

Pacific Journal of Mathematics

SECOND ORDER DIFFERENTIAL OPERATORS WITH SELF-ADJOINT EXTENSIONS

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Let \mathcal{H} denote the Hilbert space of square summable analytic functions on the unit disk, and consider those formal differential operators

$$L = p_2 \frac{d^2}{dz^2} + p_1 \frac{d}{dz} + p_0$$

which give rise to symmetric operators in \mathcal{H} . Examples have been given where the symmetric operators associated with these formal operators have defect indices $(0, 0)$ and $(2, 2)$ and hence are either self-adjoint or have self-adjoint extensions in \mathcal{H} . In this note a class of symmetric operators with defect indices $(1, 1)$ is given.

Let \mathcal{A} denote the space of functions analytic on the unit disk and \mathcal{H} the subspace of square summable functions in \mathcal{A} with inner product

$$(f, g) = \iint_{|z| < 1} f(z) \overline{g(z)} \, dx dy.$$

A complete orthonormal set for \mathcal{H} is obtained by normalizing the powers of z . From this it follows that \mathcal{H} is identical with the space of power series $\sum_{n=0}^{\infty} a_n z^n$ which satisfy

$$(1.1) \quad \sum_{n=0}^{\infty} |a_n|^2 / (n+1) < \infty.$$

Let L be such that it maps polynomials into \mathcal{H} and has the property $(Lz^n, z^m) = (z^n, Lz^m)$, $n, m = 0, 1, 2, \dots$. Let \mathcal{D}_0 be the subspace of polynomials and set $T_0 f = Lf$ for f in \mathcal{D}_0 . Then T_0 is symmetric and the defect indices m^+ and m^- of its closure, S , are just the number of linearly independent solutions of $Lu = iu$ and $Lu = -iu$ respectively which are in \mathcal{H} . See [2]. In [2] and [3] examples of such symmetric operators S with defect indices $(0, 0)$ and $(2, 2)$ are provided. We now give a class of operators with defect indices $(1, 1)$.

2. Consider the operator L ,

$$(2.1) \quad L = (c_1 z^3 + \bar{c}_1 z) \frac{d^2}{dz^2} + ((c_2 + 3c_1)z^2 + \bar{c}_2) \frac{d}{dz} + 2c_2 z.$$

In [3] it is shown that L gives rise to symmetric T_0 . Concerning the defect indices of its closure S , we have the following.

THEOREM 2.1. *Let L be the operator of (2.1) then S has defect indices $m^+ = m^- = 1$.*

Proof. The idea of the proof is to show that the equation $L\phi = \pm i\phi$ has precisely one power series solution $\phi(z) = \sum_{j=0}^{\infty} a_j z^j$ and that there exists a $K > 0$ and a positive integer p such that $|a_j| \leq K j^{-1/p}$ for j sufficiently large. Consequently the series $\sum_{j=0}^{\infty} |a_j|^2 / (j+1)$ converges and ϕ belongs to \mathcal{H} , and $m^+ = m^- = 1$.

Dividing $L\phi = \pm i\phi$ by c_1 we have the differential equation

$$(2.2) \quad (z^3 + \omega z)\phi'' + [(3 + \alpha)z^2 + \beta]\phi' + 2\alpha z\phi = \lambda\phi,$$

where $\omega = \bar{c}_1/c_1$, $\alpha = c_2/c_1$, $\beta = \bar{c}_2/c_1$, and $\lambda = \pm i/c_1$.

Substituting $\sum_{j=0}^{\infty} a_j z^j$ into (2.2) we obtain

$$(2.3) \quad \beta a_1 + \sum_{j=1}^{\infty} [(j+1)(\omega j + \beta)a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1}]z^j \\ = \lambda a_0 + \sum_{j=1}^{\infty} \lambda a_j z^j \quad \lambda \neq 0.$$

If $\beta = 0$ we have $a_0 = 0$ and (2.3) can be solved recursively for a_2, a_3, \dots , in terms of a_1 since $\omega j + \beta$ never vanishes. Thus we have but one analytic solution

$$\phi(z) = z(1 + a_2 z^2 + \dots).$$

If $\beta \neq 0$, we have $a_1 = \lambda a_0 / \beta$ and (2.3) can be solved recursively for a_2, a_3, \dots , provided that $(\omega j + \beta)$ never vanishes for $j = 1, 2, \dots$. Thus we are able to obtain the single formal power series solution $\phi(z) = 1 + a_1 z + a_2 z^2 + \dots$. The case when $(\omega j + \beta)$ vanishes for some positive integer j presents some complications and will be considered later in the proof. Solving (2.3) for a_{j+1} we have

$$(2.4) \quad a_{j+1} = \frac{1}{\omega} \left\{ \frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + \left(1 + \frac{\beta}{\omega}\right)j + \frac{\beta}{\omega}} \right\}.$$

