SUBHARMONICITY AND HULLS

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For $X$ a compact set in $C^2$, $h(X)$ denotes the polynomially convex hull of $X$. We are concerned with the existence of analytic varieties in $h(X) \setminus X$. $X$ is called "invariant" if $(z, w)$ in $X$ implies $(e^{i\theta}z, e^{-i\theta}w)$ is in $X$, for all real $\theta$. $X$ is called an "invariant disk" if there is a continuous complex-valued function $a$ defined on $\mathbb{D}$ with $a(0) = a(1) = 0$, such that $X = \{(z, w) \mid z \leq 1, w = a(|z|)/z\}$. Let $X$ be an invariant set and put $f(z, w) = zw$. Let $\Omega$ be an open disk in $C \setminus f(X)$ and put $f^{-1}(\Omega) = \{(z, w) \in h(X) \mid zw \in \Omega\}$. In Theorem 2 we show that if $f^{-1}(\Omega)$ is not empty, then $f^{-1}(\Omega)$ contains an analytic variety. Let now $X$ be an invariant disk, with certain hypotheses on the function $a$. Then we show in Theorem 3 that $f^{-1}(\Omega)$ is the union of a one-parameter family of analytic varieties. A key tool in the proofs is a general subharmonicity property of certain functions associated to a uniform algebra. This property is given in Theorem 1.

1. Let $X$ be a compact Hausdorff space, let $A$ be a uniform algebra on $X$ and let $M$ be the maximal ideal space of $A$.

Fix $f \in A$. For each $\zeta \in C$ put $f^{-1}(\zeta) = \{p \in M \mid f(p) = \zeta\}$ and for each subset $\Omega$ of $C$, put $f^{-1}(\Omega) = \{p \in M \mid f(p) \in \Omega\}$. Consider an open subset $\Omega$ of $C \setminus f(X)$. Supposing $f^{-1}(\Omega)$ to be nonempty, what can be said about the structure of $f^{-1}(\Omega)$? Work of Bishop [2] and Basener [1] yields that if $f^{-1}(\zeta)$ is at most countable for each $\zeta \in \Omega$, then $f^{-1}(\Omega)$ contains analytic disks. On the other hand, Cole [4] has given an example where no analytic disk is contained in $f^{-1}(\Omega)$. In §2 we prove.

THEOREM 1. Let $\Omega$ be an open subset of $C \setminus f(X)$. Choose $g \in A$. Define $Z(\zeta) = \sup_{f^{-1}(\zeta)}|g|$, $\zeta \in \Omega$. Then $\log Z$ is subharmonic in $\Omega$.

This theorem is proved by a method of Oka in [5].

In §3 we apply Theorem 1 to the following situation: $X$ is a compact set in $C^2$, $A$ is the uniform closure on $X$ of polynomials in $z$ and $w$. Here $M = h(X)$, the polynomially convex hull of $X$. We assume that $X$ is invariant under the map $T_\theta$:

$$(z, w) \rightarrow (e^{i\theta}z, e^{-i\theta}w) \quad \text{for} \quad 0 \leq \theta < 2\pi.$$
Put $f = zw$. Let $\Omega$ be an open disk contained in $\mathbb{C} \setminus f(X)$ with $0 \not\in \Omega$. Here $f^{-1}(\Omega) = \{(z, w) \in h(X) \mid zw \in \Omega\}$.

**Theorem 2.** If $f^{-1}(\Omega)$ is not empty, then $f^{-1}(\Omega)$ contains an analytic disk.

In §4, we consider the case when $X$ is a disk in $\mathbb{C}^2$, defined:

$$X = \left\{(z, w) \mid |z| \leq 1, \ w = \frac{a(|z|)}{z}\right\},$$

where $a$ is a continuous complex valued function defined on $0 \leq r \leq 1$, with $a(r) = o(r)$.

$X$ is evidently invariant under $T_\theta$ for all $\theta$. In Theorem 3 we give an explicit description of $h(x)$ for a certain class of such disks $X$.

**2. Proof of Theorem 1.** (Cf. [5], §2.) Fix $\zeta_0 \in \Omega$ and let $\zeta_n \to \zeta_0$. Assume $Z(\zeta_n) \to t$. We claim $Z(\zeta_0) \geq t$. For choose $p_n$ in $f^{-1}(\zeta_n)$ with $|g(p_n)| = Z(\zeta_n)$. Let $p$ be an accumulation point of $\{p_n\}$. Then $|g(p)| \geq t$, whence $Z(\zeta_0) \geq t$, as claimed. Thus $Z$ is upper-semicontinuous at $\zeta_0$, and so $Z$ is upper-semicontinuous in $\Omega$.

