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RINGS WHOSE FAITHFUL LEFT IDEALS ARE COFAITHFUL

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A left module M over a ring R is cofaithful in case there is an embedding of R into a finite product of copies of M. Our main result states that a semiprime ring R is left Goldie, that is, has a semisimple Artinian left quotient ring, if and only if R satisfies (i) every faithful left ideal is cofaithful and (ii) every nonzero left ideal contains a nonzero uniform left ideal. The proof is elementary and does not make use of the Goldie and Lesieur-Croisot theorems. We show that (i) and (ii) are Morita invariant. Moreover, (ii) is invariant under polynomial extensions, and so is (i) for commutative rings. Absolutely torsion-free rings are studied.

The ring Q is a left classical quotient ring for the ring $R \subseteq Q$ if every regular element (nondivisor of zero) of R is invertible in Q and if every element of Q is of the form $b^{-1}a$ where $a, b \in R$ and b is regular; in this case we also say that R is a left order in Q. A ring is said to be left Goldie if it has the ascending chain condition on left annihilators and has finite uniform dimension. (A left R-module has finite uniform dimension if it has no infinite direct sum of nonzero submodules, and it is said to be uniform if it is nonzero and any two nonzero submodules have a nontrivial intersection.) A theorem of Goldie **[8, 9]** and Lesieur and Croisot **[12]** states that a ring is a left order in a semisimple Artinian ring if and only if it is semiprime and left Goldie. It is known that the ascending chain condition on left annihilators is not preserved under an equivalence of categories (Morita invariant); in fact, it does not go up to matrix rings. It is unknown whether being left Goldie is Morita invariant.

In section two we give a proof of the theorem stated in the abstract, and in the prime case we give a proof which shows directly that such a ring is an order in a full matrix ring over a division ring. We also weaken the hypothesis of an important theorem on semiprime PI rings. In the third section we use these techniques to study absolutely torsion-free rings. In particular, we show that an absolutely torsion-free ring is Goldie if and only if it has a uniform left ideal, and that the endomorphism ring of a finitely generated projective module over an absolutely torsion-free ring is absolutely torsion-free.

1. Some general results. All rings will be associative and have an identity element; all modules will be unital. Let R be a ring and S a subset of R. Then the right annihilator of S in R is $\ell_R(S) = \{r \in R \mid Sr = 0\}$ and the left annihilator is $\ell_R(S)$. If X is a subset of a left R-module M, then $\mathrm{Ann}_R(X) = \{r \in R \mid rX = 0\}$. If there is no ambiguity we write $\ell(S)$ instead of $\ell_R(S)$, etc. $Z_R(M)$ will denote the singular submodule of M, the set of elements of M whose annihilator is essential in R.

A module ${}_RM$ is said to be cofaithful if there exist elements $m_1, m_2, \cdots, m_k \in M$ such that $\bigcap_{i=1}^{k} \mathrm{Ann}(m_i) = 0$, or equivalently, if for some direct sum M^k of k copies of M there exists an exact sequence $0 \to R \to M^k$. Every cofaithful module is faithful. On the other hand, every faithful left R-module is cofaithful if and only if R contains an essential Artinian left ideal (see Beachy **[1]**), in which case we say R is essentially left Artinian. A ring R is essentially left Artinian if and only if R has an essential and finitely generated left socle. We study the weaker condition that every faithful left ideal of R is cofaithful. Recall that ${}_RM$ is torsionless if for each $0 \neq m \in M$ there exists $f \in \mathrm{Hom}_R(M, R)$ with $f(m) \neq 0$.

PROPOSITION (1.1). *The following conditions are equivalent for a ring R.*

(a) *Every faithful left ideal of R is cofaithful.*

(b) *Every ideal of R which is faithful as a left ideal is cofaithful.*

(c) *Every faithful, torsionless left R-module is cofaithful.*

Proof. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are immediate.

(b) $\Rightarrow$ (c). Let ${}_RM$ be a faithful torsionless module and A be the sum in R of the homomorphic images of M. If $0 \neq r \in R$, then since M is faithful there exists $m \in M$ such that $rm \neq 0$, and since M is torsionless there exists $f \in \mathrm{Hom}_R(M, R)$ with $rf(m) = f(rm) \neq 0$, which shows that the ideal A is faithful. Thus A is cofaithful and so $\bigcap_{i=1}^{n} \ell_R(a_i) = 0$ for some $a_i \in A$, $1 \leqq i \leqq n$. Since $a_i \in A$, $a_i = \Sigma_j f_{ij}(m_{ij})$ for $m_{ij} \in M$ and $f_{ij} \in \mathrm{Hom}_R(M, R)$, and then $rm_{ij} = 0$ for all i, j implies $ra_i = 0$ for all i, so $\bigcap_{i,j} \mathrm{Ann}_R(m_{ij}) = 0$ and M is cofaithful.

COROLLARY (1.2). *The condition that every faithful left ideal of a ring is cofaithful is Morita invariant.*

Proof. By Beachy **[2]** a module is faithful if and only if it cogenerates every projective module and it is cofaithful if and only if it generates every injective module. A module is torsionless if and only if it is cogenerated by every faithful module. Since the classes of faithful, cofaithful and torsionless modules are all invariant under an equivalence of module categories, the result follows from condition (c) above.

The next two propositions show that our condition implies certain finiteness conditions, although it is much weaker than the descending chain condition for left annihilators. In particular, a commutative, semiprime ring satisfying the condition has finite uniform dimension, and so it must be Goldie.

PROPOSITION (1.3). *Let R be a ring such that every faithful left ideal is cofaithful.*

(a) *R is not a direct product of infinitely many (nontrivial) rings.*

(b) *If R is semiprime, then it contains no infinite direct sum of nonzero ideals.*

Proof. (a) Suppose that R is an infinite direct product of rings. Let A be the set of all elements which are zero in all but finitely many components. Then A is faithful but not cofaithful.

(b) Assume that $A = A_1 \oplus A_2 \oplus \cdots$ is an infinite direct sum of ideals. If R is semiprime, then $A \cap \ell(A) = 0$ and $A \oplus \ell(A)$ is faithful, so by assumption there exist $x_1, \cdots, x_k \in A \oplus \ell(A)$ such that $\bigcap_{i=1}^{k} \ell(x_i) = 0$. But there exists an integer n such that $x_i \in A_1 \oplus \cdots \oplus A_n \oplus \ell(A)$ for all i, and so for any $0 \neq y \in A_{n+1}$ we have $y \in \bigcap_{i=1}^{k} \ell(x_i)$, a contradiction.

PROPOSITION (1.4) (Faith [5]). *A ring R has the descending chain condition on left annihilators if and only if for every subset S of R there exists a finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq S$ such that $\ell_R(x_1, \cdots, x_n) = \ell_R(S)$.*

Proof. If R satisfies the descending chain condition on left annihilators, choose $\{x_1, x_2, \cdots, x_n\}$ so that $\ell(x_1, \cdots, x_n)$ is minimal in the set of all left annihilators of finite subsets of S.

Conversely let $A_1 \supseteq A_2 \supseteq \cdots$ be a descending chain of left annihilators and let $S = \bigcup \ell(A_i)$. Then there exists a subset $\{x_1, x_2, \cdots, x_n\} \subseteq S$ so that $\ell(x_1, x_2, \cdots, x_n) = \ell(S)$. There exists a positive integer k such that $\ell(A_i) \supseteq \{x_1, x_2, \cdots, x_n\}$ for all $i \geqq k$. But for $i \geqq k$, $A_i = \ell(\ell(A_i)) \subseteq \ell(x_1, \cdots, x_n) = \ell(S) \subseteq A_i$, so $A_i = \ell(S)$ and the chain terminates at A_k.

We remark that Handelman and Lawrence [11] have given an example of a prime ring in which every (faithful) left ideal is cofaithful but which does not have the analogous property for right ideals. We next give some examples to show the relationship between this condition and various other finiteness or chain conditions.

A left Noetherian ring need not satisfy our condition, as is shown by the following example due to Small **[16]**. Let R be a simple left Noetherian domain which is not a division ring, let F be the field which is the center of R, and let K be a nonzero left ideal of R. Set $M = R/K$ and let S be the ring of all matrices $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ where $a \in F$, $b \in M$ and $c \in R$. It is easily seen that S is left Noetherian, and following Small one can show that given any finite subset of M, say $\{m_1, \cdots, m_t\}$, there exists $0 \neq d \in R$ such that $dm_i = 0$ for $i = 1, \cdots, t$. Thus $I = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| \, a \in F, b \in M \right\}$ is a faithful left ideal of S which is not cofaithful.

On the other hand, a left Noetherian ring which is integral over its center has the property that every faithful left ideal is cofaithful since it is a subring of a left Artinian ring (see Blair **[4]**). Also in the positive direction, if R is left Noetherian and $Z(R) = 0$ (e.g. if R is left hereditary) then our condition holds.

The ring $R = \{(n, a) \mid n \in \mathbf{Z}, a \in \mathbf{Z}_{p^\infty}\}$, where $\mathbf{Z}_{p^\infty}$ is Prufer's quasi-cyclic group and multiplication is given by $(n, a)(m, b) = (nm, nb + ma)$, provides an example of a commutative ring with finite uniform dimension for which every faithful ideal is cofaithful (since R is essentially Artinian), but it can be checked that R does not satisfy the chain condition on annihilators.

The next proposition provides many more examples.

PROPOSITION (1.5). *If R has a left classical quotient ring Q which is essentially left Artinian, then R has finite uniform dimension and every faithful left ideal is cofaithful.*

Proof. Let A be an essential Artinian left ideal of Q. If $B_1 \oplus B_2 \oplus \cdots \oplus B_k$ is a direct sum of left ideals of R, then by standard quotient ring techniques $QB_1 \oplus QB_2 \oplus \cdots \oplus QB_k$ is a direct sum of left ideals of Q and so if each $QB_i \neq 0$ then $(QB_1 \cap A) \oplus (QB_2 \cap A) \oplus \cdots \oplus (QB_k \cap A)$ is a direct sum of nonzero left ideals in A. Since A is left Artinian such direct sums must be finite and thus R has finite uniform dimension.

If B is a faithful left ideal of R, then $\bigcap_{b \in B} \ell_Q(b) = 0$ since $\bigcap_{b \in B} \ell_Q(b) \cap R = \bigcap_{b \in B} \ell_R(b) = 0$. Since A is Artinian,

$$\bigcap_{i=1}^{n} (A \cap \ell_Q(b_i)) = 0$$

for some finite subset $b_1, b_2, \cdots, b_n$ of B, and then $\bigcap_{i=1}^{n} \ell_R(b_i) \subseteq \bigcap_{i=1}^{n} \ell_Q(b_i) = 0$ since A is essential.

We say that a ring has enough uniforms if every nonzero left ideal contains a uniform left ideal. If a ring has finite uniform dimension then it has enough uniforms. An infinite direct product of copies of $\mathbf{Z}$ shows that a ring may have enough uniforms without having finite uniform dimension.

PROPOSITION (1.6). *If R is a ring with enough uniforms, then every nonzero submodule of a free R-module has a uniform submodule.*

Proof. Let F be a free R-module and $M \neq 0$ a submodule. Without loss of generality we may assume M is cyclic, in which case we may also assume F is finitely generated. Let $F = R^n$, and p_n be the projection onto the last summand. If the restriction of p_n to M is a monomorphism then an isomorphic copy of M is contained in R and so M contains a uniform submodule. If p_n restricted to M is not a monomorphism then $M \cap R^{n-1} \neq 0$ and we complete the proof by induction.

COROLLARY (1.7). *The condition that a ring have enough uniforms is Morita invariant.*

In the course of proving that finite uniform dimension goes up to polynomial rings, Shock **[15]** showed that if U is a uniform left ideal of R then $U[x]$ is a uniform left ideal of $R[x]$. With only slight modification Shock's proof shows, in fact, that if ${}_RM$ is a uniform R-module then $M[x] (\simeq R[x] \otimes_R M)$ is a uniform left $R[x]$-module.

PROPOSITION (1.8). *If R has enough uniforms, then $R[x]$ has enough uniforms.*

Proof. Let I be an ideal of $R[x]$, where R has enough uniforms. As an R-submodule of the free R-module $R[x]$, I contains a uniform R-submodule M by Proposition 1.6. Thus there exists $q(x) \in I$ such that $Rq(x)$ is a uniform R-submodule of I. By multiplying $q(x)$ by appropriate elements of R we may assume that the left annihilators of all the nonzero coefficients of $q(x)$ are the same. This "new" $q(x)$ is also an element of $M \subseteq I$, so $Rq(x)$ remains a uniform R-module. Since the left annihilators of the nonzero coefficients of $q(x)$ are all the same we have $R[x]q(x) \simeq R[x] \otimes_R Rq(x)$. By the remarks before the theorem this shows that $R[x]q(x)$ is a uniform left ideal of $R[x]$ contained in I.

