SUBHARMONICITY AND HULLS

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For $X$ a compact set in $C^2$, $h(X)$ denotes the polynomially convex hull of $X$. We are concerned with the existence of analytic varieties in $h(X) \setminus X$. $X$ is called "invariant" if $(z, w)$ in $X$ implies $(e^{i\theta}z, e^{-i\theta}w)$ is in $X$, for all real $\theta$. $X$ is called an "invariant disk" if there is a continuous complex-valued function $a$ defined on $0 \leq r \leq 1$ with $a(0) = a(1) = 0$, such that $X = \{(z, w) \mid z \leq 1, w = a(|z|/z)\}$. Let $X$ be an invariant set and put $f(z, w) = zw$. Let $\Omega$ be an open disk in $C \setminus f(X)$ and put $f^{-1}(\Omega) = \{(z, w) \in h(X) \mid zw \in \Omega\}$. In Theorem 2 we show that if $f^{-1}(\Omega)$ is not empty, then $f^{-1}(\Omega)$ contains an analytic variety. Let now $X$ be an invariant disk, with certain hypotheses on the function $a$. Then we show in Theorem 3 that $f^{-1}(\Omega)$ is the union of a one-parameter family of analytic varieties. A key tool in the proofs is a general subharmonicity property of certain functions associated to a uniform algebra. This property is given in Theorem 1.

1. Let $X$ be a compact Hausdorff space, let $A$ be a uniform algebra on $X$ and let $M$ be the maximal ideal space of $A$.

Fix $f \in A$. For each $\zeta \in C$ put $f^{-1}(\zeta) = \{p \in M \mid f(p) = \zeta\}$ and for each subset $\Omega$ of $C$, put $f^{-1}(\Omega) = \{p \in M \mid f(p) \in \Omega\}$. Consider an open subset $\Omega$ of $C \setminus f(X)$. Supposing $f^{-1}(\Omega)$ to be nonempty, what can be said about the structure of $f^{-1}(\Omega)$? Work of Bishop [2] and Basener [1] yields that if $f^{-1}(\zeta)$ is at most countable for each $\zeta \in \Omega$, then $f^{-1}(\Omega)$ contains analytic disks. On the other hand, Cole [4] has given an example where no analytic disk is contained in $f^{-1}(\Omega)$. In §2 we prove.

**Theorem 1.** Let $\Omega$ be an open subset of $C \setminus f(X)$. Choose $g \in A$. Define $Z(\zeta) = \sup_{f^{-1}(\zeta)} |g|$, $\zeta \in \Omega$. Then $\log Z$ is subharmonic in $\Omega$.

This theorem is proved by a method of Oka in [5].

In §3 we apply Theorem 1 to the following situation: $X$ is a compact set in $C^2$, $A$ is the uniform closure on $X$ of polynomials in $z$ and $w$. Here $M = h(X)$, the polynomially convex hull of $X$. We assume that $X$ is invariant under the map $T_\theta$:

$$(z, w) \rightarrow (e^{i\theta}z, e^{-i\theta}w) \quad \text{for} \quad 0 \leq \theta < 2\pi.$$
Put \( f = zw \). Let \( \Omega \) be an open disk contained in \( C \setminus f(X) \) with \( 0 \notin \Omega \). Here \( f^{-1}(\Omega) = \{(z, w) \in h(X) \mid zw \in \Omega\} \).

**Theorem 2.** If \( f^{-1}(\Omega) \) is not empty, then \( f^{-1}(\Omega) \) contains an analytic disk.

In §4, we consider the case when \( X \) is a disk in \( C \), defined:

\[
X = \left\{(z, w) \mid |z| \leq 1, \ w = \frac{a(z)}{z}\right\},
\]

where \( a \) is a continuous complex valued function defined on \( 0 \leq r \leq 1 \), with \( a(r) = o(r) \).

\( X \) is evidently invariant under \( T_\theta \) for all \( \theta \). In Theorem 3 we give an explicit description of \( h(x) \) for a certain class of such disks \( X \).