Theorem 1.6.3 of [6] gives that an upper-semicontinuous function $u$ in $\Omega$ is subharmonic provided for each closed disk $D \subset \Omega$ and each polynomial $P$ we have

$$u \leq \text{Re} \ P \text{ on } \partial D \text{ implies } u \leq \text{Re} \ P \text{ on } D.$$

Fix a closed disk $D$ contained in $\Omega$ and let $\hat{D}$ be its interior. Choose a polynomial $P$ such that $\log Z \leq \text{Re} \ P$ on $\partial D$. Then

$$Z(\zeta) |\exp(-P(\zeta))| \leq 1 \text{ on } \partial D.$$ 

Hence for each $\zeta$ in $\partial D$, if $x$ is in $f^{-1}(\zeta)$, then

$$|g(x)| \cdot |\exp(-P(f(x)))| \leq 1, \text{ or }$$

$$|g \cdot \exp(-P(f))| \leq 1 \text{ at } x.$$ 

Now $g \cdot \exp(-P(f))$ is in $A$. Put $N = f^{-1}(\hat{D})$. The boundary of $N$ is contained in $f^{-1}(\partial D)$. Hence by the Local Maximum Modulus Principle for uniform algebras, for each $y$ in $N$ we can find $x$ in $f^{-1}(\partial D)$ with

$$|g \exp(-P(f))(y)| \leq |g \cdot \exp(-P(f))(x)|,$$
whence by (2) we have

\[(3) \quad |g \cdot \exp(-P(f))(y)| \leq 1.\]

Fix $\zeta_0$ in $D$. Choose $y$ in $f^{-1}(\zeta_0)$ with $|g(y)| = Z(\zeta_0)$. Applying (3) to this $y$, we get

\[(4) \quad Z(\zeta_0) |\exp(-P(\zeta_0))| \leq 1.\]

Hence $\log Z(\zeta_0) \leq \text{Re} P(\zeta_0)$. So (1) is satisfied, and so $\log Z$ is subharmonic in $\Omega$, as desired.

3. Proof of Theorem 2. Since $X$ is invariant under the maps $T_\theta, h(X)$ is invariant under each $T_\theta$. Fix $\zeta \in \Omega$. There are two possibilities:

(a) $|z|$ is constant on $f^{-1}(\zeta)$.

(b) $\exists r_1, r_2$ with $0 < r_1 < r_2$ and $\exists (z_1, w_1), (z_2, w_2) \in f^{-1}(\zeta)$ with $|z_1| = r_1, |z_2| = r_2$.

Suppose (b) occurs. Then the circles: $z = r_1 e^{i\theta}, w = \zeta/r_1 e^{i\theta}, 0 \leq \theta \leq 2\pi$ and $z = r_2 e^{i\theta}, w = \zeta/r_2 e^{i\theta}, 0 \leq \theta \leq 2\pi$ both lie in $h(X)$. Hence the analytic annulus: $r_1 < |z| < r_2$, $w = \zeta/z$ lies in $f^{-1}(\zeta)$. Thus if (b) occurs at any point $\zeta$ in $\Omega$, $f^{-1}(\Omega)$ does contain an analytic disk. Hence to prove the Theorem, we may assume that (a) holds for each $\zeta \in \Omega$. Define, for $\zeta \in \Omega$, $Z(\zeta) = \sup_{f^{-1}(\zeta)} |z|$, $W(\zeta) = \sup_{f^{-1}(\zeta)} |w|$. Fix $(z_0, w_0) \in f^{-1}(\zeta)$. Since we have case (a), $Z(\zeta) = |z_0|$. Hence $W(\zeta) = |w_0|$ and so $Z(\zeta) W(\zeta) = |\zeta|$, whence

$$\log Z(\zeta) + \log W(\zeta) = \log |\zeta|.$$ 

Since $\log Z$ and $\log W$ are subharmonic in $\Omega$ while $\log |\zeta|$ is harmonic, $\log Z, \log W$ are in fact harmonic in $\Omega$. Put $U = \log Z$ and let $V$ be the harmonic conjugate of $U$ in $\Omega$. Put $\phi(\zeta) = e^{U+iV}(\zeta)$. Then $\phi$ is analytic in $\Omega$ and $|\phi| = Z$ in $\Omega$.