PROPOSITION (1.9). *Let R be a commutative ring in which every faithful ideal is cofaithful. Then every faithful ideal of $R[x]$ is cofaithful.*

Proof. Let I be a faithful ideal of the ring $R[x]$ and let I_0 be the ideal of R generated by the coefficients of the polynomials in I. Clearly I_0 is a faithful ideal of R. Thus there exist $a_1, a_2, \cdots, a_t \in I_0$ such that $\ell_R\{a_1, a_2, \cdots, a_t\} = 0$. Let $f_i(x)$ be a polynomial of I in which a_i appears. Let $\deg f_i(x) = n_i$ and set $n_0 = 0$. We show $\ell_{R[x]}(f_1(x), \cdots, f_t(x)) = 0$. If not, and say $h(x)f_i(x) = 0$ for $i = 1, \cdots, t$, then set $m_i = \Sigma_{j=0}^{i-1}(n_i + 1)$ and $g(x) = \Sigma_{i=1}^{t} f_i(x)x^{m_i}$. Now $h(x)g(x) = 0$, and since R is commutative, 6.13 of Nagata **[13]** shows that there exists $0 \neq c \in R$ such that $cg(x) = 0$, and so $ca_i = 0$ for $i = 1, \cdots, t$, a contradiction.

We remark that Theorems 1.8 and 1.9 are true for polynomial rings in a finite number of indeterminants by induction, and then due to the "local" nature of the conditions the results hold for polynomial rings in an arbitrary number of indeterminants.

2. Orders in semisimple Artinian rings.

THEOREM (2.1). *The ring R is semisimple (simple) Artinian if and only if R is semiprime (prime), every nonzero left ideal contains a minimal left ideal, and every faithful left ideal is cofaithful.*

Proof. Assume that R is semiprime, every faithful left ideal is cofaithful, and every nonzero left ideal contains a minimal left ideal, and let S be the sum of all minimal left ideals of R. Then by assumption S is essential in R, and hence faithful since R is semiprime. Thus S must be cofaithful, and so there exists an exact sequence $0 \to R \to S^k$ for some positive integer k. This shows that ${}_RR$ is completely reducible, and therefore semisimple Artinian.

In analogous fashion we are able to characterize orders in semisimple Artinian rings by merely requiring enough uniform left ideals instead of enough minimal left ideals as in Theorem 2.1. We study the prime case first. (Recall that a ring is prime if and only if every nonzero left ideal is faithful.)

THEOREM (2.2). *The ring R is a left order in a simple Artinian ring if and only if R is prime, contains a uniform left ideal, and every nonzero left ideal is cofaithful.*

Proof. If R is an order in a simple Artinian ring then every left ideal is cofaithful and R contains a uniform left ideal by Proposition 1.5.

Conversely, if $Z(R)$, the singular ideal of R, is nonzero, then there is an exact sequence $0 \to R \to Z(R)^k$ for some positive integer k. This implies $Z(R) = R$, a contradiction. Since $Z(R) = 0$, in order to show that R is left Goldie it suffices to show that R has finite uniform dimension. To see this, let U be a uniform left ideal of R. Then R has finite uniform dimension, since for some positive integer k there exists an exact sequence $0 \to R \to U^k$.

We next give a proof of Theorem 2.2 which avoids Goldie's Theorem and simultaneously produces the full matrix ring over a division ring in which the ring is a left order. The proof is inspired by the proof of Faith's Theorem 34 **[6]**. We first observe that if the left uniform dimension of R is n and $0 \to R \to M^m$ is exact for some positive integer m, then there exists an exact sequence $0 \to R \to M^k$ with $k \leqq n$.

THEOREM (2.2 bis). *If the ring R is prime, contains a uniform left ideal, and every nonzero left ideal is cofaithful, then $Q_{cl}(R)$, the left classical quotient ring of R, is an $n \times n$ matrix ring over a division ring.*

Proof. As in the proof of Theorem 2.2, $Z(R) = 0$ and R has finite uniform dimension, say $\dim R = n$. Furthermore, there exists an exact sequence $0 \to R \to U^n$ where U is a uniform left ideal of R. Let V be the quasi-injective hull of U. Since $Z(R) = 0$, $Z(U) = 0$ and U is strongly uniform in the sense of Storrer **[17]**. By Lemma 7.4 of Storrer **[17]**, $D = \mathrm{End}_R(V)$ is a division ring. By Proposition 13 of Faith **[6]**, V is in fact injective. We claim that V has dimension n as a vector space over D. There exists an exact sequence $0 \to R \to V^n$, and if $(v_1, v_2, \cdots, v_n)$ is the image of $1 \in R$ in V^n, we show $\{v_1, \cdots, v_n\}$ is a basis for V over D. Let $v \in V$ and $f: R \to V$ be the map given by $f(r) = rv$. By the injectivity of V^n this extends to a map f': $V^n \to V$ with components $d_i \in D$. Hence $v = f(1) = f'(v_1, \cdots, v_n) = \sum_{i=1}^n d_i v_i$. Thus $v_1, \cdots, v_n$ span V. If, on the other hand, $\sum_{i=1}^n d_i v_i = 0$, with say $d_j v_j \neq 0$, then $rv_i = 0$ for $i \neq j$ implies $rv_j = 0$ and there is a monomorphism from R into V^{n-1}, so, since V is a uniform R-module, this contradicts the fact that $\dim(R) = n$. If $Q = \mathrm{End}_D(V)$, then Q is the ring of $n \times n$ matrices over D^{opp}, and there is a natural embedding $R \subseteq Q$, since V is faithful. Now V is isomorphic as a Q-module to a minimal left ideal of Q and ${}_QQ \simeq {}_RV^n$, which implies that R is essential in Q. Furthermore, if V_i is the intersection of R and the ith component of Q, then V_i is nonzero for otherwise we have an embedding of R in V^{n-1} and contradict $\dim(R) = n$.

Let A be an essential left ideal of R, and let $A_i = A \cap V_i \neq 0$. Then $A_1 A_i \neq 0$, since R is prime, so there exists $a_i \in A_i$ with $A_1 a_i \neq 0$. Thus we may define a nonzero homomorphism $f: A_1 \to A_i \subseteq V$ by $f(a) = aa_i$ for $a \in A_1$. Since U is essential in V and $Z(U) = 0$, we have $Z(V) = 0$. If $\ker(f) \neq 0$, then $\ker(f)$ is an essential submodule of V and if $x \in A_1 \subseteq V$ then $\{r \in R \mid rf(x) = 0\}$ is an essential left ideal of R, which implies $f(x) \in Z(V) = 0$ and $f = 0$, a contradiction. Hence there exists an exact sequence

$$0 \to R \to A_1^n \to \bigoplus_{i=1}^{n} A_i \subseteq A,$$

and so there exists $x \in A$ such that $\ell_R(x) = 0$. Since R is essential in Q, $\ell_Q(x) = 0$, and since Q is left Artinian x must be invertible in Q and hence regular in R. This shows that every essential left ideal of R contains a regular element, so if $q \in Q$, then $(R: q) = \{r \in R \mid rq \in R\}$ is an essential left ideal of R since R is essential in Q, and thus $(R: q)$ contains a regular element x. Hence $xq = r \in R$ and so $q = x^{-1}r$. Thus R is a left order in Q.

LEMMA (2.3). *Let R be a semiprime ring and U a uniform left ideal of R. Then $P = \ell_R(U)$ is a prime ideal of R, and the image of U in R/P is a uniform left ideal of R/P.*

Proof. Let A and B be left ideals of R such that $AB \subseteq P$. Then $ABU = 0$ and so $BUA = 0$ for otherwise $(BUA)^2 = 0$, while $BUA \neq 0$. Hence $BU \cdot AU = 0$ and so $BU \cap AU = 0$ since R is semiprime. Since U is uniform $BU = 0$ or $AU = 0$ and so $A \subseteq P$ or $B \subseteq P$. Since $P \cap U = 0$, it is easy to see that the image of U in R/P is again uniform.

LEMMA (2.4). *Let R be a semiprime ring in which every faithful left ideal is cofaithful and let S be a left ideal of R. If $A = \ell_R(S)$, then every faithful left ideal of R/A is cofaithful.*

Proof. Let I be the ideal SR. Then $A = \ell_R(I)$, and since R is semiprime, $A = \ell_R(I)$. Let B/A be a faithful left ideal in R/A. If $C = \ell_R(B)$ then $CB = 0$ and so $CB \subseteq A$; hence $C \subseteq A$. Thus $C^2 = 0$, since $C \subseteq A \subseteq B$. Since R is semiprime, $C = 0$ and B is a faithful left ideal of R. By hypothesis there exist $b_1, \cdots, b_t \in B$ such that $rb_i = 0$ for $i = 1, \cdots, t$ implies that $r = 0$. Let $\bar{b}_i$ be the image of b_i in

B/A. Suppose $\bar{r} \in R/A$ is such that $\bar{r}b_i = 0$ for $i = 1, \cdots, t$. Then $rb_i \in A$ for $i = 1, \cdots, t$ and $rb_iI = 0$, so $Irb_i = 0$ for $i = 1, \cdots, t$. Hence $Ir = 0$ and $r \in \ell_R(I) = A$. Thus $\bar{r} = 0$ and $\ell_{R/A}(\bar{b}_1, \cdots, \bar{b}_t) = 0$.

THEOREM (2.5). *The ring R is a left order in a semisimple Artinian ring if and only if R is semiprime, has enough uniform left ideals, and every faithful left ideal is cofaithful.*

Proof. Assume R is semiprime, has enough uniforms and every faithful left ideal is cofaithful. Let A be the sum in R of all uniform left ideals and let M be the external direct sum of these left ideals. Since R is semiprime and A is essential, A is a faithful left ideal. Thus M is a faithful torsionless left R-module and by Proposition 1.1 it is cofaithful. Hence there exists an exact sequence $0 \to R \xrightarrow{f} M^k$ for some k. Since $1 \in R, f(1)$ belongs to a finite direct sum of uniform R-modules and so R is isomorphic to a submodule of a finite dimensional module. This shows that R has finite uniform dimension.

Let $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ be a maximal direct sum of uniform left ideals of R. It is clear that U is a faithful left ideal of R and $\bigcap_{i=1}^m P_i = 0$, where $\{P_i\}$ is the set of distinct elements of $\{\ell(U_i)\}$. Thus the canonical map $\phi: R \to R/P_1 \oplus \cdots \oplus R/P_m$ is a monomorphism. By Lemmas 2.3 and 2.4, R/P_i is a prime ring which contains a uniform left ideal and in which every left ideal is cofaithful. By Theorem 2.2, R/P_i is a left order in a simple left Artinian ring S_i. One can show directly that $S = S_1 \oplus \cdots \oplus S_m$ is the left classical quotient ring of $\phi(R)$, or else since R is a subring of the left Artinian ring S, R has the ascending chain condition on left annihilators and we can apply Goldie's theorem.

The Gabriel dimension of a ring is defined by Gordon and Robson **[10]** in terms of localizing Serre subcategories. From Corollary 2.10 of **[10]**, one can easily show that a ring with Gabriel dimension has enough uniforms. Thus we are able to state the following corollary of Theorem 2.5.

COROLLARY (2.6). *A semiprime ring with Gabriel dimension is left Goldie if and only if every faithful left ideal is cofaithful.*

We end this section with a theorem on semiprime rings with a polynomial identity. Our result weakens the hypothesis of theorems of Armendariz-Steinberg, Formanek, Rowen and Small.

THEOREM (2.7). *Let R be a semiprime ring with a polynomial identity and center C, where C satisfies the condition that every faithful ideal is cofaithful. Let S be the set of regular elements of C. Then R is an order in a semisimple Artinian ring and*

(1) $S^{-1}C = F_1 \oplus \cdots \oplus F_k$, *a finite direct product of fields.*

(2) $S^{-1}R = Q_1 \oplus \cdots \oplus Q_k$, *where Q_i is a finite-dimensional central simple algebra with center F_i.*

Proof. Apply Proposition 1.3(b) to the semiprime ring C to see that C satisfies the ascending chain condition on annihilators. The result now follows from Theorem 9 of Formanek [7].

