A MAP OF $E^3$ ONTO $E^3$ TAKING NO DISK ONTO A DISK

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An example is given of an u.s.c. decomposition in which no disk in $E^3$ maps onto a disk under the natural projection map $P$, and, furthermore, the decomposition space $E^3/G$ is homeomorphic to $E^3$. Each nondegenerate element is a tame arc. The image $P(H)$ of the set of nondegenerate elements is 0-dimensional, although $\text{Cl} P(H)$ is $E^3$. The basic construction used is called a knit Cantor set of nondegenerate elements.

Bing and Borsuk [3] have given an example of a 3-dimensional absolute retract $R$ containing no disk. They define a particular u.s.c. decomposition of $E^3$ that yields $R$ as the decomposition space. Hence, their example is a closed map of $E^3$ taking no disk onto a disk, but, of course, their image is not $E^3$.

In [8] the author defined a set $X \subset E^3$ to be the $P$-lift of a set $Y$ contained in the decomposition space $E^3/G$ if and only if $X$ and $Y$ are homeomorphic and the image of $X$ under the natural projection is $Y$. A disk is said to be $P$-liftable if and only if it has a $P$-lift. Using this terminology, the example that is constructed in this note has no $P$-liftable disk in the image space.

In [1] Armentrout asked whether there exists a pointlike decomposition $G$ of $E^3$ such that there is a 2-sphere $S$ in $E^3/G$ that can not be approximated by a $P$-liftable sphere. This was first answered by the author in [9] by giving an example of a space $E^3/G$ containing such a 2-sphere. In the decomposition space of this note no 2-sphere is $P$-liftable. Hence, this space is another answer to Armentrout's query.

The construction we describe in this note is based on a knit example in the author's papers [6], [7], and [9]. It is assumed that the reader is familiar with this example and the notations in [6]. We also need the following definitions.

**Definition.** Let $J_1$ be the circle in the $x-y$ plane with radius 1 and center at the origin, and $J_2$ be the circle in the $y-z$ plane with radius 1 and center at $y = -1, z = 0$. Any two tame simple closed curves $J_1$ and $J_2$ in $E^3$ are said to simply link if and only if there is a homeomorphism of $E^3$ onto itself taking $J_1$ and $J_2$ onto $J_1$ and $J_2$, respectively. Two disjoint compact sets $S_1$ and $S_2$ are said to simply link if and only if there exist simple closed curves $J_1 \subset S_1$ and $J_2 \subset S_2$ such that $J_1$ and $J_2$ simply link.
DEFINITION. Let \( \{ M_i \} \) be a defining sequence for a decomposition \( G \). The sequence \( M_1, M_2, M_3, M_4, \ldots \), denoted by \( \{ M_i \} \), is called a **compound defining sequence** for \( G \) if and only if (1) \( i = 1, 2, 3, \ldots \); (2) \( 1 \leq j \leq i \); (3) \( \{ M_i \} \) with the lexicographic order indicated above on the indices is a defining sequence; (4) \( M_i^{-1} \) is a regular neighborhood of \( M_i \); and (5) \( M_i = M_i \). Given any decomposition of \( E^3 \) with a defining sequence, there exists a compound defining sequence.