2. Proof of Theorem 1. (Cf. [5], §2.) Fix \( \zeta_0 \in \Omega \) and let \( \zeta_n \to \zeta_0 \). Assume \( Z(\zeta_n) \to t \). We claim \( Z(\zeta_0) \geq t \). For choose \( p_n \) in \( f^{-1}(\zeta_n) \) with \( g(p_n) = Z(\zeta_n) \). Let \( p \) be an accumulation point of \( \{p_n\} \). Then \( |g(p)| \geq t \), whence \( Z(\zeta_0) \geq t \), as claimed. Thus \( Z \) is upper-semicontinuous at \( \zeta_0 \), and so \( Z \) is upper-semicontinuous in \( \Omega \).

Theorem 1.6.3 of [6] gives that an upper-semicontinuous function \( u \) in \( \Omega \) is subharmonic provided for each closed disk \( D \subset \Omega \) and each polynomial \( P \) we have

(1) \( u \leq \text{Re } P \) on \( \partial D \) implies \( u \leq \text{Re } P \) on \( D \).

Fix a closed disk \( D \) contained in \( \Omega \) and let \( \hat{D} \) be its interior. Choose a polynomial \( P \) such that \( \log Z \leq \text{Re } P \) on \( \partial D \). Then

\[
Z(\zeta) \exp(-P(\zeta)) \leq 1 \quad \text{on } \partial D.
\]

Hence for each \( \zeta \) in \( \partial D \), if \( x \) is in \( f^{-1}(\zeta) \), then

(2) \[ |g(x)| \cdot |\exp(-P(f))(x)| \leq 1, \quad \text{or} \]

\[
|g \cdot \exp(-P(f))| \leq 1 \quad \text{at } x.
\]

Now \( g \cdot \exp(-P(f)) \) is in \( A \). Put \( N = f^{-1}(\hat{D}) \). The boundary of \( N \) is contained in \( f^{-1}(\partial D) \). Hence by the Local Maximum Modulus Principle for uniform algebras, for each \( y \) in \( N \) we can find \( x \) in \( f^{-1}(\partial D) \) with

\[
|g \exp(-P(f))(y)| \leq |g \cdot \exp(-P(f))(x)|,
\]
whence by (2) we have

\[(3) \quad |g \cdot \exp(-P(f))(y)| \leq 1.\]

Fix \(\zeta_0\) in \(\hat{D}\). Choose \(y\) in \(f^{-1}(\zeta_0)\) with \(|g(y)| = Z(\zeta_0)\). Applying (3) to this \(y\), we get

\[(4) \quad Z(\zeta_0)|\exp(-P(\zeta_0))| \leq 1.\]

Hence \(\log Z(\zeta_0) \leq \Re P(\zeta_0)\). So (1) is satisfied, and so \(\log Z\) is subharmonic in \(\Omega\), as desired.

3. Proof of Theorem 2. Since \(X\) is invariant under the maps \(T_\theta, h(X)\) is invariant under each \(T_\theta\). Fix \(\zeta \in \Omega\). There are two possibilities:

(a) \(|z|\) is constant on \(f^{-1}(\zeta)\).
(b) \(\exists r_1, r_2\) with \(0 < r_1 < r_2\) and \(\exists (z_1, w_1), (z_2, w_2) \in f^{-1}(\zeta)\) with \(|z_1| = r_1, |z_2| = r_2\).

Suppose (b) occurs. Then the circles: \(z = r_1 e^{i\theta}, w = \zeta |z| e^{i\theta}\), \(0 \leq \theta \leq 2\pi\) and \(z = r_2 e^{i\theta}, w = \zeta |w| e^{i\theta}\), \(0 \leq \theta \leq 2\pi\) both lie in \(h(X)\). Hence the analytic annulus: \(r_1 < |z| < r_2\), \(w = \zeta \) lies in \(f^{-1}(\zeta)\). Thus if (b) occurs at any point \(\zeta\) in \(\Omega\), \(f^{-1}(\Omega)\) does contain an analytic disk. Hence to prove the Theorem, we may assume that (a) holds for each \(\zeta \in \Omega\). Define, for \(\zeta \in \Omega\), \(Z(\zeta) = \sup_{f^{-1}(\zeta)} |z|, W(\zeta) = \sup_{f^{-1}(\zeta)} |w|\). Fix \((z_0, w_0) \in f^{-1}(\zeta)\). Since we have case (a), \(Z(\zeta) = |z_0|\). Hence \(W(\zeta) = |w_0|\) and so \(Z(\zeta)W(\zeta) = |\zeta|\), whence

\[\log Z(\zeta) + \log W(\zeta) = \log |\zeta| .\]