3. Absolutely torsion-free rings. A left exact subfunctor of the identity Id on ${}_R\mathcal{M}$, the category of left R-modules, is called a torsion preradical. For torsion preradicals ρ and σ we write $\rho \leqq \sigma$ if $\rho(M) \subseteq \sigma(M)$ for all $M \in {}_R\mathcal{M}$. Observe that $\sigma(R) = R$ if and only if $\sigma = Id$. For $M \in {}_R\mathcal{M}$, let Rad^M be the smallest torsion preradical σ such that $\sigma(M) = M$. Then for $X \in {}_R\mathcal{M}$, $\mathrm{Rad}^M(X) = \{x \in X \mid x = \sum_{i=1}^n f_i(m_i) \text{ for } m_i \in M,\ f_i \in \mathrm{Hom}_R(Rm_i, X)\}$, and it follows that $\mathrm{Rad}^M = Id$ if and only if M is cofaithful by Beachy [3].

We recall that a module is said to be prime if for all nonzero submodules $M' \subseteq M$, $AM' = 0$ implies $AM = 0$ for all left ideals A of R. A submodule will be called fully invariant if it is invariant under all endomorphisms. The injective envelope of a module M is denoted $E(M)$.

PROPOSITION (3.1). *The following are equivalent for $M \in {}_R\mathcal{M}$.*

(a) *For all torsion preradicals σ of ${}_R\mathcal{M}$, either $\sigma(M) = 0$ or $\sigma(M) = M$.*

(b) *M is contained in every nonzero fully invariant submodule of $E(M)$.*

(c) *For all $0 \neq x \in M$ and $y \in M$ there exist $r_1, r_2, \cdots, r_n \in R$ such that $\bigcap_{i=1}^n \mathrm{Ann}(r_i x) \subseteq \mathrm{Ann}(y)$.*

(d) *M is prime and if $0 \neq M' \subseteq M$ then for all $y \in M$ there exist $x_1, x_2, \cdots, x_n \in M'$ such that $\bigcap_{i=1}^n \mathrm{Ann}(x_i) \subseteq \mathrm{Ann}(y)$.*

Proof. (a) $\Rightarrow$ (b). If $0 \neq N \subseteq E(M)$ is fully invariant, then $\mathrm{Rad}^N(E(M)) = N$, so $\mathrm{Rad}^N(M) = M \cap \mathrm{Rad}^N(E(M)) = M \cap N \neq 0$, and thus we have $\mathrm{Rad}^N(M) = M$ and so $M \subseteq N$.

(b) $\Rightarrow$ (c). For $0 \neq x \in M$, let N be the sum in $E(M)$ of the homomorphic images of Rx. Then N is fully invariant, so by assumption $M \subseteq N$ and thus $y = \sum_{i=1}^n f_i(r_i x)$ for $r_i \in R$ and $f_i \in \mathrm{Hom}_R(Rx, E(M))$. Therefore $ar_i x = 0$ for all i implies $ay = 0$.

(c) $\Rightarrow$ (d). If $0 \neq M' \subseteq M$, and $AM' = 0$ for some $A \subseteq R$, then let $0 \neq x \in M'$. By assumption for any $y \in M$ there exist $r_1, r_2, \cdots, r_n \in R$ such that $A \subseteq \bigcap_{i=1}^{n} \mathrm{Ann}(r_i x) \subseteq \mathrm{Ann}(y)$, so $AM = 0$ and M is prime. The second condition follows immediately from (c).

(d) $\Rightarrow$ (a). If $0 \neq \sigma(M) = N$ for some torsion preradical σ, then for any $y \in M$ there exist $x_1, x_2, \cdots, x_k \in N$ such that $\bigcap_{i=1}^{k} \mathrm{Ann}(x_i) \subseteq \mathrm{Ann}(y)$. Thus for $x = (x_1, \cdots, x_k) \in N^k$, the mapping $f\colon Rx \to Ry$ defined by $f(ax) = ay$ is a well-defined homomorphism, and since $x \in \sigma(N^k)$ we must have $y = f(x) \in \sigma(M)$. This shows that $\sigma(M) = M$, completing the proof.

Taking $y = 1 \in R$ in condition (d) of Proposition 3.1 shows that ${}_R R$ satisfies the equivalent conditions of the proposition if and only if R is prime and every (faithful) left ideal is cofaithful. Condition (a) is satisfied if and only if $\sigma(R) = 0$ for every torsion preradical σ such that $\sigma \neq Id$; such rings are the absolutely torsion-free rings studied by Rubin **[14]**. Taking $y = 1$ in condition (c) shows that R is absolutely torsion-free if and only if for all $0 \neq r \in R$ there exist $r_1, \cdots, r_n \in R$ such that $sr_i r = 0$ for all i implies $s = 0$, and this gives the condition studied by Handelman and Lawrence **[11]**. It also gives the equivalent condition that every nonzero left ideal is cofaithful, as shown by Viola-Prioli **[18]**, Theorem 1.1.

Many of Rubin's results on absolutely torsion-free rings are easier to prove in the light of Proposition 3.1. A prime left Goldie ring is absolutely torsion-free on the left and right (Rubin **[14]**, Theorem 1.11) since it satisfies the descending chain condition on both left and right annihilators. Since being prime and having every faithful left ideal cofaithful are both Morita invariant, so is being absolutely torsion-free (Rubin **[14]**, Theorem 1.12). Applied to ${}_R R$ condition (b) of Proposition 3.1 states that $E(R)$ has no nontrivial invariant submodules. If $S \supseteq R$ is a subring of the complete ring of quotients of R, then $E({}_S S) = E({}_R R)$ and the condition implies that S is absolutely torsion-free whenever R is (Rubin **[14]**, Theorem 1.15).

PROPOSITION (3.2). *A ring R is left absolutely torsion-free if and only if R is prime, $Z(R) = 0$, and every nonsingular quasi-injective left R-module is injective.*

Proof. Assume that R is left absolutely torsion-free and that $0 \neq {}_R M$ is quasi-injective with $Z(M) = 0$. Then $\mathrm{Rad}^M(M) \neq 0$ implies that $\mathrm{Rad}^M = Id$ by Violi-Prioli **[18]** Theorem 1.1, so M is cofaithful and hence injective.

Conversely, let M be a fully invariant submodule of $E(R)$. Then M is quasi-injective and nonsingular since by assumption $Z(R)=0$, so M must be injective and thus a direct summand of $E(R)$, say $E(R)=M\oplus N$. But $M\cap R$ is an ideal since M is fully invariant in $E(R)$, so $(M\cap R)\cdot(N\cap R)=0$ and this implies that $N\cap R=0$ since R is prime. Thus $N=0$ since R is essential in $E(R)$, so $M=E(R)$ and it follows from Proposition 3.1 that R is absolutely torsion-free.

Finally, as a consequence of Proposition 3.1 we have the following restatement of Theorem 2.2.

THEOREM (3.3). *A left absolutely torsion-free ring is left Goldie if and only if it has a uniform left ideal.*

We call a module M semicompressible if for all nonzero submodules $N\subseteq M$ there exists an exact sequence $0\to M\to N^k$ for some positive integer k. (Note that a semicompressible module satisfies the conditions of Proposition 3.1.) The following proposition can be generalized easily to quasi-projective semicompressible modules.

PROPOSITION (3.4). *The endomorphism ring of a projective, semicompressible left R-module is left absolutely torsion-free.*

Proof. Let ${}_RM$ be semicompressible and projective and let $\mathrm{End}_R(M)$ act on the left of M. We will show that $\mathrm{End}_R(M)$ satisfies condition (c) of Proposition 3.1. Let $f,g\in\mathrm{End}_R(M)$, $f\neq 0$, $g\neq 0$. Since $g(M)\neq 0$ and M is semicompressible, there exists a positive integer k and a monomorphism $j: M\to(g(M))^k$. Let $p: M^k\to(g(M))^k$ be the homomorphism with components $p_i=g$. Since M is projective, j lifts to a map $h: M\to M^k$ with components h_i:

$$\begin{tikzcd} & M \arrow[dl, dashed, "h"'] \arrow[d, "j"] & \\ M^k \arrow[r] & (g(M))^k \arrow[r] & 0. \end{tikzcd}$$

Then $phf=fj\neq 0$ since j is monic and $f\neq 0$, so $gh_if=(ph)_if\neq 0$ for some component $(ph)_i$ of ph. Hence $\ell((ph)_if)\subseteq\ell(g)$.

THEOREM (3.5). *The ring of endomorphisms of a finitely generated projective module over a left absolutely torsion-free ring is left absolutely torsion-free.*

Proof. Let R be left absolutely torsion-free. Then ${}_RR$ is semicompressible, so the result will follow from Proposition 3.4 if we can show that any finitely generated free module over R is semicompressible, since a submodule of a semicompressible module is semicompressible. More generally, we show that if ${}_RM$ is semicompressible, then M^n is also. For this purpose let $0 \neq N \subseteq M^n$ and let p_n be the projection of M^n onto the last component. If p_n is monic when restricted to N, then since M is semicompressible there exists k such that $0 \to M \to (p_n(N))^k \simeq N^k$ is exact and so $0 \to M^n \to (N^k)^n$ is exact. If p_n is not monic on N then $N \cap M^{n-1} \neq 0$ and the above argument can be applied to $N \cap M^{n-1}$. Continuing we see that there exists an embedding $0 \to M^n \to N^t$ for some t and M^n is semicompressible.

REFERENCES

1. J. A. Beachy, *On quasi-Artinian rings*, J. London Math. Soc., (2), **3** (1971), 449–452.
2. ———, *Generating and cogenerating structures*, Trans. Amer. Math Soc., **158** (1971), 75–92.
3. ———, *A generalization of injectivity*, Pacific J. Math., **41** (1972), 313–327.
4. W. D. Blair, *Right Noetherian rings integral over their centers*, J. Algebra, **27** (1973), 187–198.
5. C. Faith, *Rings with ascending condition on annihilators*, Nagoya Math. J., **27** (1966), 179–191.
6. ———, *Modules finite over endomorphism ring*, Lecture Notes in Math. 246 (1972), 145–181.
7. E. Formanek, *Noetherian PI-rings*, Comm. Algebra, **1** (1974), 79–86.
8. A. W. Goldie, *The structure of prime rings under ascending chain conditions*, Proc. London Math. Soc., **8** (1958), 589–608.
9. ———, *Semiprime rings with maximum condition*, Proc. London Math. Soc., **10** (1960), 201–220.
10. R. Gordon and J. C. Robson, *Krull dimension*, Mem. Amer. Math. Soc., **133** (1973).
11. D. Handelman and J. Lawrence, *Strongly prime rings*, preprint.
12. L. Lesieur and R. Croisot, *Sur les anneaux premiers noetheriens à gauche*, Ann. Sci. École Norm. Sup., **76** (1959), 161–183.
13. M. Nagata, *Local Rings*, Interscience, New York, 1962.
14. R. A. Rubin, *Absolutely torsion-free rings*, Pacific J. Math., **46** (1973), 503–514.
15. R. C. Shock, *Polynomial rings over finite dimensional rings*, Pacific J. Math., **42** (1972), 251–258.
16. L. W. Small, *The embedding problem for Noetherian rings*, Bull. Amer. Math. Soc., **75** (1969), 147–148.
17. H. H. Storrer, *On Goldman's primary decomposition*, Lecture Notes in Math., **246** (1972), 617–661.
18. J. Viola-Prioli, *On absolutely torsion-free rings and kernel functors*, Ph.D. thesis, Rutgers--The State University, New Brunswick, New Jersey, 1973.

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WALLMAN RINGS

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In 1964 Frink defined a normal base. He hypothesized that every Hausdorff compactification of a Tychonoff space X may be realized as a compactification $w(\mathcal{F})$ of Wallman type obtained from a normal base $\mathcal{F}$ on X, where $\mathcal{F}$ is the family of zero sets for some subring of $C(X)$. Later Biles formally defined a Wallman Ring on a Tychonoff space to be a subring of $C(X)$ whose zero sets form a normal base on X.

The problem in this paper is to study examples of Wallman Rings and develop properties of Wallman Rings. For a locally compact space with a given compactification and a certain type of retract map, a Wallman Ring is defined which induces the given compactification.

General algebraic and topological properties of Wallman Rings are considered. Among the results obtained are "Every Wallman Ring is equivalent to one which contains all rational constant functions" and "An ideal of a Wallman Ring which is itself a Wallman Ring is equivalent to the superring."

I. Introduction. In 1938 H. Wallman [23] gave a method for associating a compact T_1-space $w(\mathcal{F})$ with a distributive lattice $\mathcal{F}$; $w(\mathcal{F})$ is the space of all $\mathcal{F}$-ultrafilters and the topology of $w(\mathcal{F})$ has a base for closed sets a lattice $\mathcal{F}^*$ which is isomorphic to the lattice $\mathcal{F}$. Wallman applied this procedure to the case when $\mathcal{F}$ is the lattice of all closed subsets of a T_1-space X to arrive at a T_1 compactification $w(\mathcal{F})$ of X which is now called "the Wallman Compactification" of X.