In [6] and [9] the author gave an example of a knit decomposition \( G_0 \) of \( E^3 \), a 2-complex \( X \) in \( E^3 \), and an \( \varepsilon > 0 \) such that \( P(X) \subset E^3/G_0 \) is a disk \( D \) having the property that no disk \( D_\varepsilon \) which is \( \varepsilon \)-homeomorphic to \( D \) is \( P \)-liftable. This decomposition used "knit Cantor sets of non-degenerate elements". In the figure two countably infinite sets of arcs knit from the point \( p \) to the point \( q \) are indicated. Each arc pictured represents a Cantor set of arcs. These Cantor sets of arcs and the limiting arc \( g_p \) containing \( p \) and \( q \) are the nondegenerate elements of the decomposition \( G_0 \). The 2-complex \( X \) consists of eight squares that do not form a disk. Notice that each arc except \( g_p \) pierces \( X \) in a point. Since in \( E^3/G_0 \) the arc \( g_p \) has an image that is a point, the image of \( X \) is a disk \( D \). The decomposition is a modification of two \((2,1)\) toroidal decompositions. The entwining of the nondegenerate elements caused by the \((2,1)\) toroidal decompositions is not indicated in the figure. It would be above and below the portions shown. The result
of the entwining is that $\text{Bd } T_0$ is not homotopic to a point in $E^3 - H^*$. It was shown that $E^3/G_0$ is homeomorphic to $E^3$. In the proof only disks that are $\epsilon$-homeomorphic to $D$ were considered. Using similar arguments, it can be shown that there is no disk $\delta$ in $E^3$ such that $P(\delta)$ is a disk in $E^3/G_0$ and $\text{Bd } X$ is homotopic in $E^3 - M_1$ to $\text{Bd } \delta$. (In [6] $M_1$ is the first manifold in a particular defining sequence for $G_0$.) Hence, for this decomposition $G_0$, there is a regular neighborhood $\Gamma_0$ of $\text{Bd } X$ in $E^3$ which is an unknotted polyhedral solid torus such that no simple closed curve homotopic in $T_0$ to its core bounds a disk $\delta$ that has an image in $E^3/G_0$ that is a disk.

Simply linking $T_0$, there is an unknotted polyhedral solid torus $S_0$ that contains $M_1$, which in turn contains all the nondegenerate elements of $G_0$. Let $T_c$ and $S_c$ be any pair of simply linked unknotted polyhedral solid tori. There is a homeomorphism of $E^3$ onto itself that takes $T_0$ and $S_0$ onto $T_c$ and $S_c$, respectively. Given $d > 0$, this homeomorphism can be chosen so that the diameter of each nondegenerate element is less than $d$. (This follows from the proof that $E^3/G_0$ is homeomorphic to $E^3$.) Hence, given simply linked unknotted polyhedral solid tori $T_c$ and $S_c$ and given $d > 0$, there is a decomposition $G_c$ of $E^3$ with nondegenerate elements $H_c$ such that (1) $H_c^* \subset S_c$; (2) for each $g \in H_c$, diam $g < d$; and (3) no disk $\delta$ with $\text{Bd } \delta$ homotopic in $T_c$ to its core has an image in $E^3/G_c$ which is a disk.

We now construct a family $\mathcal{T}$ of solid tori $T_c$. Associated with each $T_c$ there are an $S_c$ and $H_c$ having the above properties. The family is dense in $E^3$ and so chosen that for any disk $D$ in $E^3$ there are a solid torus $T_c \in \mathcal{T}$ and a tame simple closed curve $J \subset D \cap T_c$ such that $J$ is homotopic in $T_c$ to its core. Let $G$ be the union of $H = \{g \in H_c: H_c$ is associated with some $T_c \in \mathcal{T}\}$ and points in $E^3 - H$. Then no disk projects onto a disk under the mapping $P: E^3 \rightarrow E^3/G$.

The family $\mathcal{T}$ is constructed in stages. To define the first stage, we start with the set of points $\mathcal{V}_1 = \{(p/2, q/2, r/2): p, q, \text{ and } r \text{ are integers}\}$. Associated with $\mathcal{V}_1$ is the set $\mathcal{C}_1$ of all unknotted polygonal simple closed curves having vertices in $\mathcal{V}_1$ and diameters not greater than one.

For any tame unknotted simple closed curve $J$, let $L_j = \text{lub}\{d: \text{there is a polygonal simple closed curve } K \text{ in the unbounded component of } E^3 - N_d(J) \text{ such that } K \text{ simply links } J\}$. (Here $N_d(J)$ denotes the $d$-neighborhood of $J$.) For each $C \in \mathcal{C}_1$, choose a polygonal simple closed curve $K_C$ that simply links $C$ and lies in the unbounded component of the complement of the $L_c/2$-neighborhood of $C$. These can certainly be chosen so that the diameter of each $K_c$ is less than four and each $K_c$ fails to intersect the union of the other such simple closed curves associated with elements of $\mathcal{C}_1$. For each $K_c$,
choose a polyhedral solid torus $S_c$ with core $K_c$ and contained in the $L_c/4$-neighborhood of $K_c$. This implies that $C$ simply links $S_c$. The set of these solid tori $S_c$ can be chosen to be mutually disjoint.