But $\beta/\omega = \bar{c}_2/\bar{c}_1 = \bar{\alpha}$, hence (2.4) becomes

$$(2.4) \quad a_{j+1} = \frac{1}{\omega} \left\{ \frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}} \right\}.$$

Thus we obtain the estimate

$$(2.5) \quad |a_{j+1}| \leq \frac{1}{|\omega|} \left| \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}} \right| |a_{j-1}| + \frac{|\lambda|}{|\omega|} \frac{1}{|j^2 + (1 + \bar{\alpha})j + \bar{\alpha}|} |a_j|.$$

Since $|\omega| = 1$ we have

$$(2.6) \quad |a_{j+1}| \leq |u_1(j)| |a_{j-1}| + |u_2(j)| |a_j|,$$

where

$$u_1(j) = \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}},$$

and

$$u_2(j) = \frac{\lambda}{j^2 + (1 + \bar{\alpha})j + \bar{\alpha}}.$$

We now estimate $|u_1(j)|$ and $|u_2(j)|$ for large j . Since $|u_2(j)|$ tends to zero as j^{-2} it follows that there exists an $M > 0$ such that

$$(2.7) \quad |u_2(j)| \leq \frac{M}{j^2}, \quad \text{for } j \text{ sufficiently large.}$$

Concerning $|u_1(j)|$ we obtain, upon dividing,

$$u_1(j) = \left(1 - \frac{1}{j}\right) + \frac{2}{j} \operatorname{Im}(\alpha)i + O(j^{-2}).$$

Thus $|u_1(j)|^2 = 1 - 2/j + O(j^{-2})$, and hence by a direct calculation,

$$|u_1(j)| = 1 - \frac{1}{j} + O(j^{-2}).$$

For $\xi > 0$, we note that $|u_1(j)| \leq 1 - \xi j^{-1}$ for j sufficiently large if and only if $-1 < -\xi$, or $\xi < 1$. Hence we have

$$(2.8) \quad |u_i(j)| \leq 1 - \frac{\xi}{j}, \quad \text{for } j \text{ sufficiently large}$$

and $0 < \xi < 1$.

Using (2.6), (2.7), and (2.8) we obtain, for j sufficiently large,

$$\begin{aligned} |a_{j+1}| &\leq (1 - \xi j^{-1})|a_{j-1}| + Mj^{-2}||a_j| \\ &\leq (1 - \xi j^{-1} + Mj^{-2})M(j), \quad 0 < \xi < 1, \end{aligned}$$

where $M(j) = \max\{|a_{j-1}|, |a_j|\}$.

Thus, for sufficiently large j , we have

$$(2.9) \quad |a_{j+1}| \leq (1 - \gamma j^{-1})M(j),$$

where $0 < \gamma = \xi/2 < \frac{1}{2}$.

Now consider the expression $(1 - \gamma j^{-1})(j - 1)^{-1/p}$, where p is a positive integer. This is dominated by $(j + 1)^{-1/p}$ for j sufficiently large if and only if

$$j^{p+1} + (-p\gamma + 1)j^p + \dots \leq j^{p+1} - j^p.$$

Hence, if and only if $-p\gamma + 1 < -1$ or $-p\gamma < -2$. Since $\gamma > 0$, $p > 2/\gamma$. Thus we have

$$(2.10) \quad (1 - \gamma j^{-1})(j - 1)^{-1/p} \leq (j + 1)^{-1/p}, \quad p > \frac{2}{\gamma}.$$

We now show that there exists a positive constant K for which $|a_j| \leq Kj^{-1/p}$ for $j \geq 1$. Let j_1 be such that (2.9) and (2.10) hold for $j > j_1$. Let $K = \max_{j \leq j_1} |a_j|j^{1/p}$ so that $|a_j| \leq Kj^{-1/p}$ for $j \leq j_1$. Using (2.9) it follows that

$$|a_{j+1}| \leq (1 - \gamma j^{-1})M(j),$$

where

$$\begin{aligned} M(j) &= \text{Max}(Kj_1^{-1/p}, K(j_1 - 1)^{-1/p}) \\ &= K(j_1 - 1)^{-1/p}. \end{aligned}$$