Several later mathematicians applied Wallman's construction to certain types of lattices which are sublattices of the lattice of all closed sets of a T_1-space. Among these were Sanin [18], Banaschewski [3], and Frink [12]. These techniques give rise to certain classes of compactifications. In 1964, Frink [12] asked whether each Hausdorff compactification of a Tychonoff space X can be realized as a compactification $w(\mathcal{F})$ of Wallman type obtained from a normal base $\mathcal{F}$ for the closed sets of X. (A normal base is a lattice which is a base for closed sets and which satisfies certain separation properties.) This question remains unanswered to the present day, but a great deal of effort has been exerted by many mathematicians in an endeavor to solve the problem.

Partial solutions have been obtained, e.g. Steiner and Steiner have shown that any product of compact metric spaces is a Wallman type compactification (determined by some normal base) of any of its dense subspaces. It has become customary to call a compactification of the form $w(\mathcal{F})$, where $\mathcal{F}$ is some normal base, a "Wallman compactification" of X and we shall use this terminology in the sequel.

Frink observed that the family $Z(X)$ of all zero sets of continuous real valued functions on a Tychonoff space X is a normal base on X which gives rise to a compactification $w(Z(X))$ equivalent to the Stone-Cech compactification βX of X. He also observed that if Y is a given compactification of X, then $Z(E(X, Y))$, the zero sets of continuous real valued functions on X which are continuously extendible to Y, is a normal base. Biles **[8]** later called a subring $\mathcal{A}$ of the ring $C(X)$, of all real valued continuous functions on X, a Wallman ring on X whenever $Z(\mathcal{A})$, the zero sets of functions in $\mathcal{A}$, form a normal base.

Frink wondered whether every Hausdorff compactification of a Tychonoff space X is of the form $w(Z(\mathcal{A}))$ where $\mathcal{A}$ is some Wallman ring on X. This question is still unanswered although many partial results have been obtained.

Biles **[8]** studied relationships between the Gelfand and the Wallman compactifications determined by a Wallman ring $\mathcal{A}$ on a Tychonoff space X.

In this paper we study examples of Wallman rings and develop properties of Wallman rings.

In §IV we will consider a locally compact space and a compactification of that space such that there is a certain type of retract map on the compactification. We establish a Wallman ring on this locally compact space which induces the given compactification. From this result we are able to define a Wallman ring which yields the Alexandroff compactification and a Wallman ring on the open unit disc which induces a compactification equivalent to the closed unit disc.

In section III we consider general properties of Wallman rings on spaces with more than one element. We find that a Wallman ring cannot be an integral domain; that every Wallman ring is equivalent to one which is inverse closed; that an ideal of a Wallman ring which is itself a Wallman ring is equivalent to the larger ring. We also examine the relationship between a Wallman ring being the direct sum of nontrivial ideals and its associated compactification being disconnected. We present some results linking Wallman rings to sublattices of $C(X)$, and we pose the question: "Is every Wallman ring on X equivalent to one which is a sublattice of $C(X)$?"

II. Preliminary notations and definitions.

Throughout this paper, all topological spaces are Tychonoff (completely regular and Hausdorff), and contain at least two points. If X is a topological space, then a compact space Y is said to be a compactification of X if there is a homeomorphism h from X into Y such that $h[X]$ is dense in Y. The function h is called an embedding. To simplify notation, embeddings will be taken to be inclusions.

We will use the customary ordering of compactifications: $Y_1 \leqq Y_2$ if there is a continuous map f: $Y_2 \rightarrow Y_1$ which leaves X pointwise fixed (i.e. $f(x) = x$ for all $x \in X$). Also, Y_1 and Y_2 are equivalent ($Y_1 \cong Y_2$) if $Y_1 \leqq Y_2$ and $Y_2 \leqq Y_1$. As is well known, $Y_1 \cong Y_2$ if and only if there is a homeomorphism f: $Y_1 \rightarrow Y_2$ which leaves X pointwise fixed.

We will use the following notation throughout the paper.

$\mathcal{N}$ — the set of natural numbers

$\mathcal{R}$ — the field of real numbers

$C(X)$ — the ring of all real valued continuous functions on the space X.

$C^*(X)$ — the ring of all bounded functions in $C(X)$.

$\mathcal{A}^*$ — the subset of bounded functions of a collection $\mathcal{A} \subseteq C(X)$, $\mathcal{A}^* = \{f \in \mathcal{A} : f \in C^*(X)\}$

$Z(f)$ — the zero set of a real valued function f on X, $Z(f) = \{x \in X : f(x) = 0\}$.

$Z[\mathcal{A}]$ – the zero sets of a collection $\mathcal{A}$ of real valued functions on X, $Z[\mathcal{A}] = \{Z(f) : f \in \mathcal{A}\}$.

$Z(X)$ — the zero sets of $C(X)$, $Z(X) = Z[C(X)] = Z[C^*(X)]$.

For basic concepts regarding the ring $C(X)$, we refer the reader to Gillman and Jerison **[13]**.

Following the terminology of Biles **[8]** and Frink **[12]** we give the following definitions.

2.1. Definition. If $\mathcal{F}$ is a collection of subsets of X, then $\mathcal{F}$ is a lattice on X if:

(1) $\phi, X \in \mathcal{F}$

(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$.

2.2. Definition. The lattice $\mathcal{F}$ on X is a normal base on X if $\mathcal{F}$ is:

(1) a base for the closed subsets of X,

(2) a disjunctive lattice on X (i.e. if $A \in \mathcal{F}$ and $x \in X - A$, then there exists $B \in \mathcal{F}$ such that $x \in B$ and $A \cap B = \phi$),

(3) a normal lattice on X i.e. for each A, $B \in \mathcal{F}$ such that $A \cap B = \phi$, there exists $C, D \in \mathcal{F}$ such that $A \cap D = \phi$, $B \cap C = \phi$ and $C \cup D = X$.

The space $w(\mathcal{F})$ consisting of the set of all $\mathcal{F}$-ultrafilters on X is a Hausdorff compactification of X. The topology of $w(\mathcal{F})$ is defined as follows. If $A \in \mathcal{F}$, A^* is the set of all $\mathcal{F}$-ultrafilters having A as a member. A base for the closed subsets of $w(\mathcal{F})$ is the set of all A^* such that $A \in \mathcal{F}$.

X is embedded in $w(\mathcal{F})$ by the map which sends a point $x \in X$ into the $\mathcal{F}$-ultrafilter consisting of all $\mathcal{F}$-sets which contain x.

2.3. DEFINITION. $w(\mathcal{F})$ is said to be a Wallman compactification of X.

As was mentioned earlier a subring $\mathcal{A}$ of $C(X)$ is a Wallman ring on X if $Z[\mathcal{A}]$ is a normal base on X. So if $\mathcal{A}$ is a Wallman ring on X, then $w(Z[\mathcal{A}])$ is a Wallman compactification of X.

We will content ourselves here with the above statements on Wallman compactifications. The reader is referred to the literature (e.g. Frink **[12]**, Biles **[8]**, Steiner **[19]**, Alo and Shapiro **[2]**) for proofs of the above statements.

Being interested in ordering of compactifications, we are led to the following concept which is due to Steiner **[19]**.

2.4. DEFINITION. Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets. Then:

(1) $\mathcal{F} \leqq \mathcal{G}$ ($\mathcal{G}$ separates $\mathcal{F}$) if and only if for each F_1, $F_2 \in \mathcal{F}$, $F_1 \cap F_2 = \phi$ implies there exist G_1, $G_2 \in \mathcal{G}$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \phi$.

(2) $\mathcal{F} \cong \mathcal{G}$ ($\mathcal{F}$ is equivalent to $\mathcal{G}$) if and only if $\mathcal{F} \leqq \mathcal{G}$ and $\mathcal{G} \leqq \mathcal{F}$.

2.5. THEOREM. *The relation "$\leqq$" defined in* 2.4 *is transitive and reflexive. The relation "$\cong$" is an equivalence relation.*

We will now look at an application of this concept. By a sublattice of $C(X)$ we mean a subset of $C(X)$ which contains the supremum and infimum of each pair of its elements. By a closed subring of $C(X)$ we mean a subring of $C(X)$ which is closed in the uniform topology on $C(X)$.

2.6. DEFINITION. $\mathcal{A}$ is an inverse closed subset of $C(X)$ if and only if for each $f \in \mathcal{A}$ and for each $g \in \mathcal{A}$ such that $Z(g) = \phi$, $f/g \in \mathcal{A}$.

2.7. THEOREM. *Let $\mathcal{A}$ be a Wallman subring and sublattice of $C(X)$, then f, $g \in \mathcal{A}$ implies $\{x \in X: f(x) \leqq g(x)\} \in Z[\mathcal{A}]$.*

Proof. $\{x \in X: f(x) \leqq g(x)\} = Z((f-g) \vee 0)$.

In a restricted situation, the following theorem gives some insight into the separating relation of Steiner defined in 2.4. This theorem also is closely related to generalizations of the Stone-Weierstrass Theorem (for details see Taylor and Bentley **[22]**).

2.8. THEOREM. *Let $\mathscr{A}$ be a sublattice of $C(X)$ which contains the constant functions. Let $\mathscr{B}$ be an inverse closed subset of $C(X)$ which is also a closed subring of $C(X)$. Then $Z[\mathscr{A}] \leqq Z[\mathscr{B}]$ if and only if $\mathscr{A}^* \subseteq \mathscr{B}$.*

Proof. Half of the proof is obvious. We will show only the other half. We borrow our method of proof from Hager **[14]**.

Let $g \in \mathscr{A}^*$. There is a positive real number r such that $|g| \leqq r$. Choose $\epsilon > 0$. Then there is a natural number n such that $1/n < \epsilon/2r$. Let $I = \{-n-1, -n, \cdots, -1, 0, 1, \cdots, n-1\}$.

For $i \in I$, let

$$F_i = \{x \in X: g(x) \leqq ir/n\} \cup \{x \in X: (i+2)r/n \leqq g(x)\}.$$

Then $F_i \in Z[\mathscr{A}]$ and $\bigcap_{i \in I} F_i = \phi$. As was shown by H. L. Bentley **[4]** there are functions $f_i \in \mathscr{B}$ for each $i \in I$ such that $F_i \subseteq Z(f_i)$ and $\bigcap_{i \in I} Z(f_i) = \phi$. For each $j \in I$, let

$$h_j = \frac{f_j^2}{\sum_{i \in I} f_i^2}.$$

So $\Sigma_{j \in I} h_j = 1$ and since $\mathscr{B}$ is inverse closed $h_j \in \mathscr{B}$ for each $j \in I$.

Let $u = \Sigma_{i \in I} irh_i/n$. Since $\mathscr{B}$ is inverse closed, is a closed subring of $C(X)$, and contains a function whose zero set is empty, namely $\Sigma_{i \in I} f_i^2$, $\mathscr{B}$ contains all the constant functions. Therefore $u \in \mathscr{B}$.

Now let $x \in X$ and let j be as small as possible so that $x \notin F_j$; x cannot be in each of the F_i since $\bigcap_{i \in I} F_i = \phi$. $x \notin F_j$ implies $jr/n < g(x) < (j+2)r/n$ so $|g(x) - (j+1)r/n| < r/n$ and

$$x \in (\bigcap \{F_i: i \in I, i < j\}) \cap (\bigcap \{F_i: i \in I, i > j+1\}).$$

$$\begin{aligned}
\text{Now} \quad u(x) &= \sum_{i \in I} \frac{i}{n} rh_i(x) \\
&= \frac{j}{n} rh_j(x) + \frac{j+1}{n} rh_{j+1}(x) \\
&= \frac{j}{n} r(h_j(x) + h_{j+1}(x)) + \frac{r}{n} h_{j+1}(x) \\
&= \frac{j}{n} r\Big(\sum_{i \in I} h_i(x)) + \frac{r}{n} h_{j+1}(x)\Big) \\
&= \frac{j}{n} r + \frac{r}{n} h_{j+1}(x).
\end{aligned}$$

Since $0 \leqq h_{j+1}(x) \leqq 1$ this yields

$$\frac{j}{n} r \leqq u(x) \leqq \frac{j}{n} r + \frac{r}{n} = \frac{j+1}{n} r.$$

Therefore $|u(x) - (j+1)r/n| \leqq r/n$ and

$$|g(x) - u(x)| \leqq \left| g(x) - \frac{j+1}{n} \right| + \left| u(x) - \frac{j+1}{n} \right| \leqq \frac{2r}{n} < \epsilon.$$

For $\epsilon > 0$, we have shown there is a function $u \in \mathcal{B}$ such that $|g(x) - u(x)| < \epsilon$ for each $x \in X$. Therefore, since $\mathcal{B}$ is closed in $C(X)$, $g \in \mathcal{B}$.

The following theorem is due to A. K. Steiner and E. F. Steiner **[21]**.