Since \(\log Z\) and \(\log W\) are subharmonic in \(\Omega\) while \(\log |\zeta|\) is harmonic, \(\log Z, \log W\) are in fact harmonic in \(\Omega\). Put \(U = \log Z\) and let \(V\) be the harmonic conjugate of \(U\) in \(\Omega\). Put \(\phi(\zeta) = e^{U+iV}(\zeta)\). Then \(\phi\) is analytic in \(\Omega\) and \(|\phi| = Z\) in \(\Omega\).

Assertion. The variety \(z = \phi(\zeta), w = \zeta / \phi(\zeta), \zeta \in \Omega\), is contained in \(h(X)\).

Fix \(\zeta \in \Omega\). Choose \((z_1, w_1) \in f^{-1}(\zeta)\). Then \(Z(\zeta) = |z_1|\), so \(|\phi(\zeta)| = |z_1|\), i.e., \(\exists\) real \(\alpha\) with \(z_1 = \phi(\zeta)e^{i\alpha}\). Then \(w_1 = \zeta / \phi(\zeta)e^{i\alpha}\). But \((e^{-i\alpha} z_1, e^{i\alpha} w_1) \in h(X)\). Hence \((\phi(\zeta), \zeta / \phi(\zeta)) \in h(X)\). The Assertion is proved, and Theorem 2 follows.

Note. Questions related to the result just proved are studied by J. E. Björk in [3].
4. Invariant disks in $C^2$. Let $P$ be a polynomial with complex coefficients, $P(t) = \sum_{n=1}^{N} c_n t^n$, which is one-one on the unit interval with endpoints identified, i.e., we assume that $P(1) = P(0) = 0$ and $P(t_1) \neq P(t_2)$ if $0 \leq t_1 < t_2 < 1$. Also assume $P'(t) \neq 0$ for $0 \leq t \leq 1$. Then the curve $\beta$ given parametrically: $\zeta = P(t)$, $0 \leq t \leq 1$, is a simple closed analytic curve in the $\zeta$-plane whose only singularity is a double-point at the origin. Denote by $\theta$ the angle between the two arcs of $\beta$ meeting at $0$. Assume $\theta < \pi$. Define $\alpha(r) = P(r^2)$, i.e.,

$$a(r) = \sum_{n=1}^{N} c_n r^{2n}.$$  

Let $X$ be the disk in $C^2$ defined

$$X = \left\{ \left( z, \frac{a(|z|)}{z} \right) \biggm| |z| \leq 1 \right\}.$$  

The function $f = zw$ maps $X$ on $\beta$. Denote by $\Omega$ the interior of $\beta$.

**Theorem 3.** \exists function $\phi$ analytic in $\Omega$ such that $h(X)$ is the union of $X$ and $\{(z,0) \biggm| |z| \leq 1\}$ and

$$\{(z,w) \biggm| zw \in \Omega \quad \text{and} \quad |z| = |\phi(zw)|\}.$$  

**Corollary.** Every point of $h(X) \setminus X$ lies on some analytic disk contained in $h(X)$.

**Notation.** $A(\Omega)$ denotes the class of functions $F$ defined and continuous in $\bar{\Omega}$ and analytic in $\Omega$.

$F$ denotes the algebra of functions on $|z| \leq 1$ which are uniformly approximable by polynomials in $-z$ and $a(|z|)/z$.

**Lemma 1.** Let $G \in C[0,1]$. If $G(|z|) \in F$, then $\exists F \in A(\Omega)$ such that $G(r) = F(a(r))$ for $0 \leq r \leq 1$.

**Proof.** Let $g$ be a polynomial in $z$ and $a(|z|)/z$. Calculation gives that there is a polynomial $\tilde{g}$ in one variable with

$$\frac{1}{2\pi} \int_{0}^{2\pi} g(re^{i\theta}) d\theta = \tilde{g}(a(r)), \quad 0 \leq r \leq 1.$$  

Choose a sequence $\{g_n\}$ of polynomials in $z$ and $a(|z|)/z$ approaching
$G(|z|)$ uniformly on $|z| \leq 1$. Then $g_n(a(r)) \to G(r)$ uniformly on $0 \leq r \leq 1$. Hence $\exists F \in A(\Omega)$ with $g_n \to F$ uniformly on $\beta$, so $G(r) = F(a(r))$.