Assertion. The variety $z = \phi(\zeta), w = \zeta/\phi(\zeta), \zeta \in \Omega$, is contained in $h(X)$.

Fix $\zeta \in \Omega$. Choose $(z_1, w_1) \in f^{-1}(\zeta)$. Then $Z(\zeta) = |z_1|$, so $|\phi(\zeta)| = |z_1|$, i.e., $\exists$ real $\alpha$ with $z_1 = \phi(\zeta)e^{i\alpha}$. Then $w_1 = \zeta/\phi(\zeta)e^{i\alpha}$. But $(e^{-i\alpha}z_1, e^{i\alpha}w_1) \in h(X)$. Hence $(\phi(\zeta), \zeta/\phi(\zeta)) \in h(X)$. The Assertion is proved, and Theorem 2 follows.

Note. Questions related to the result just proved are studied by J. E. Björk in [3].
4. Invariant disks in \( C^2 \). Let \( P \) be a polynomial with complex coefficients, \( P(t) = \sum_{n=0}^{N} c_n t^n \), which is one-one on the unit interval with endpoints identified, i.e., we assume that \( P(1) = P(0) = 0 \) and \( P(t_1) \neq P(t_2) \) if \( 0 \leq t_1 < t_2 < 1 \). Also assume \( P'(t) \neq 0 \) for \( 0 \leq t \leq 1 \). Then the curve \( \beta \) given parametrically: \( \zeta = P(t), \ 0 \leq t \leq 1 \), is a simple closed analytic curve in the \( \zeta \)-plane whose only singularity is a double-point at the origin. Denote by \( \theta \) the angle between the two arcs of \( \beta \) meeting at 0. Assume \( \theta < \pi \). Define \( a(r) = P(r^2) \), i.e.,

\[
\begin{equation}
(5) \quad a(r) = \sum_{n=1}^{N} c_n r^{2n}.
\end{equation}
\]

Let \( X \) be the disk in \( C^2 \) defined

\[
(6) \quad X = \left\{ \left( z, \frac{a(|z|)}{z} \right) \mid |z| \leq 1 \right\}.
\]

The function \( f = zw \) maps \( X \) on \( \beta \). Denote by \( \Omega \) the interior of \( \beta \).

**Theorem 3.** \( \exists \) function \( \phi \) analytic in \( \Omega \) such that \( h(X) \) is the union of \( X \) and \( \{(z, 0) \mid |z| \leq 1\} \) and

\[
\{(z, w) \mid zw \in \Omega \quad \text{and} \quad |z| = |\phi(zw)|\}.
\]

**Corollary.** Every point of \( h(X) \setminus X \) lies on some analytic disk contained in \( h(X) \).

**Notation.** \( A(\Omega) \) denotes the class of functions \( F \) defined and continuous in \( \overline{\Omega} \) and analytic in \( \Omega \).

\( \mathfrak{A} \) denotes the algebra of functions on \( |z| \leq 1 \) which are uniformly approximable by polynomials in \( z \) and \( a(|z|)/z \).

**Lemma 1.** Let \( G \in C[0, 1] \). If \( G(|z|) \in \mathfrak{A} \), then \( \exists F \in A(\Omega) \) such that \( G(r) = F(a(r)) \) for \( 0 \leq r \leq 1 \).

**Proof.** Let \( g \) be a polynomial in \( z \) and \( a(|z|)/z \). Calculation gives that there is a polynomial \( \tilde{g} \) in one variable with

\[
\frac{1}{2\pi} \int_{0}^{2\pi} g(re^{i\theta})d\theta = \tilde{g}(a(r)), \quad 0 \leq r \leq 1.
\]

Choose a sequence \( \{g_n\} \) of polynomials in \( z \) and \( a(|z|)/z \) approaching
\[ G(|z|) \text{ uniformly on } |z| \leq 1. \text{ Then } g_n(a(r)) \rightarrow G(r) \text{ uniformly on } 0 \leq r \leq 1. \text{ Hence } \exists F \in A(\Omega) \text{ with } g_n \rightarrow F \text{ uniformly on } \beta, \text{ so } G(r) = F(a(r)). \]