2.9. THEOREM. *If $\mathcal{F}$ and $\mathcal{G}$ are normal bases on X, then*:
(1) $\mathcal{F} \leqq \mathcal{G}$ *if and only if* $w(\mathcal{F}) \leqq w(\mathcal{G})$.
(2) $\mathcal{F} \cong \mathcal{G}$ *if and only if* $w(\mathcal{F}) \cong w(\mathcal{G})$.

2.10. THEOREM. *If $\mathcal{F} \cong \mathcal{G}$ and $\mathcal{F}$ is normal, then $\mathcal{G}$ is normal.*

Proof. Let G_1 and G_2 be disjoint elements of $\mathcal{G}$. $\mathcal{G} \leqq \mathcal{F}$ implies there are sets F_1, $F_2 \in \mathcal{F}$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \phi$. Since $\mathcal{F}$ is normal there exist F_1', $F_2' \in \mathcal{F}$ such that $F_1 \cap F_2' = \phi$, $F_2 \cap F_1' = \phi$ and $F_1' \cup F_2' = X$. Now, since $\mathcal{G}$ separates $\mathcal{F}$ there exist G_1', G_2', G_a, $G_b \in \mathfrak{G}$ such that $F_1 \subseteq G_a$, $F_2' \subseteq G_2'$, $F_2 \subseteq G_b$, $F_1' \subseteq G_1'$, $G_a \cap G_2' = \phi$ and $G_b \cap G_1' = \phi$.

Now $G_1 \subseteq F_1 \subseteq G_a$ and $G_a \cap G_2' = \phi$ implies $G_1 \cap G_2' = \phi$.
$G_2 \subseteq F_2 \subseteq G_b$ and $G_b \cap G_1' = \phi$ implies $G_2 \cap G_1' = \phi$.
and $F_1' \subseteq G_1'$ and $F_2' \subseteq G_2'$ implies $G_1' \cup G_2' = X$.
Therefore $\mathcal{G}$ is normal.

2.11. THEOREM. *If $\mathcal{F}$ is a disjunctive base for the closed sets of X, $\mathcal{G}$ is a collection of closed sets of X and $\mathcal{F} \leqq \mathcal{G}$, then $\mathcal{G}$ is a disjunctive base for the closed sets of X.*

Proof. X and $\phi \in \mathcal{G}$ since X and ϕ, as elements of $\mathcal{F}$, must be separated by elements of $\mathcal{G}$.

If C is a closed subset of X and x is an element of X not in C, then there exists $F_1 \in \mathcal{F}$ such that $C \subseteq F_1$ and $x \notin F_1$. Consequently there exists $F_2 \in \mathcal{F}$ such that $x \in F_2$ and $F_1 \cap F_2 = \phi$.

Since $\mathscr{F} \leqq \mathscr{G}$, there are $G_1, G_2 \in \mathscr{G}$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \phi$. This implies $\mathscr{G}$ is a base for the closed sets of X since $x \notin G_1$ and $C \subseteq G_1$.

If $G \in \mathscr{G}$ and $x \notin G$, then as above there are $F_1, F_2 \in \mathscr{F}$ such that $x \in F_2$, $G \subseteq F_1$ and $F_1 \cap F_2 = \phi$; and consequently disjoint $G_1, G_2 \in \mathscr{G}$ such that $x \in G_2$, $G \subseteq G_1$. Since $x \in G_2$ and $G \cap G_2 = \phi$, $\mathscr{G}$ is a disjunctive base for the closed sets in X.

2.12. THEOREM. *If $\mathscr{F}$ is a normal base on X, $\mathscr{G}$ is a lattice of closed subsets of X and $\mathscr{F} \cong \mathscr{G}$, then $\mathscr{G}$ is a normal base on X and $w(\mathscr{F}) \cong w(\mathscr{G})$.*

We will now translate this result from normal bases to Wallman rings.

2.13. DEFINITION. If $\mathscr{A}$ and $\mathscr{B}$ are subrings of $C(X)$ then:
(1) $\mathscr{A} \leqq \mathscr{B}$ if and only if $Z[\mathscr{A}] \leqq Z[\mathscr{B}]$.
(2) $\mathscr{A} \cong \mathscr{B}$ if and only if $\mathscr{A} \leqq \mathscr{B}$ and $\mathscr{B} \leqq \mathscr{A}$.

2.14. THEOREM. *If $\mathscr{A}$ ı nd $\mathscr{B}$ are Wallman subrings of $C(X)$ then:*
(1) *$\mathscr{A} \leqq \mathscr{B}$ if and only if $w(Z[\mathscr{A}]) \leqq w(Z[\mathscr{B}])$*
(2) *$\mathscr{A} \cong \mathscr{B}$ if and only if $w(Z[\mathscr{A}]) \cong w(Z[\mathscr{B}])$.*

2.15. THEOREM. *$\mathscr{A}$ is a Wallman ring on X, $\mathscr{B}$ is a subring of $C(X)$ and $\mathscr{A} \cong \mathscr{B}$, then $\mathscr{B}$ is a Wallman ring on X and $w(Z[\mathscr{A}]) \cong w(Z[\mathscr{B}])$.*

III. Properties of Wallman Rings and Some Questions. Since a Wallman ring is a ring in the usual algebraic sense, it is natural for us to investigate which properties of rings Wallman rings possess. Along this line we have discovered that a Wallman ring cannot be an integral domain, and that each Wallman ring is equivalent to a Wallman ring which is inverse closed. We have investigated the problem of when a Wallman ring is equivalent to a Wallman ring which is a sublattice of $C(X)$, but have only partial results.

We also investigated relationships between algebraic properties of Wallman rings and topological properties of the induced compactifications. Our main result along this line is one involving the relationship between a Wallman ring being the direct sum of nontrivial ideals and the induced compactification being disconnected.

Our first result is the following.

3.1. THEOREM. *A Wallman ring cannot be an integral domain.*

Proof. Let $\mathscr{A}$ be a Wallman ring on a space X and let x and y be distinct elements of X. Since $Z[\mathscr{A}]$ is a disjunctive base for the closed sets of X, there are functions f and $g \in \mathscr{A}$ such that $x \in Z(f)$, $y \in Z(g)$ and $Z(f) \cap Z(g) = \phi$. By the normality of $Z[\mathscr{A}]$ there are functions f' and $g' \in \mathscr{A}$ such that $Z(f) \cap Z(g') = \phi$, $Z(g) \cap Z(f') = \phi$ and $Z(f') \cup Z(g') = X$. Now we have $f'(y) \neq 0$, $g'(x) \neq 0$ but $f'g'$ is the zero function.

We will now show that every Wallman ring is equivalent to a Wallman ring which is inverse closed and therefore to a Wallman ring which contains all the rational constants.

3.2. LEMMA. *Every Wallman ring contains at least two functions whose zero sets are pairwise disjoint and nonempty, and a function whose zero set is empty.*

Proof. Let $\mathscr{A}$ be a Wallman ring over X. Let x and y be distinct elements of X. Then there are functions f_1 and $f_2 \in \mathscr{A}$ such that $x \in Z(f_1)$, $y \in Z(f_2)$ and $Z(f_1) \cap Z(f_2) = \phi$. Then $f_1^2 + f_2^2$ is a function from $\mathscr{A}$ whose zero set is empty.

3.3. THEOREM. *Every Wallman ring is equivalent to a Wallman ring which is inverse closed.*

Proof. Let $\mathscr{A}$ be a Wallman ring on X and let $\mathscr{B} = \{f/g : f, g \in \mathscr{A}, Z(g) = \phi\}$. If $f \in \mathscr{A}$ then $Z(f) = Z(f/g)$ where g is some function from $\mathscr{A}$ such that $Z(g) = \phi$; so $Z[\mathscr{A}] \subseteq Z[\mathscr{B}]$. Similarly $Z[\mathscr{B}] \subseteq Z[\mathscr{A}]$. $\mathscr{B}$ is a subring of $C(X)$ and $\mathscr{A} \cong \mathscr{B}$ so by Theorem 2.15 $\mathscr{B}$ is a Wallman ring and $w(Z[\mathscr{A}]) \cong w(Z[\mathscr{B}])$.

3.4. COROLLARY. *Every Wallman ring is equivalent to a Wallman ring which contains all the rational constants.*

Proof. Let $\mathscr{A}$ be a Wallman ring on X. Let $\mathscr{B}$ be the Wallman ring $\{f/g : f, g \in \mathscr{A}, Z(g) = \phi\}$. Let $g \in \mathscr{A}$ such that $Z(g) = \phi$. Then if m and n are integers, $n \neq 0$, $mg/ng \in \mathscr{B}$ and $(mg/ng) = (m/n)$.

From this proof we observe that every inverse closed Wallman ring contains all the rational constant functions.

The Wallman ring $C^*(X)$ is equivalent to the inverse closed Wallman ring $C(X)$, for any given space X. $C^*(X)$ itself need not be inverse closed.

3.5. DEFINITION. If Y is a compactification of X, then $E(X, Y)$ is the set of all real valued continuous functions on X which are continuously extendable to Y.

Frink **[12]** was the first to observe that $E(X, Y)$ is a Wallman ring on X. Proofs of this were later given by Hager **[14]** and Biles **[8]**.

In the next theorem we give conditions for $C^*(X)$ to be inverse closed.

3.6. THEOREM. *The following are equivalent:*

(1) *X is pseudocompact,*

(2) *Every nonempty zero set of βX meets X,*

(3) *For any compactification Y of X, every nonempty zero set of Y meets X,*

(4) *For any compactification Y of X, $E(X, Y)$ is inverse closed,*

(5) *$C^*(X)$ is inverse closed.*

Proof. $1 \Rightarrow 2$. Let $f \in C(\beta X)$ and suppose $Z(f|X) = \phi$. Then $(1/f|X) \in C(X) = C^*(X)$. So So there exists $g \in C(\beta X)$ such that $g|X = (1/f|X)$. Now $(gf)|X = 1$ so $gf = 1$ and $Z(f) = \phi$. So $Z(f|X) = \phi$ implies $Z(f) = \phi$ or equivalently $Z(f) \neq \phi$ implies $Z(f|X) \neq \phi$.

$2 \Rightarrow 3$. Let α be a continuous mapping of βX into Y which leaves X pointwise fixed. The existence of such a function is guaranteed by Stone's Theorem **[14]**. Let $f \in C(Y)$ such that $Z(f) \neq \phi$ and suppose $Z(f|X) = \phi$. Let $g = f|X$. $g \in C(X)$ so g has an extension to a continuous function g^β in $C(\beta X)$, $g^\beta = f \circ \alpha$. Since $Z(g) = \phi$, $Z(g^\beta) = \phi$. Let $x \in Z(f)$. There exists $y \in \beta X$ such that $x = \alpha(y)$. Therefore $0 = f(x) = f(\alpha(y)) = g^\beta(y)$, which contradicts the fact that $Z(g^\beta) = \phi$. Therefore $Z(f) \neq \phi$ implies $Z(f|X) \neq \phi$.

$3 \Rightarrow 4$. . Let $f \in E(X, Y)$ such that $Z(f) = \phi$. There is a function $g \in C(Y)$ such that $f = g|X$. Since $Z(g|X) = \phi$, $Z(g) = \phi$ and $1/g \in C(Y)$. Now $1/f = 1/g|X$, so $1/f \in E(X, Y)$ and $E(X, Y)$ is inverse closed.

$4 \Rightarrow 5$. $C^*(X) = E(X, \beta X)$

$5 \Rightarrow 1$. Let $f \in C(X)$. Let $g = |f| \vee 1$. f will be bounded if and only if g is. $Z(g) = \phi$ so $1/g \in C(X)$. But $|1/g| = 1/g \leqq 1$, so $1/g \in C^*(X)$. Of course $Z(1/g) = \phi$. Therefore $g = 1/(1/g) \in C^*(X)$, and $C(X) = C^*(X)$.

Parts of this theorem are problems in Gillman and Jerison. $1 \Leftrightarrow 2$ is problem 6I, $1 \Leftrightarrow 5$ is problem 15Q.

We now investigate what happens when $\mathscr{A}$ is a Wallman ring and $\mathscr{B}$ is an ideal of $\mathscr{A}$.