**Lemma 2.** If $f = zw$, then $f^{-1}(\Omega)$ is not empty.

**Proof.** Fix $\zeta_0 \in \Omega$. If $f^{-1}(\Omega)$ is empty, then $f - \zeta_0 \neq 0$ on $h(X)$ and so $(zw - \zeta_0)^{-1}$ lies in the closure of the polynomials in $z$ and $w$ on $X$. Then $(a(|z| - \zeta_0)^{-1}) \in \mathbb{A}$. By Lemma 1, $\exists F \in A(\Omega)$ with $F(a(r)) = (a(r) - \zeta_0)^{-1}$. Then $(\zeta - \zeta_0)^{-1} \in A(\Omega)$, which is false. So $f^{-1}(\Omega)$ is not empty.

**Lemma 3.** Fix $\zeta \in \beta \setminus \{0\}$. Let $(z_0, w_0)$ be a point in $h(X)$ with $z_0w_0 = \zeta$. Then $(z_0, w_0) \in X$.

**Proof.** Assume $(z_0, w_0) \not\in X$. Let $r$ be the point in $(0, 1)$ with $a(r) = \zeta$. Put, for each $r$, $\gamma_r = \{(re^{i\theta}, (a(r)/re^{i\theta})) | 0 \leq \theta < 2\pi\}$. Then $\gamma$ is a polynomially convex circle contained in $X$. Hence $\exists$ polynomial $P$ with $|P(z_0, w_0)| > 2$, $|P| < 1$ on $\gamma_r$. Choose a neighborhood $N$ of $\gamma_r$ on $X$ where $|P| < 1$. The image of $X \setminus N$ under the map $(z, w) \to zw$ is a closed subarc $\beta$, of $\beta$ which excludes $\zeta$. Choose $F \in A(\Omega)$ with $F(\zeta) = 1$, $|F| < 1$ on $\beta \setminus \{\zeta\}$. Then $\exists \delta > 0$ such that $|F| < 1 - \delta$ on $\beta$. Hence $|F(zw)| < 1 - \delta$ on $X \setminus N$. Also $|F(zw)| \leq 1$ on $X$. Fix $n$ and put

$$Q = F(zw)^n \cdot P(z, w).$$

$|Q(z_0, w_0)| > 2$. On $N$, $|Q| \leq |P| < 1$. On $X \setminus N$, $|Q| < (1 - \delta)^n \cdot \max_X |P|$, and so $|Q| < 1$ on $X \setminus N$ for large $n$. Then $|Q| < 1$ on $X$. Since $F$ is a uniform limit on $\beta$ of polynomials in $\zeta, Q$ is a uniform limit on $X \cup \{(z_0, w_0)\}$ of polynomials in $z$ and $w$. This contradicts that $(z_0, w_0) \in h(X)$. Thus $(z_0, w_0) \in X$. We are done.

**Note.** Since $f$ maps $X$ on $\beta$ and $C \setminus f(X)$ is the union of the interior and exterior of $\beta$, we conclude from the last Lemma that $h(X)$ is the union of $X$ and $f^{-1}(\{0\})$ and $f^{-1}(\Omega)$.

We need some notation now. For each $\zeta \in \beta \setminus \{0\}$, denote by $r(\zeta)$ the unique $r$ in $(0, 1)$ with $a(r) = \zeta$.

Since $a$ is a polynomial in $r$ vanishing at 0, there is a constant $d > 0$ such that

$$r(\zeta) > d \cdot |\zeta|, \quad \text{all } \zeta \in \beta.$$
For $\zeta_0 \in \Omega$, denote by $\mu_{\zeta_0}$ harmonic measure at $\zeta_0$ relative to $\Omega$. Since $\beta$ consists of analytic arcs, with one jump-discontinuity for the tangent at $\zeta = 0$, $\mu_{\zeta_0} = K_{\zeta_0} ds$, where $K_{\zeta_0}$ is a bounded functions on $\beta$ and $ds$ is arc-length. Define

$$U(\zeta_0) = \int_{\beta} \log r(\zeta) d\mu_{\zeta_0}(\zeta).$$

Since (7) holds, this integral converges absolutely. $U$ is a harmonic function in $\Omega$, bounded above, and continuous at each boundary point $\zeta \in \beta \setminus \{0\}$ with boundary value $\log r(\zeta)$ at $\zeta$.