The method for choosing the solid torus $T_c$ that simply links $S_c$ depends on the fact that the union of two solid tori with common boundaries and disjoint interiors is the 3-sphere obtained as the union of $E^3$ and the point at infinity. Since simply linked tori do not have common boundaries, for each $C$ enlarge $S_c$ slightly: choose a solid torus $S'_c$ that satisfies the definition of $S_c$ and contains $S_c$ in its interior. Let $N$ be the $L_c/4$-neighborhood of $C$. Let $A_c$ be the complement of a polyhedral 3-ball containing $N \cup S_c$ and having diameter less than eight. Let $B_c$ be a polyhedral 3-ball (a tubular neighborhood of a polygonal arc) in $E^3 - (A_c \cup N \cup S'_c)$ connecting $A_c$ and $S'_c$ in such a way that $Cl(E^3 - (A_c \cup B_c \cup S'_c))$ is a polyhedral unknotted solid torus having $C$ as a core. Denote this solid torus with core $C$ by $T_c$. Observe that $T_c$ contains the $L_c/4$-neighborhood of $C$ and simply links $S_c$. Let $\mathcal{T}_1 = \{T_c : C \in \mathcal{C}\}$. This is the first stage of the construction of the family $\mathcal{T}$.

For each $T_c$ and $S_c$, we choose a decomposition $G_c$, having the above properties with respect to $T_c$ and $S_c$ and having no nondegenerate element with diameter greater than one. It can be assumed that each nondegenerate element is polygonal.

From the definition of a compound defining sequence it follows that each component which is a solid torus is one of a finite nest of solid tori which are regular neighborhoods of the innermost one of the nest. We assume that all solid tori in a nest are tubular neighborhoods of the same polygonal simple closed curve.

To define the $n$th stage, let $\mathcal{V}_n = \{(p/(2^n), q/(2^n), r/(2^n)) : p, q, r \text{ are integers}\}$. The family $\mathcal{C}_n$ is the set of all unknotted polygonal simple closed curves having vertices in $\mathcal{V}_n$ and diameters again not greater than one. Complete choices of $T_c$ and $S_c$ as in the first stage with the added requirement that each $K_c$ miss all nondegenerate elements from previous stages.

We next determine the size requirement for nondegenerate elements at the $n$th stage. For any $C \in \mathcal{C}_n$, the associated solid torus $S_c$ intersects the compound defining sequences of only a finite number of the sets $H_c$ previously defined. Call them $H_k, 1 \leq k \leq k_c$, where $k_c$ is the appropriate integer. For each $H_k$, let $\{(M_k)_i\}$ be the compound defining sequence. Because $S_c$ misses each set $H^+_k$, there are only a finite number of $(M_k)_i$ whose boundaries intersect $S_c$. For each $C$, let $x_c = \min\{d : d \text{ is the distance between two sets } Bd(M_k)_i \cap S_c \text{ for some values of } i, j, \text{ and } k\}$. This $x_c$ is strictly positive. We require that each nondegenerate element in $H_c$ have diameter less than $x_c$ and less than $1/n$. There is a decomposition $G_c$ satisfying this and the conditions
above with respect to $S_C$ and the corresponding $T_C$. Again assume that each image of a nondegenerate element is polygonal and that manifolds in compound defining sequences are unions of prisms. This completes the construction of the $n$th stage.

Let $G$ be the union of $H = \{g \in H_C: C \in \mathcal{C}_n \text{ for some positive integer } n\}$ and points in $E^3 - H^*$. This decomposition $G$ defines the map claimed in the title.

The proof is based on McAuley's countably shrinkable theorem [4], as slightly revised by Reed [5]. To use the theorem it is necessary to shrink certain elements without permitting others to grow too much. Some of the shrinking is based on Bing's shrinking of the (2,1) toroidal decomposition [2]. Recall that, in the construction, elements at a later stage are not permitted to intersect boundaries of more than two manifold stages in previous compound defining sequences. This allows growth of later stage nondegenerate elements to be controlled during the shrinking of a particular stage. The proof is tedious, but straightforward.

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