3.7. THEOREM. *If $\mathcal{A}$ is a Wallman ring on X, $\mathcal{B}$ is an ideal of $\mathcal{A}$ and $Z[\mathcal{B}]$ is a disjunctive base on X, then $\mathcal{A} \leqq \mathcal{B}$.*

Proof. Let f and $g \in \mathcal{A}$ such that $Z(f) \cap Z(g) = \phi$. Assume $Z(f) \neq \phi$ and $Z(g) \neq \phi$ since otherwise the conclusion follows from $X \in Z[\mathcal{B}]$ and $\phi \in Z[\mathcal{B}]$. Let $x \in Z(f)$. Then $x \notin Z(g)$ so there is a function $f_x \in \mathcal{B}$ such that $x \in Z(f_x)$ and $Z(f_x) \cap Z(g) = \phi$. Now let $y \in Z(g)$; $y \notin Z(ff_x)$ so there is a function $g_y \in \mathcal{B}$ such that $y \in Z(g_y)$ and $Z(g_y) \cap (ff_x) = \phi$. Consequently ff_x and gg_y are functions from $\mathcal{B}$ whose zero sets separate the zero sets of f and g.

3.8. COROLLARY. *If $\mathcal{A}$ and $\mathcal{B}$ are Wallman rings on X and $\mathcal{B}$ is an ideal of $\mathcal{A}$ then $\mathcal{A} \cong \mathcal{B}$,* i.e., $w(Z[\mathcal{A}]) \cong e(Z[\mathcal{B}])$.

The next corollary tells us that a Wallman ring cannot be an ideal of $C(X)$ unless the Wallman ring is $C(X)$ itself.

3.9. COROLLARY. *If $\mathcal{A}$ is an inverse closed Wallman ring on X, then $\mathcal{A}$ has no proper nontrivial ideals whose zero sets are disjunctive; consequently $\mathcal{A}$ has no proper ideals which are Wallman rings.*

Proof. Let $\mathcal{B}$ be an ideal of $\mathcal{A}$ such that $Z[\mathcal{B}]$ is disjunctive. By Lemma 3.2 there are at least two disjoint non-empty zero sets of $\mathcal{A}$, say F_1 and F_2. By Theorem 3.7 there are functions f_1 and $f_2 \in \mathcal{B}$ such that $F_1 \subseteq Z(f_1)$, $F_2 \subseteq Z(f_2)$ and $Z(f_1) \cap Z(f_2) = \phi$. Let $f = f_1^2 + f_2^2 \cdot f \in \mathcal{R}$ and $Z(f) = \phi$. Since $\mathcal{A}$ is inverse closed $g/f \in \mathcal{A}$ for each $g \in \mathcal{A}$. Therefore $g = f\, g/f \in \mathcal{B}$ for each $g \in \mathcal{A}$. The only nontrivial ideal of $\mathcal{A}$ whose zero sets are disjunctive is $\mathcal{A}$ itself.

If we were to eliminate the hypothesis in 3.9 that $\mathcal{A}$ be inverse closed, then the conclusion of the corollary would not necessarily follow, as is illustrated in this example.

3.10. EXAMPLE. Let $X = (0,1]$, $\mathcal{A} = C^*(0,1]$,

$$\mathcal{B} = \{f \in \mathcal{A} : \lim_{x\to 0} f(x) = 0\}.$$

Then $\mathcal{A}$ is a Wallman ring on X which is not inverse closed, $\mathcal{B}$ is a proper ideal of $\mathcal{A}$ and as we shall now show, $Z(\mathcal{B})$ is disjunctive. Let $f \in \mathcal{B}$ and let $y \in X - Z(f)$. Then for some $a \in X$, $0 < a < y$ and $[a, y] \cap Z(f) = \phi$. Let $H = ([a,1] \cap Z(f)) \cup \{a\}$. Then $y \notin H$ and H is closed in $[a,1]$. So, there exists a function $g \in C([a,1])$ so that $g(y) = 0$ and $g(H) = 1$. Observe that $g(a) = 1$. Extended g to a function $h \in \mathcal{B}$

by defining $h(x) = x/a$ for each $x \in (0, a)$. Clearly $y \in Z(h)$ and $Z(f) \cap Z(h) = \phi$ so that $Z(\mathscr{B})$ is disjunctive.

We now divert our attention to the topology of a space with relation to the zero sets of a Wallman ring on the space and observe that the following properties hold.

3.11. THEOREM. *If $\mathscr{A}$ is a Wallman ring on X, then every neighborhood of a point $x \in X$ contains a $Z[\mathscr{A}]$ — neighborhood of x.*

Proof. Let A be a neighborhood of x, $x \in X$. $\{X - Z(f): f \in \mathscr{A}\}$ is a base for the open sets of X. So for some $f \in \mathscr{A}$, $x \in (X - Z(f)) \subseteq A$. Since $Z[\mathscr{A}]$ is disjunctive, there is a function $g \in \mathscr{A}$ such that $g(x) = 0$ and $Z(g) \cap Z(f) = \phi$. By the normality of $Z[\mathscr{A}]$, there are functions h and $k \in \mathscr{A}$ such that $Z(f) \cap Z(k) = \phi$, $Z(g) \cap Z(h) = \phi$, and $Z(h) \cup Z(k) = X$. This yields $x \in Z(g) \subseteq X - Z(h) \subseteq X - Z(f) \subseteq A$. So $Z(k)$ is a neighborhood of x and $Z(k) \subseteq A$.

3.12. COROLLARY. *If $\mathscr{A}$ is a Wallman ring on X, then the weak topology generated by $\mathscr{A}$ is the given topology on X.*

In the case of the Wallman ring $C^*(X)$ it is known that its Wallman compactification βX is connected if and only if $C^*(X)$ cannot be expressed as the direct sum of nontrivial ideals. With slight modifications, this theorem can be generalized to arbitrary Wallman rings.

3.13. THEOREM. *If $\mathscr{A}$ is a Wallman ring such that $\mathscr{A} = \mathscr{B} \oplus \mathscr{C}$ where $\mathscr{B}$ and $\mathscr{C}$ are proper ideals of $\mathscr{A}$ and if*

$$\mathscr{A}' = \left\{\frac{f}{g}: f, g \in \mathscr{A}, \quad Z(g) = \phi\right\}$$

$$\mathscr{B}' = \left\{\frac{f}{g}: f \in \mathscr{B}, \quad g \in \mathscr{A}, \quad Z(g) = \phi\right\}$$

$$\mathscr{C}' = \left\{\frac{f}{g}: f \in \mathscr{C}, \ g \in \mathscr{A}, \ Z(g) = \phi\right\}$$

then $\mathscr{B}'$ and $\mathscr{C}'$ are proper ideals of $\mathscr{A}$ and $\mathscr{A}' = \mathscr{B}' \oplus \mathscr{C}'$.

3.14. THEOREM. *If $\mathscr{A}$ is a Wallman ring such that $\mathscr{A} = \mathscr{B} \oplus \mathscr{C}$ where $\mathscr{B}$ and $\mathscr{C}$ are proper ideals of $\mathscr{A}$, then $w(Z[\mathscr{A}]$ is disconnected.*

Proof. Define $\mathscr{A}'$, $\mathscr{B}'$ and $\mathscr{C}'$ as in the previous theorem. Then $1 \in \mathscr{A}'$ implies there are function $f \in \mathscr{B}'$ and $g \in \mathscr{C}'$ such that $1 = f + g$

and $fg = 0$. Since $\mathscr{B}'$ and $\mathscr{C}'$ are proper ideals, they contain no functions whose zero sets are empty. In particular $Z(f) \neq \phi$ and $Z(g) \neq \phi$.

$$\begin{aligned} \text{Now } Z(f) \cap Z(g) &= Z(f^2 + g^2) \\ &= Z(f^2 + 2fg + g^2) \\ &= Z(1) \\ &= \phi \end{aligned}$$

and $Z(f) \cup Z(g) = Z(fg) = X$.

Therefore $\mathrm{Cl}_{w(Z[\mathscr{A}'])} Z(f) \cap \mathrm{Cl}_{w(Z[\mathscr{A}'])} Z(g) = \phi$ and

$$\mathrm{Cl}_{w(Z[\mathscr{A}'])} Z(f) \cup \mathrm{Cl}_{w(Z[\mathscr{A}'])} Z(g) = w(Z[\mathscr{A}'])$$

where $\mathrm{Cl}_{w(Z[\mathscr{A}'])} Z(f)$ and $\mathrm{Cl}_{w(Z[\mathscr{A}'])} Z(g)$ are nonempty closed sets in $w(Z[\mathscr{A}'])$. This means $w(Z[\mathscr{A}'])$ is disconnected. Therefore $w(Z[\mathscr{A}])$ is disconnected, since it is homeomorphic to $W(Z[\mathscr{A}'])$.

Whether or not the converse of this theorem is valid is an open question. We do however have a partial converse.

3.15. Definition. If $\mathscr{A} \subseteq C(X)$, then $\mathscr{A}$ is sectionally replete if it satisfies the following condition:

$$\text{If} \qquad X = \bigcup_{i=1}^{n} Y_i,$$

where $Y_i \in Z[\mathscr{A}]$ for $i = 1, \cdots, n$ and if there are functions $g_i \in \mathscr{A}$ such that $f \mid Y_i = g_i \mid Y_i$, for $i = 1, \cdots, n$, then $f \in \mathscr{A}$.

3.16. Theorem. *If $\mathscr{A}$ is a sectionally replete Wallman ring and $w(Z[\mathscr{A}])$ is disconnected, then there are proper ideals $\mathscr{B}$ and $\mathscr{C}$ of $\mathscr{A}$ such that $\mathscr{A} = \mathscr{B} \oplus \mathscr{C}$.*

Proof. Since $w(Z[\mathscr{A}])$ is disconnected, there exist nonempty, disjoint closed subsets H and L of $w(Z[\mathscr{A}])$ such that $H \cup L = w(Z[\mathscr{A}])$. Therefore, there exist sets B, $C \in Z[\mathscr{A}]$ such that $H \subseteq \mathrm{Cl}_{w(Z[\mathscr{A}])} B$, $L \subseteq \mathrm{Cl}_{w(Z[\mathscr{A}])} C$ and $B \cap C = \phi$ (Bentley **[4]**). Let $\mathscr{B} = \{f \in \mathscr{A}: f = 0 \text{ on } B\}$ and $\mathscr{C} = \{f \in \mathscr{A}: f = 0 \text{ on } C\}$. $\mathscr{B}$ and $\mathscr{C}$ are ideals of $\mathscr{A}$ whose intersection is the zero ideal.

For each $f \in \mathscr{A}$ let

$$f_1(x) = \begin{cases} 0 & \text{if } \; x \in B \\ f(x) & \text{if } \; x \in C \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x) & \text{if } \; x \in B \\ 0 & \text{if } \; x \in C. \end{cases}$$

Since $\mathscr{A}$ is sectionally replete f_1 and $f_2 \in \mathscr{A}$. $f = f_1 + f_2$, $f_1 \in \mathscr{B}$, and $f_2 \in \mathscr{C}$. Therefore $\mathscr{A} = \mathscr{B} \oplus \mathscr{C}$.

The final topic in this section is the sublattice problem. The question of when a sublattice of $C(X)$ is a Wallman ring was answered by Biles **[8]**. Any sublattice of $C(X)$ whose zero sets form a base for the closed sets of X is a Wallman ring. It is well-known that every closed subring of $C(X)$ is a closed sublattice of $C(X)$. So every uniformly closed Wallman ring is a sublattice of $C(X)$.

A question related to this, namely, "Given an arbitrary Wallman ring, is it possible to construct an equivalent Wallman ring which is a sublattice of $C(X)$?" does not appear to be easily answered.

Along this line we do have the following results.

3.17. THEOREM. *Let $\mathscr{A}$ be a Wallman ring on X, let $\mathscr{B}$ be the ring generated by $\{|f|: f \in \mathscr{A}\}$. Then $\mathscr{B}$ is a Wallman ring on X and $\mathscr{A} \cong \mathscr{B}$.*

Proof. $Z(|f| \pm |g|) = Z(f^2 \pm g^2)$ and $Z(|f|\ |g|) = Z(fg)$ so $Z[\mathscr{A}] = Z[\mathscr{B}]$.

Clearly $\mathscr{B}$ is not necessarily a sublattice of $C(X)$. With further hypotheses we can get a little closer to a sublattice of $C(X)$.

3.18. THEOREM. *Let $\mathscr{A}$ be a Wallman ring over X, let $1 \in \mathscr{A}$, and let $Z(f \wedge 0) \in Z[\mathscr{A}]$ for each $f \in \mathscr{A}$. Then*

$$\mathscr{B} = \left\{\sum_{i=1}^{n} f_i |g_i|: \quad f_i, g_i \in \mathscr{A}, \quad n \in \mathscr{N}\right\}$$

is a Wallman ring on X, $\mathscr{A} \cong \mathscr{B}$, and $\mathscr{B}$ contains the supremum and infimum of any two functions from $\mathscr{A}$.

Proof. (1) $Z[\mathscr{A}] \subseteq Z[\mathscr{B}]$ since $\mathscr{A} \subseteq \mathscr{B}$.