For $\zeta \in \Omega$, define

$$Z(\zeta) = \sup_{f^{-1}(\zeta)} |z|, \quad W(\zeta) = \sup_{f^{-1}(\zeta)} |w|.$$

**Lemma 4.** For all $\zeta \in \Omega$, $\log Z(\zeta) \leq U(\zeta)$ and $\log W(\zeta) \leq \log |\zeta| - U(\zeta)$.

**Proof.** Fix $\zeta \in \beta \setminus \{0\}$, choose $\xi_n \in \Omega$ with $\xi_n \to \zeta$ and suppose $Z(\xi_n) \to \lambda$. Choose $p_n \in f^{-1}(\xi_n)$ with $Z(\xi_n) = |z(p_n)|$. Without loss of generality, $p_n \to p$ for some point $p \in h(X)$. Then $f(p) = \zeta$. By Lemma 3, $p \in X$, i.e., $p = (re^{i\theta}, (a(r)/re^{i\theta}))$ for some $r, \theta$. Also $a(r) = \zeta$ and so $r = r(\zeta)$, whence $|z(p_n)| \to r(\zeta)$ and so $\lambda = r(\zeta)$. Thus $Z(\xi') \to r(\zeta)$ as $\xi' \to \zeta$ from within $\Omega$, and so $\log Z$ assumes the same boundary values as $U$, continuously on $\beta \setminus \{0\}$.

For each positive integer $k$, let $\Omega_k = \{\zeta \in \Omega \mid |\zeta| > 1/k\}$. $\partial \Omega_k$ is the union of a closed subarc $\beta_k$ of $\beta \setminus \{0\}$ and an arc $\alpha_k$ on the circle $|\zeta| = 1/k$.

Fix $\zeta_0 \in \Omega$. For large $k$, $\zeta_0 \in \Omega_k$. Denote by $\mu_{\zeta_0}^{(k)}$ the harmonic measure at $\zeta_0$ relative to $\Omega_k$. An elementary estimate gives that there is a constant $C_{\zeta_0}$ independent of $k$ such that

$$\mu_{\zeta_0}^{(k)}(\alpha_k) \leq C_{\zeta_0} \cdot \frac{1}{\sqrt{k}} \text{ for all } k.$$  

(8)

Let $S$ be any function subharmonic in $\Omega$ and assuming continuous boundary values, again denoted $S$, on $\beta \setminus \{0\}$. Assume $\exists$ constant $M$ with $S \leq M$ in $\Omega$. Then for all $k$,

$$S(\zeta_0) \leq \int_{\beta_k} Sd\mu_{\zeta_0}^{(k)} + \int_{\alpha_k} M\mu_{\zeta_0}^{(k)}, \text{ whence}$$

$$S(\zeta_0) \leq \int_{\beta_k} Sd\mu_{\zeta_0}^{(k)} + M \cdot C_{\zeta_0} \cdot \frac{1}{\sqrt{k}}.$$  

(9)
Applying (9) with $S = \log Z$, we get

$$\log Z(\zeta_0) \leq \int_{\beta_k} Ud\mu_{\zeta_0}^{(k)} + MC_{\zeta_0} \cdot \frac{1}{\sqrt{k}},$$

since as we saw earlier, $\log Z = U$ on $\beta \setminus \{0\}$.

By (7), if $\zeta' \in \alpha_k$,

$$U(\zeta') = \int_\beta \log r(\zeta) d\mu_{\zeta}(\zeta) > C + \int_\beta \log |\zeta| d\mu_{\zeta}(\zeta),$$

where $C$ is a constant, so

$$U(\zeta') > C + \log |\zeta'| = C + \log \frac{1}{k}.$$  
Hence

$$U(\zeta_0) = \int_{\beta_k} Ud\mu_{\zeta_0}^{(k)} + \int_{\alpha_k} Ud\mu_{\zeta_0}^{(k)}$$
$$\geq \int_{\beta_k} Ud\mu_{\zeta_0}^{(k)} + \left( C + \log \frac{1}{k} \right) \frac{C_{\zeta_0}}{\sqrt{k}}.$$

Combining this with (10) and letting $k \to \infty$, we get that $\log Z(\zeta_0) \leq U(\zeta_0)$, as desired. A parallel argument gives the assertion regarding $W$. We are done.