Hence,

$$|a_{j+1}| \leq (1 - \gamma j^{-1})K(j_1 - 1)^{-1/p},$$

and using (2.10) we have

$$(2.11) \quad |a_{j_1+1}| \leq K(j_1 + 1)^{-1/p}.$$

We now proceed inductively to establish

$$(2.12) \quad |a_{j_1+k}| \leq K(j_1 + k)^{-1/p}, \quad k = 2, 3, \dots.$$

Let

$$\begin{aligned} K_1 &= \max_{j \leq j_1+1} |a_j| j^{1/p} \\ &= \max \{K, K(j_1 + 1)^{-1/p}\} \leq K, \end{aligned}$$

making use of (2.11). Using (2.9) we have

$$|a_{j_1+2}| \leq (1 - \gamma(j_1 + 1)^{-1})M(j_1 + 1),$$

where,

$$\begin{aligned} M(j_1 + 1) &= \text{Max}(|a_{j_1+1}|, |a_{j_1}|) \\ &= \text{Max}(K(j_1 + 1)^{-1/p}, K(j_1)^{-1/p}) \\ &= K(j_1)^{-1/p}. \end{aligned}$$

It follows from (2.10) that

$$\begin{aligned} |a_{j_1+2}| &\leq (1 - \gamma(j_1 + 1)^{-1})K(j_1)^{-1/p} \\ &\leq K(j_1 + 2)^{-1/p}. \end{aligned}$$

Continuing on in this manner we establish (2.12). Hence any solution $\sum_{j=0}^{\infty} a_j z^j$ whose coefficients satisfy (2.4) is in \mathcal{H} . To complete the proof we have only to deal with the case where $j\omega + \beta$ vanishes for some positive integer j .

We now consider the case when $j\omega + \beta$ vanishes for some positive integer n . The analytic solution obtained from (2.3) by taking $a_0 = a_1 = \dots = a_n = 0$, and solving recursively for a_{n+2}, a_{n+3}, \dots , in terms of a_{n+1} is, as we have seen, in \mathcal{H} . If there were a second analytic solution corresponding to $a_0 \neq 0$ it would be in \mathcal{H} as well, and $m^+(m^-)$ would be 2. We now show that this is not the case, i.e., $m^+ = m^- = 1$. To do this we make use of the following result.

Let μ be such that $\text{Im}(\mu) > 0$ and let \mathcal{D}_μ^+ be the nullspace of the operator $S^* - \mu$. Then the dimension of \mathcal{D}_μ^+ is equal to m^+ . Similarly,

let $\text{Im}(\mu) < 0$ and let \mathcal{D}_μ^- be the nullspace of the operator $S^* - \mu$, then the dimension of \mathcal{D}_μ^- is equal to m^- , [1, p. 1232].

Using this we see that m^+ is just the number of linearly independent solutions of $L\phi = \mu\phi$ in \mathcal{H} for any μ such that $\text{Im}(\mu) > 0$. Similarly, m^- is the number of linearly independent solutions of $L\phi = \mu\phi$ in \mathcal{H} for any μ such that $\text{Im}(\mu) < 0$. Hence, if we can show that there exist μ such that $\text{Im} \mu > 0$ ($\text{Im} \mu < 0$) for which there is no analytic solution corresponding to $a_0 \neq 0$ we will have shown that $m^+ = m^- = 1$.

Consider (2.3), where λ is now μ/c_2 , and suppose that $\beta = -n\omega$. Taking $j = 1, 2, \dots, n$ we obtain the following set of $n + 1$ linear equations in a_0 thru a_n :

$$\begin{aligned} -n\omega a_1 &= \lambda a_0 \\ (j+1)(j-n)\omega a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1} &= \lambda a_j, \\ & j = 1, 2, \dots, n-1 \\ (n^2 + n\alpha + \alpha - 1)a_{n-1} &= \lambda a_n. \end{aligned}$$

Thus we are led to consider the homogeneous system

$$\begin{aligned} -\lambda a_0 - n\omega a_1 &= 0 \\ 2\alpha a_0 - \lambda a_1 + 2(2-n)\omega a_2 &= 0 \\ (n^2 + n\alpha - 2n)a_{n-2} - \lambda a_{n-1} - n\omega a_n &= 0 \\ (n^2 + n\alpha + \alpha - 1)a_{n-1} - \lambda a_n &= 0 \end{aligned}$$

Since the parameter $\lambda = \mu/c_2$ appears only on the diagonal the system determinant $D_n(\lambda)$ is a polynomial in λ of degree $n + 1$,

$$D_n(\lambda) = (-1)^{n+1} \lambda^{n+1} + \dots$$

Thus $D_n(\lambda)$ vanishes at most $n + 1$ points in the complex plane, and we can find μ in the upper half-plane and lower half-plane for which $D_n(\mu/c_2) \neq 0$. Thus $a_0 = a_1 = \dots = a_n = 0$ and there is only one analytic solution of $L\phi = \mu\phi$.

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Received March 27, 1974 and in revised form May 24, 1974.

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