(2) $Z[\mathscr{B}] \subseteq Z[\mathscr{A}]$ since

$$Z\left(\sum_{i=1}^{n} f_i |g_i|\right) = \cup \{P_\alpha : \alpha \subseteq \{1, \cdots, n\}\}$$

where

$$S_\alpha = \bigcap_{i \in \alpha} Z(g_i \wedge 0), \qquad T_\alpha = \bigcap_{i \notin \alpha} Z((-g_i) \wedge 0),$$

$$V_\alpha = Z\left(\sum_{i \in \alpha} f_i g_i - \sum_{i \notin \alpha} f_i g_i\right)$$

and

$$P_\alpha = S_\alpha \cap T_\alpha \cap V_\alpha.$$

(3) $\mathcal{B}$ is a subring of $C(X)$.
Therefore $\mathcal{B}$ is a Wallman ring on X and $\mathcal{B} \cong \mathcal{A}$.
(4) If $f, g \in \mathcal{A}$, then

$$f \wedge g = g + [(f-g) \wedge 0] = \frac{f+g-|f-g|}{2}$$

and

$$f \vee g = g + [(f-g) \vee 0] = \frac{f+g+|f+g|}{2}.$$

So $f \wedge g$ and $f \vee g \in \mathcal{B}$.

The following example illustrates that in trying to find a sublattice of $C(X)$ which is a Wallman ring equivalent to a given Wallman ring we cannot in general look at the sublattice generated by our given Wallman ring.

3.19. Example. Let $Y = [-1, 1]$, $X = Y - \{0\}$, $\mathcal{M} = \{f \in C(X)$: for some compact set $H \subseteq X$, f is constant on $X - H\}$, $j: X \to R$ be the inclusion map, $\mathcal{A}$ be the subring of $C(X)$ generated by $\mathcal{M} \cup \{j\}$, and $\mathcal{B}$ be the sublattice generated by $\mathcal{A}$. Then (1) $Z[\mathcal{A}] = Z[\mathcal{M}]$ so $\mathcal{A}$ is a Wallman ring and (2) $Z[\mathcal{B}] \not\leq Z[\mathcal{A}]$.

Proof. (1) $\mathcal{A} = \{\Sigma_{n=0}^m g_n j^n : g_n \in \mathcal{M}, m \in \mathcal{N}\}$, and $\mathcal{M} \subseteq \mathcal{A}$ so $Z[\mathcal{M}] \subseteq Z[\mathcal{A}]$.

To show $Z[\mathcal{A}] \subseteq Z[\mathcal{M}]$ let $f \in \mathcal{A}$, $f = \Sigma_{n=0}^m g_n j^n$ where $g_n \in \mathcal{M}$. If $m = 0$, then $f = g_0 \in \mathcal{M}$ and we are through. Therefore suppose $m > 0$.

For $n = 0, \cdots, m$, there exist compact subsets H_n of X such that g_n is constant on $X - H_n$.

Let $t > 0$ such that $[-t, t] \subseteq Y - \cup_{n=0}^m H_n$. Let $a_n = g_n(t)$ for $n = 0, \cdots, m$. Then $a_n = g_n(x)$ for $x \in [-t, t] \cap X$, $n = 0, \cdots, m$. For $x \in [-t, t] \cap X$, $f(x) = \Sigma_{n=0}^m g_n(x) j^n(x) = \Sigma_{n=0}^m a_n x^n$ which is a polynomial over $[-t, t] \cap X$. So f has finitely many zeros in $[-t, t] \cap X$. $Z(f) = [Z(f) \cap ([-1, -t] \cup [t, 1]) \cup [Z(f) \cap [-t, t]]$ so $Z(f)$ is compact. Therefore $Z(f) \in Z[\mathcal{M}]$.

(2) Suppose $Z[\mathcal{B}] \leq Z[\mathcal{A}] = Z[\mathcal{M}]$. Let

$$j^+ = j \vee 0, \quad \text{and} \quad j^- = (-j)^+.$$

Then j^+ and $j^- \in \mathscr{B}$, and $Z(j^+) \cap Z(j^-) = \phi$. Now since $Z[\mathscr{B}] \leq Z[\mathscr{M}]$ there exist functions $f, g \in \mathscr{M}$ such that $Z(j^+) \subseteq Z(f)$, $Z(j^-) \subseteq Z(g)$, and $Z(f) \cap Z(g) = \phi$. But $Z(j^+) = [-1, 0)$ and $Z(j^-) = (0, 1]$. So $\mathrm{Cl}_Y[-1, 0) \cap \mathrm{Cl}_Y(0, 1] \subseteq \mathrm{Cl}_Y Z(f) \cap \mathrm{CL}_Y Z(g) = \phi$ since $Y \cong w(Z[\mathscr{M}])$. But $\mathrm{Cl}_Y[-1, 0) \cap \mathrm{Cl}_Y(0, 1] = \{0\}$ so we have a contradiction and it follows that $Z[\mathscr{B}] \not\leq Z[\mathscr{A}]$.

Henriksen and Isbell **[16]** showed that $\mathscr{B}$ is a ring but we do not know if it is a Wallman ring.

This example shows us that there is a Wallman ring $\mathscr{A}$ such that if $\mathscr{C}$ is a Wallman ring on X, $\mathscr{C}$ is a sublattice of $C(X)$ and $\mathscr{A} \subseteq \mathscr{C}$ then $\mathscr{A}$ and $\mathscr{C}$ are not equivalent, since any such Wallman ring would contain $\mathscr{B}$.

Example 3.19 eliminates the obvious procedure for attacking another problem. Given an arbitrary Wallman ring we have shown there is an equivalent Wallman ring which contains all the rational constants. If we could state that the uniform closure of a Wallman ring is a Wallman ring equivalent to the original, we would have shown that every Wallman ring is equivalent to one which contains all the real constants. However, a uniformly closed subring of $C(X)$ is a sublattice of $C(X)$ and by example 3.19 there is a Wallman ring which is not contained in any equivalent Wallman ring which is a sublattice of $C(X)$. This means that in general the answer to getting the real constants in a Wallman ring does not lie in taking uniform closures. For now, we cannot answer the question, "For an arbitrary Wallman ring, is there an equivalent Wallman ring which contains all the real valued constant functions?"

IV. Examples of Wallman rings on locally compact spaces. In this section, we present a method for constructing examples of Wallman Rings on locally compact spaces. These Wallman rings are determined by compactifications of the space which can be mapped by a certain kind of retract map onto the remainder. H. L. Bentley **[6]** has shown that these compactifications are Wallman; we show how they arise from Wallman rings.

Throughout this section, X is assumed to be a locally compact space.

4.1. DEFINITION. A closed subset L of X is called co-compact provided $\mathrm{Cl}_Y(X - L)$ is compact.

4.2. DEFINITION. If $\mathscr{H}$ is a family of closed subsets of X, then the *compact modification* of $\mathscr{H}$ is the family

$$CM(\mathscr{H}) = \{(H \cap L) \cup B : H \in \mathscr{H}, L \text{ is co-compact, and } B \text{ is compact}\}.$$

4.3. DEFINITION. (Borsuk **[10]**.) A continuous map $f\colon Y \to K$ is called a *retract map* provided K is a subspace of Y and $f(x) = x$ for each $x \in K$.

4.4. DEFINITION. (Bentley **[6]**.) If Y is a compactification of X, $K = Y - X$, and $f\colon Y \to K$ is a continuous map, then *f maps onto K at* ∞ provided $f[L] = K$ for each co-compact subset L of X.

4.5. THEOREM. (Bentley **[6]**.) *Let Y be a compactification of the locally compact space X, let* $K = Y - X$, *and let* $f\colon Y \to K$ *be a retract map which maps onto K at* ∞. *Let* $\mathcal{H} = \{X \cap f^{-1}(E)\colon E$ *is a closed subset of* $K\}$. *Then* $CM(\mathcal{H})$ *is a normal base on X and* $w(CM(\mathcal{H})) \cong Y$.

Our objective is to exhibit a Wallman ring $\mathcal{A}$ on X for which $w(Z[\mathcal{A}]) \cong Y$, with Y as in the preceding theorem.

4.6. THEOREM. *Let Y be a compactification of the locally compact space X, let* $K = Y - X$ *and let* $f\colon Y \to K$ *be a retract map which maps onto K at* ∞. *Let* $\mathcal{B}$ *be the set of all* $h \in C(Y)$ *for which there exists a co-compact set* $L \subseteq X$ *such that for all* $z \in K$, $h[L \cap f^{-1}(\{z\})] = \{h(z)\}$. *Let* $\mathcal{A}$ *be the set of all restrictions* $h\,|X$ *with* $h \in \mathcal{B}$. *Then* $\mathcal{A}$ *is a Wallman ring on X and* $w(Z[\mathcal{A}]) \cong Y$.

The proof of this theorem will depend on the following lemmas. Let Y, K, f, $\mathcal{B}$ and $\mathcal{A}$ be as in Theorem 4.6 and let $\mathcal{H}$ be as in Theorem 4.5.

4.7. LEMMA. $Z[\mathcal{A}] \leqq CM(\mathcal{H})$

Proof. Let $h \in \mathcal{A}$, then h has an extension $h' \in C(Y)$ such that for some co-compact set L in X

$$h'[L \cap f^{-1}(\{z\})] = h'(z) \quad \text{for} \quad \text{all} \quad z \in K.$$

So
$$\begin{aligned} Z(h) &= Z(h') \cap X. \\ &= [Z(h') \cap \mathrm{Cl}_X(X - L)] \cup [Z(h') \cap L]. \\ &= [Z(h') \cap \mathrm{Cl}_X(X - L)] \cup [f^{-1}(Z(h') \cap K) \cap L]. \\ &= [Z(h') \cap \mathrm{Cl}_X(X - L)] \cup [(f^{-1}(Z(h') \cap K) \cap X) \cap L]. \end{aligned}$$

Now $Z(h') \cap \mathrm{Cl}_X(X - L)$ is compact, L is co-compact, and $Z(h') \cap K$ is a closed subset of K, so $Z(h) \in CM(\mathcal{H})$ and $Z[\mathcal{A}] \leqq CM(\mathcal{H})$.

4.8. LEMMA. $CM(\mathcal{H}) \leqq Z[\mathcal{A}]$.

Proof. Let $F_1, F_2 \in CM(\mathcal{H})$ such that $F_1 \cap F_2 = \phi$. There are sets L_1 and L_2 which are closed and co-compact in X, subsets B_1 and B_2 of X which are compact, and subsets E_1 and E_2 of K which are compact, such that $F_i = (L_i \cap f^{-1}(E_i)) \cup B_i$, for $i = 1,2$. Also $F_1 \cap F_2 = \phi$ implies $[L_1 \cap f^{-1}(E_1)] \cap [L_2 \cap f^{-1}(E_2)] = \phi$. So have $L_1 \cap L_2 \cap f^{-1}(E_1 \cap E_2) = \phi$. Since f maps onto K at ∞, and $L_1 \cap L_2$ is co-compact, we conclude $E_1 \cap E_2 = \phi$.

K is completely regular so there exists $u_1, u_2 \in C^*(K)$, u_1: $K \to [0,1]$, u_2: $K \to [0,1]$, such that $E_1 \subseteq Z(u_1)$, $E_2 \subseteq Z(u_2)$, and $Z(u_1) \cap Z(u_2) = \phi$. Let $f'_1 = u_1 \circ f$, and $f'_2 = u_2 \circ f$. Then $f'_1, f'_2 \in C(Y)$, f'_1: $Y \to [0,1]$, f'_2: $Y \to [0,1]$.

If $x \in L_i \cap f^{-1}(E_i)$, then $f(x) \in E_i$; and since $u_i = 0$ on E_i, it follows that $x \in Z(f'_i$ for $i = 1,2$. Also $Z(f'_i) \cap Z(f'_2) = \phi$.

Now $\{[L_1 \cap f^{-1}(E_1)] \cup B_1\} \cap \{[L_2 \cap f^{-1}(E_2)] \cup B_2\} = \phi$ and these two sets are elements of a normal base for Y so

$$\mathrm{Cl}_Y\{[L_1 \cap f^{-1}(E_1)] \cup B_1\} \cap \mathrm{Cl}_Y\{[L_2 \cap f^{-1}(E_2)] \cup B_2\} = \phi.$$

Therefore there exist closed sets G_1 and G_2 in Y such that $\mathrm{Cl}_Y\{[L_i \cap f^{-1}(E_i)] \cup B_i\} \subseteq \mathrm{Int}_Y G_i$, for $i = 1,2$. B_1 and B_2 are disjoint compact sets in X, so there are disjoint compact sets C_1 and C_2 in X such that $B_i \subseteq \mathrm{Int}_X C_i$, $i = 1,2$. Then $C_i \cap G_i$ is a compact set in X which contains B_i in its interior and is disjoint from $[C_j \cap G_j] \cup [L_j \cap f^{-1}(E_j)]$ for $i \neq j$; $i = 1,2$; $j = 1,2$.