**Lemma 5.**  With $Z$ defined as above, $\log Z(\zeta) = U(\zeta)$ for all $\zeta \in \Omega$, and $\log W(\zeta) = \log |\zeta| - U(\zeta)$.

**Proof.**  Suppose either equality fails at some point $\zeta_0$. By the last Lemma, this implies that

$$\log Z(\zeta_0) + \log W(\zeta_0) < \log |\zeta_0|.$$

Fix $p \in f^{-1}(\zeta_0)$. Then $|z(p)| \leq Z(\zeta_0), |w(p)| \leq W(\zeta_0)$, so

$$\log |z(p)w(p)| < \log |\zeta_0|.$$

But $z(p)w(p) = \zeta_0$, so we have a contradiction, proving the Lemma.

**Proof of Theorem 3.**  Let $V$ denote the harmonic conjugate of $U$ in $\Omega$ and put $\phi = e^{U+iV}$. Fix $(z_0, w_0) \in f^{-1}(\Omega)$ and put $\zeta_0 = z_0 \cdot w_0$. Unless $|z_0| = Z(\zeta_0)$ and $|w_0| = W(\zeta_0)$, we have

$$|\zeta_0| = |z_0| |w_0| < Z(\zeta_0) W(\zeta_0) = |\zeta_0|$$

by the last Lemma. So we must have $|z_0| = Z(\zeta_0) = |\phi(\zeta_0)|$. 
Conversely fix $\zeta_0 \in \Omega$ and let $(z_0, w_0)$ be a point in $C^2$ such that $z_0 \cdot w_0 = \zeta_0$ and $|z_0| = |\phi(\zeta_0)|$. Choose $(z_1, w_1) \in f^{-1}(\zeta_0)$. By the preceding $|z_1| = |\phi(\zeta_0)|$, so $\exists$ real $\alpha$ with $z_0 = e^{ia}z_1$, $w_0 = e^{-ia}w_1$. Hence $(z_0, w_0) \in h(X)$, so $(z_0, w_0) \in f^{-1}(\Omega)$. Thus $f^{-1}(\Omega)$ consists precisely of those points $(z, w)$ with $zw \in \Omega$ and $|z| = |\phi(zw)|$.

To finish the proof we need only identify $f^{-1}(0)$. The circle $\{(z, 0) \mid |z| = 1\}$ lies in $X$, so the disk $D: \{(z, 0) \mid |z| \leq 1\}$ is contained in $f^{-1}(0)$. If $(z_0, w_0) \in f^{-1}(0)$ and does not lie in $D$, then $z_0 = 0, w_0 \neq 0$. The same argument as was used in proving Lemma 3 shows that then $(z_0, w_0) \in h(X)$, contrary to assumption. So $f^{-1}(0) = D$, and the proof of Theorem 3 is finished.

**Remark.** As we have just seen, $f^{-1}(\Omega)$ is the union of varieties $V_\alpha$, $0 \leq \alpha < 2\pi$, where $V_\alpha$ is defined:

$$z = e^{ia}\phi(\zeta), \quad w = e^{-ia}\frac{\zeta}{\phi(\zeta)}, \quad \zeta \in \Omega.$$ 

What does the boundary of such a variety $V_\alpha$ in $h(X)$ look like? It splits into two sets:

$$S = \{(z, w) \in \partial V_\alpha \mid zw \in \beta \setminus \{0\}\} \quad \text{and} \quad T = \{(z, w) \in \partial V_\alpha \mid zw = 0\}.$$ 

It is easy to see that $S$ is an arc on $X$ cutting each circle: $\{(z, w) \in X \mid |z| = r\}, \ 0 < r < 1$, exactly once while $T$ is a closed subset of the disk $D = \{(z, 0) \mid |z| \leq 1\}$.

It is remarkable that even though $X$ is itself very regular, the rest of the hull of $X$ is attached to $X$ in a very complicated way.

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**References**


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