**Lemma 2.** If \( f = zw \), then \( f^{-1}(\Omega) \) is not empty.

**Proof.** Fix \( \zeta_0 \in \Omega \). If \( f^{-1}(\Omega) \) is empty, then \( f - \zeta_0 \neq 0 \) on \( h(X) \) and so \( (zw - \zeta_0)^{-1} \) lies in the closure of the polynomials in \( z \) and \( w \) on \( X \). Then \( a(|z|) - \zeta_0)^{-1} \) lies in \( \mathfrak{a} \). By Lemma 1, \( \exists F \in A(\Omega) \) with \( F(a(r)) = (a(r) - \zeta_0)^{-1} \). Then \( (\zeta - \zeta_0)^{-1} \in A(\Omega) \), which is false. So \( f^{-1}(\Omega) \) is not empty.

**Lemma 3.** Fix \( \zeta \in \beta \setminus \{0\} \). Let \( (z_0, w_0) \) be a point in \( h(X) \) with \( z_0w_0 = \zeta \). Then \( (z_0, w_0) \in X \).

**Proof.** Assume \( (z_0, w_0) \notin X \). Let \( r \) be the point in \((0, 1)\) with \( a(r) = \zeta \). Put, for each \( r \), \( \gamma_r = \{(re^{i\theta}, (a(r)/re^{i\theta})) | 0 \leq \theta < 2\pi\} \). Then \( \gamma_r \) is a polynomially convex circle contained in \( X \). Hence \( \exists \) polynomial \( P \) with \( |P(z_0, w_0)| > 2, |P| < 1 \) on \( \gamma_r \). Choose a neighborhood \( N \) of \( \gamma_r \) on \( X \) where \( |P| < 1 \). The image of \( X \setminus N \) under the map \( (z, w) \rightarrow zw \) is a closed subarc \( \beta_1 \) of \( \beta \) which excludes \( \zeta \). Choose \( F \in A(\Omega) \) with \( F(\zeta) = 1, |F| < 1 \) on \( \beta \setminus \{\zeta\} \). Then \( \exists \delta > 0 \) such that \( |F| < 1 - \delta \) on \( \beta_1 \). Hence \( |F(zw)| < 1 - \delta \) on \( X \setminus N \). Also \( |F(zw)| \leq 1 \) on \( X \). Fix \( n \) and put

\[ Q = F(zw)^n \cdot P(z, w). \]

\[ |Q(z_0, w_0)| > 2. \text{ On } \mathfrak{N}, |Q| \leq |P| < 1. \text{ On } X \setminus N, |Q| < (1 - \delta)^n \cdot \max_x |P|, \text{ and so } |Q| < 1 \text{ on } X \setminus N \text{ for large } n. \text{ Then } |Q| < 1 \text{ on } X. \text{ Since } F \text{ is a uniform limit on } \beta \text{ of polynomials in } \zeta, Q \text{ is a uniform limit on } X \cup \{(z_0, w_0)\} \text{ of polynomials in } z \text{ and } w. \text{ This contradicts that } (z_0, w_0) \in h(X). \text{ Thus } (z_0, w_0) \in X. \text{ We are done.} \]

**Note.** Since \( f \) maps \( X \) on \( \beta \) and \( C \setminus f(X) \) is the union of the interior and exterior of \( \beta \), we conclude from the last Lemma that \( h(X) \) is the union of \( X \) and \( f^{-1}(\{0\}) \) and \( f^{-1}(\Omega) \).

We need some notation now. For each \( \zeta \in \beta \setminus \{0\} \), denote by \( r(\zeta) \) the unique \( r \) in \((0, 1)\) with \( a(r) = \zeta \).

Since \( a \) is a polynomial in \( r \) vanishing at 0, there is a constant \( d > 0 \) such that

\[ r(\zeta) > d |\zeta|, \text{ all } \zeta \in \beta. \]
For \( \zeta_0 \in \Omega \), denote by \( \mu_{\zeta_0} \) harmonic measure at \( \zeta_0 \) relative to \( \Omega \). Since \( \beta \) consists of analytic arcs, with one jump-discontinuity for the tangent at \( \zeta = 0 \), \( \mu_{\zeta_0} = K_{\zeta_0} ds \), where \( K_{\zeta_0} \) is a bounded functions on \( \beta \) and \( ds \) is arc-length. Define

\[
U(\zeta_0) = \int_\beta \log r(\zeta) d\mu_{\zeta_0}(\zeta).
\]

Since (7) holds, this integral converges absolutely. \( U \) is a harmonic function in \( \Omega \), bounded above, and continuous at each boundary point \( \zeta \in \beta \setminus \{0\} \) with boundary value \( \log r(\zeta) \) at \( \zeta \).

