as we have seen, in \(\mathcal{H}\). If there were a second analytic solution corresponding to \(a_0 \neq 0\) it would be in \(\mathcal{H}\) as well, and \(m^+ (m^-)\) would be 2. We now show that this is not the case, i.e., \(m^+ = m^- = 1\). To do this we make use of the following result.

Let \(\mu\) be such that \(\text{Im}(\mu) > 0\) and let \(\mathcal{D}_\mu^+\) be the nullspace of the operator \(S^* - \mu\). Then the dimension of \(\mathcal{D}_\mu^+\) is equal to \(m^+\). Similarly,
let \( \text{Im}(\mu) < 0 \) and let \( D_{\mu}^- \) be the nullspace of the operator \( S^* - \mu \), then the dimension of \( D_{\mu}^- \) is equal to \( m^- \), [1, p. 1232].

Using this we see that \( m^+ \) is just the number of linearly independent solutions of \( L \phi = \mu \phi \) in \( \mathcal{H} \) for any \( \mu \) such that \( \text{Im}(\mu) > 0 \). Similarly, \( m^- \) is the number of linearly independent solutions of \( L \phi = \mu \phi \) in \( \mathcal{H} \) for any \( \mu \) such that \( \text{Im}(\mu) < 0 \). Hence, if we can show that there exist \( \mu \) such that \( \text{Im} \mu > 0 \) (\( \text{Im} \mu < 0 \)) for which there is no analytic solution corresponding to \( a_0 \neq 0 \) we will have shown that \( m^+ = m^- = 1 \).

Consider (2.3), where \( \lambda \) is now \( \mu / c_2 \), and suppose that \( \beta = -n \omega \). Taking \( j = 1, 2, \ldots, n \) we obtain the following set of \( n + 1 \) linear equations in \( a_0 \) thru \( a_n \):

\[
\begin{align*}
-n \omega a_1 &= \lambda a_0 \\
(j + 1)(j - n) \omega a_{j+1} + (j^2 + j \alpha + \alpha - 1)a_{j-1} &= \lambda a_j, \\
&\quad j = 1, 2, \ldots, n - 1 \\
(n^2 + n \alpha + \alpha - 1)a_{n-1} &= \lambda a_n.
\end{align*}
\]

Thus we are led to consider the homogeneous system

\[
\begin{align*}
-\lambda a_0 - n \omega a_1 &= 0 \\
2 \alpha a_0 - \lambda a_1 + 2(2 - n) \omega a_2 &= 0 \\
(n^2 + n \alpha - 2n)a_{n-2} - \lambda a_{n-1} - n \omega a_n &= 0 \\
(n^2 + n \alpha + \alpha - 1)a_{n-1} - \lambda a_n &= 0
\end{align*}
\]

Since the parameter \( \lambda = \mu / c_2 \) appears only on the diagonal the system determinant \( D_n(\lambda) \) is a polynomial in \( \lambda \) of degree \( n + 1 \),

\[
D_n(\lambda) = (-1)^{n+1} \lambda^{n+1} + \cdots.
\]

Thus \( D_n(\lambda) \) vanishes at most \( n + 1 \) points in the complex plane, and we can find \( \mu \) in the upper half-plane and lower half-plane for which \( D_n(\mu / c_2) \neq 0 \). Thus \( a_0 = a_1 = \cdots = a_n = 0 \) and there is only one analytic solution of \( L \phi = \mu \phi \).

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