For \( \zeta \in \Omega \), define

\[
Z(\zeta) = \sup_{f^{-1}(\zeta)} |z|, \quad W(\zeta) = \sup_{f^{-1}(\zeta)} |w|.
\]

**Lemma 4.** For all \( \zeta \in \Omega \), \( \log Z(\zeta) \leq U(\zeta) \) and \( \log W(\zeta) \leq \log |z| - U(\zeta) \).

**Proof.** Fix \( \zeta \in \beta \setminus \{0\} \), choose \( \zeta_n \in \Omega \) with \( \zeta_n \to \zeta \) and suppose \( Z(\zeta_n) \to \lambda \). Choose \( p_n \in f^{-1}(\zeta_n) \) with \( Z(\zeta_n) = |z(p_n)| \). Without loss of generality, \( p_n \to p \) for some point \( p \in h(X) \). Then \( f(p) = \zeta \). By Lemma 3, \( p \in X \), i.e., \( p = (re^{i\theta}, (a(r)/re^{i\theta})) \) for some \( r, \theta \). Also \( a(r) = \zeta \) and so \( r = r(\zeta) \), whence \( |z(p_n)| \to r(\zeta) \) and so \( \lambda = r(\zeta) \). Thus \( Z(\zeta') \to r(\zeta) \) as \( \zeta' \to \zeta \) from within \( \Omega \), and so \( \log Z \) assumes the same boundary values as \( U \), continuously on \( \beta \setminus \{0\} \).

For each positive integer \( k \), let \( \Omega_k = \{ \zeta \in \Omega \mid |\zeta| > 1/k \} \). \( \partial \Omega_k \) is the union of a closed subarc \( \beta_k \) of \( \beta \setminus \{0\} \) and an arc \( \alpha_k \) on the circle \( |\zeta| = 1/k \).

Fix \( \zeta_0 \in \Omega \). For large \( k \), \( \zeta_0 \in \Omega_k \). Denote by \( \mu_{\zeta_0}^{(k)} \) the harmonic measure at \( \zeta_0 \) relative to \( \Omega_k \). An elementary estimate gives that there is a constant \( C_{\zeta_0} \) independent of \( k \) such that

\[
\mu_{\zeta_0}^{(k)}(\alpha_k) \leq C_{\zeta_0} \cdot \frac{1}{\sqrt{k}} \quad \text{for all } k.
\]

Let \( S \) be any function subharmonic in \( \Omega \) and assuming continuous boundary values, again denoted \( S \), on \( \beta \setminus \{0\} \). Assume \( \exists \) constant \( M \) with \( S \leq M \) in \( \Omega \). Then for all \( k \),

\[
S(\zeta_0) \leq \int_{\beta_k} S d\mu_{\zeta_0}^{(k)} + \int_{\alpha_k} M d\mu_{\zeta_0}^{(k)}, \quad \text{whence}
\]

\[
S(\zeta_0) \leq \int_{\beta_k} S d\mu_{\zeta_0}^{(k)} + M \cdot C_{\zeta_0} \cdot \frac{1}{\sqrt{k}}.
\]
Applying (9) with $S = \log Z$, we get

$$(10) \quad \log Z(\zeta_0) \leq \int_{\beta_k} U d \mu^{(k)}_{\zeta_0} + MC_{\zeta_0} \cdot \frac{1}{\sqrt{k}},$$

since as we saw earlier, $\log Z = U$ on $\beta \setminus \{0\}$.

By (7), if $\zeta' \in \alpha_k$,

$$U(\zeta') = \int_{\beta} \log r(\zeta) d \mu_\zeta(\zeta) > C + \int_{\beta} \log |\zeta| d \mu_\zeta(\zeta),$$

where $C$ is a constant, so

$$U(\zeta') > C + \log |\zeta'| = C + \log \frac{1}{k}.$$ Hence

$$U(\zeta_0) = \int_{\beta_k} U d \mu^{(k)}_{\zeta_0} + \int_{\alpha_k} U d \mu^{(k)}_{\zeta_0}$$

$$\leq \int_{\beta_k} U d \mu^{(k)}_{\zeta_0} + \left(C + \log \frac{1}{k}\right) \frac{C_{\zeta_0}}{\sqrt{k}}.$$

Combining this with (10) and letting $k \to \infty$, we get that $\log Z(\zeta_0) \leq U(\zeta_0)$, as desired. A parallel argument gives the assertion regarding $W$. We are done.

**Lemma 5.** With $Z$ defined as above, $\log Z(\zeta) = U(\zeta)$ for all $\zeta \in \Omega$, and $\log W(\zeta) = \log |\zeta| - U(\zeta)$.

**Proof.** Suppose either equality fails at some point $\zeta_0$. By the last Lemma, this implies that

$$\log Z(\zeta_0) + \log W(\zeta_0) < \log |\zeta_0|.$$ Fix $p \in f^{-1}(\zeta_0)$. Then $|z(p)| \leq Z(\zeta_0)$, $|w(p)| \leq W(\zeta_0)$, so

$$\log |z(p)w(p)| < \log |\zeta_0|.$$ But $z(p)w(p) = \zeta_0$, so we have a contradiction, proving the Lemma.

**Proof of Theorem 3.** Let $V$ denote the harmonic conjugate of $U$ in $\Omega$ and put $\phi = e^{U+iv}$. Fix $(z_0, w_0) \in f^{-1}(\Omega)$ and put $\zeta_0 = z_0 \cdot w_0$. Unless $|z_0| = Z(\zeta_0)$ and $|w_0| = W(\zeta_0)$, we have

$$|\zeta_0| = |z_0| |w_0| < Z(\zeta_0) W(\zeta_0) = |\zeta_0|$$

by the last Lemma. So we must have $|z_0| = Z(\zeta_0) = |\phi(\zeta_0)|$. 


Conversely fix $\zeta_0 \in \Omega$ and let $(z_0, w_0)$ be a point in $C^2$ such that $z_0 \cdot w_0 = \zeta_0$ and $|z_0| = |\phi(\zeta_0)|$. Choose $(z_1, w_1) \in f^{-1}(\zeta_0)$. By the preceding $|z_1| = |\phi(\zeta_0)|$, so $\exists$ real $\alpha$ with $z_0 = e^{i\alpha} z_1$, $w_0 = e^{-i\alpha} w_1$. Hence $(z_0, w_0) \in h(X)$, so $(z_0, w_0) \in f^{-1}(\Omega)$. Thus $f^{-1}(\Omega)$ consists precisely of those points $(z, w)$ with $zw \in \Omega$ and $|z| = |\phi(zw)|$

To finish the proof we need only identify $f^{-1}(0)$. The circle $\{(z, 0) \mid |z| = 1\}$ lies in $X$, so the disk $D: \{(z, 0) \mid |z| \leq 1\}$ is contained in $f^{-1}(0)$. If $(z_0, w_0) \in f^{-1}(0)$ and does not lie in $D$, then $z_0 = 0, w_0 \neq 0$. The same argument as was used in proving Lemma 3 shows that then $(z_0, w_0) \notin h(X)$, contrary to assumption. So $f^{-1}(0) = D$, and the proof of Theorem 3 is finished.

**Remark.** As we have just seen, $f^{-1}(\Omega)$ is the union of varieties $V_\alpha$, $0 \leq \alpha < 2\pi$, where $V_\alpha$ is defined:

$$z = e^{i\alpha} \phi(\zeta), \quad w = e^{-i\alpha} \frac{\zeta}{\phi(\zeta)}, \quad \zeta \in \Omega.$$ 

What does the boundary of such a variety $V_\alpha$ in $h(X)$ look like? It splits into two sets:

$$S = \{(z, w) \in \partial V_\alpha \mid zw \in \beta \setminus \{0\}\} \quad \text{and} \quad T = \{(z, w) \in \partial V_\alpha \mid zw = 0\}.$$ 

It is easy to see that $S$ is an arc on $X$ cutting each circle: $\{(z, w) \in X \mid |z| = r\}$, $0 < r < 1$, exactly once while $T$ is a closed subset of the disk $D = \{(z, 0) \mid |z| \leq 1\}$.

It is remarkable that even though $X$ is itself very regular, the rest of the hull of $X$ is attached to $X$ in a very complicated way.

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**References**


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