Now define $h_1 = 0$ on B_1, $h_1 = 1$ on $\mathrm{Cl}_Y(Y - (C_1 \cap G_1))$. These are disjoint closed sets in Y so we can take h_1 to be a continuous function on Y, h_1: $Y \to [0,1]$. Similarly define h_2: $Y \to [0,1]$, $h_2 = 0$ on B_2, $h_2 = 1$ on $\mathrm{Cl}_Y(Y - (C_2 \cap G_2))$. So $Z(h_1) \cap Z(h_2) = \phi$.

Now we have no assurance that $Z(f'_2) \cap Z(h_1) = \phi$ or that $Z(f'_1) \cap Z(h_2) = \phi$ so we modify f'_1 and f'_2.

$f^{-1}(Z(u_1)) \cap (C_2 \cap G_2)$ is a compact set in X disjoint from $f^{-1}(E_1) \cap L_1$, a closed set in X. Therefore since X is locally compact there exists a compact subset D_1 of X such that

$$f^{-1}(Z(u_1)) \cap C_2 \cap G_2 \subseteq \mathrm{Int}_X D_1 \subseteq D_1 \subseteq X - (f^{-1}(E_1) \cap L_1).$$

Similarly there exists a compact subset D_2 of X such that

$$f^{-1}(Z(u_2)) \cap C_1 \cap G_1 \subseteq \mathrm{Int}_X D_2 \subseteq D_2 \subseteq X - (f^{-1}(E_2) \cap L_2).$$

Since $f^{-1}[Z(u_1)] \cap C_2 \cap G_2$ and $\mathrm{Cl}_Y(Y - D_1)$ are disjoint closed subsets of Y and since Y is compact, hence normal, there exists a continuous function $g_1\colon Y \to [0,1]$ such that g_1 is 1 on $f^{-1}[Z(u_1) \cap C_2 \cap G_2]$ and g_1 is 0 on $\mathrm{Cl}_Y(Y - D_1)$. Similarly define g_2.

Now let $f_1 = f_1' \vee g_1$ and $f_2 = f_2' \vee g_2$. Then $f_1, f_2 \in C(Y)$, $f_1\colon Y \to [0,1]$, $f_2\colon Y \to [0,1]$, $Z(f_1) \cap Z(f_2) = \phi$, $Z(f_1) \cap Z(h_2) = \phi$ and $Z(f_2) \cap Z(h_1) = \phi$.

Finally let $\alpha_1 = f_1 \wedge h_1$ and $\alpha_2 = f_2 \wedge h_2$. $F_1 \subseteq Z(\alpha_1)$ since

$$f^{-1}(E_1) \cap L \subseteq Z(f_1') \cap Z(g_1) = Z(f_1) \subseteq Z(\alpha_1)$$

and $B_1 \subseteq Z(h_1) \subseteq Z(\alpha_1)$. Similarly $F_2 \subseteq Z(\alpha_2)$. $Z(\alpha_1)$ and $Z(\alpha_2)$ are disjoint since:

$$\begin{aligned} Z(\alpha_1) \cap Z(\alpha_2) &= (Z(f_1) \cup Z(h_1)) \cap (Z(f_2) \cup Z(h_2)) \\ &= (Z(f_1) \cap Z(f_2)) \cup (Z(f_1) \cap Z(h_2)) \cup (Z(h_1) \cap Z(f_2)) \\ &\quad \times \cup (Z(h_1) \cap Z(h_2)) \\ &= \phi \end{aligned}$$

$\alpha_1 \in \mathscr{B}$ since $\alpha_1 = f_1'$ on $\mathrm{Cl}_Y L_1 \cap \mathrm{Cl}_Y(Y - D_1) \cap \mathrm{Cl}_Y(Y - (C_1 \cap G_1))$, a set whose intersection with X is co-compact. Similarly $\alpha_2 \in \mathscr{B}$.

Therefore we have functions $\alpha_1|X$ and $\alpha_2|X$ whose zero sets separate F_1 and F_2, consequently $CM(\mathscr{H}) \leqq Z[\mathscr{A}]$.

4.9. LEMMA. *$Z[\mathscr{A}]$ is a lattice of closed subsets of X.*

Proof. If $f, g \in \mathscr{B}$, and L_f and L_g are the co-compact sets associated with f and g in the definition of $\mathscr{B}$ then $f - g$ and fg satisfy the condition for being elements of $\mathscr{B}$ on the co-compact set $L_f \cap L_g$. Consequently $\mathscr{B}$ is a subring of $C(Y)$ so $Z(\mathscr{B})$ is a lattice of closed subsets of Y and hence $Z(\mathscr{A})$ is a lattice of closed subsets of X.

We are now in a position to prove Theorem 4.6.

Proof. By Lemmas 4.7 and 4.8 $Z[\mathscr{A}] \cong CM(\mathscr{H})$ and by Lemma 4.9, $Z[\mathscr{A}]$ is a lattice of closed subsets of X. Therefore by Theorem 2.12 $Z[\mathscr{A}]$ is a normal base on X and $w(Z[\mathscr{A}]) \cong w(CM(\mathscr{H}))$. But $w(CM(\mathscr{H})) \cong Y$, so we have $w(Z[\mathscr{A}]) \cong Y$.

We note that the theorem is also valid when f is defined just on the closure in Y of some co-compact subset of X.

The following corollary gives a Wallman ring which generates the one point compactification of a locally compact space. This result was earlier observed by Brooks **[11]**.

4.10. COROLLARY. *If X is a locally compact space, then $\mathscr{A} = \{f \in C(X)$: there is a co-compact subset of X on which f is constant$\}$ is a Wallman ring on X and $w(Z[\mathscr{A}])$ is the Alexandroff compactification of X.*

Proof. Let Y be the Alexandroff compactification of X. There can be only one function mapping Y onto $(Y-X)$ since there is just one point in $(Y-X)$. Clearly this function maps onto $(Y-X)$ at ∞. Therefore the hypotheses of Theorem 4.6 are satisfied. By examining the set $\mathscr{B}$ as defined in the theorem, we see $\mathscr{B} = \{h \in C(Y)$: $h(L)$ is constant for some co-compact subset L of $X\}$. Consequently $\mathscr{A}$ is a Wallman ring which generates the Alexandroff compactification of X.

The zero sets of this Wallman ring are precisely those zero sets of X which are either compact or co-compact.

4.11. THEOREM. *If X is a locally compact space and $\mathscr{A} = \{f \in C(X)$: there is a co-compact subset of X on which f is constant$\}$, then $Z[\mathscr{A}] = \{F$: $F \in Z(X)$ and F is either compact or co-compact$\}$.*

Proof. Let $f \in \mathscr{A}$, then $Z(f) \in Z(X)$. Let F be the co-compact set on which f is constant. Then $Z(f) \cap F = \phi$ or $F \subseteq Z(f)$. If $Z(f) \cap F = \phi$, then $Z(f) \subseteq \mathrm{Cl}_X(X-F)$ which implies $Z(f)$ is compact. If $F \subseteq Z(f)$, then $\mathrm{Cl}_X(X - Z(f)) \subseteq \mathrm{Cl}_X(X-F)$ which implies that $\mathrm{Cl}_X(X - Z(f))$ is compact and $Z(f)$ is co-compact.

Now let $Z(f)$ be a zero set of X which is either compact or co-compact, $f \in C(X)$. If $Z(f)$ is compact, then let $f' = 1 \wedge |f|$ Then $Z(f) = Z(f')$. Since $Z(f)$ is compact and X is locally compact there is a compact subset W of X such that $Z(f) \subseteq \mathrm{Int}\, W \subseteq W \subseteq X$. Also there is a function $g \in C(X)$ such that $g[Z(f)] = \{0\}$, $g[X - \mathrm{Int}\, W] = \{1\}$, $g: X \to [0,1]$. Define $h = f' \vee g$. Then $Z(h) = Z(f)$ and h is constant on $(X - \mathrm{Int}\, W)$ which is a co-compact subset of X. Therefore $Z(f) \in Z[\mathscr{A}]$.

If $Z(f)$ is co-compact, then f is constant on the co-compact set $Z(f)$, so $f \in \mathscr{A}$, and $Z(f) \in Z[\mathscr{A}]$.

We will now define a Wallman ring $\mathscr{A}$, on the open unit disc such that $w(Z[\mathscr{A}])$ is the closed unit disc.

4.12. EXAMPLE. Let X be the open unit disc in Euclidean 2-space and let Y be the closed unit disc.

We will consider the elements of Y to be complex numbers. Let $K = \{z \in Y: |z| = 1\}$, let $J = \{z \in Y: |z| \geqq \frac{1}{2}\}$ and define a function $f: J \to K$ by

$$f(z) = \frac{z}{|z|} \quad \text{for} \quad \text{eaach} \quad z \in J.$$

Let $\mathscr{B}$ be the set of all functions $h \in C(Y)$ for which there exists a co-compact subset L of X such that for all $z \in L \cap J$, $h(z) = h(z/|z|)$. Note that $\mathscr{B}$ can equivalently be described as the set of all functions $h \in C(Y)$ for which there exists $t \in [\frac{1}{2}, 1]$ such that for all z with $t \leqq |z| \leqq 1$, $h(z) = h(z/|z|)$.

As was noted after the proof of Theorem 4.6, that theorem is valid if the retract map is defined only on the closure in Y of some co-compact subset of X. In the present situation, J is such a closure. Clearly, f maps $J \cap X$ onto K at ∞. Therefore if we let $\mathscr{A} = \{h | X: h \in \mathscr{B}\}$, then $\mathscr{A}$ is a Wallman ring on X which induces a compactification equivalent to the closed disc.

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References

1. J. M. Aarts, *Every Metric Compactification is a Wallman Type Compactification*, Proceedings of the International Symposium on Topology and its Applications, (1968), 29–34.

2. R. A. Alo, and H. L. Shapiro, *Normal bases and compactifications*, Math. Annalen, **175** (1968), 337–340.

3. B. Banaschewski, *On Wallman's method of compactification*, Math. Nachr., **27** (1963), 105–114.

4. H. L. Bentley, *Normal Bases and Compactifications*, Proceedings of the Topology Conference, Oklahoma (1972).

5. ———, *Some Wallman compactifications determined by retracts*, Trans. Amer. Math. Soc., **33** (1972), 587–593.

6. ———, *Some Wallman compactifications of locally compact spaces*, Fund. Math., **75** (1972), 13–24.

7. E. S. Berney, *On Wallman compactifications*, Notices Amer. Math. Soc., **17** (1970), 215 A-672-462.

8. C. M. Biles, *Gelfand and Wallman-Type compactifications*, Pacific J. Math., **35** (1970), 267–278.

9. ———, *Wallman-type compactifications*, Proc. Amer. Math. Soc., **25** (1970), 363–368.

10. K. Borsuk, *Theory of Retracts*, Polish Scientific Publishers, Warsaw, 1967.

11. R. M. Brooks, *On Wallman compactifications*, Fund. Math., **60** (1967), 157–173.

12. Orrin Frink, *Compactifications and semi-normal spaces*, Amer. J. Math., **86** (1964), 602–607.

13. L. Gillman, and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, 1960.

14. A. W. Hager, *On inverse-closed subalgebras of $C(X)$*, Proc. London Math. Soc., (3) **19** (1969), 233–257.

15. A. W. Hager, and D. G. Johnson, *A note on certain subalgebras of $C(X)$*, Canadian J. Math., **20** (1968), 389–393.

16. N. Henrickesen, and J. Isbell, *Lattice ordered rings and function rings*, Pacific J. Math., **12** (1962), 533–565.

17. Olav Njastad, *On Wallman-type compactifications*, Math. Zeitzchr., **91** (1966), 267–276.

18. N. A. Sanin, *On special extensions of topological spaces*, Dokl, Akad. Nauk S.S.S.R., **38** (1943), 6–9.

19. E. F. Steiner, *Wallman spaces and compactifications*, Fund. Math., **61** (1968), 295–304.

20. A. K. Steiner, and E. F. Steiner, *Products of compact metric spaces are regular Wallman*, Indag. Math., **30** (1968), 428–430.

21. ———, *Wallman and Z-compactifications*, Duke Math. J., **35** (1968), 269–276.

22. B. J. Taylor, and H. L. Bentley, *The Stone-Weierstrass Theorem for Wallman Rings*, preprint.

23. H. Wallman, *Lattices and topological spaces*, Ann. Math., **39** (1938), 112–126.

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MATRIX RINGS OVER POLYNOMIAL IDENTITY RINGS II

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If A is a ring satisfying a polynomial identity, what identity is satisfied by the matrix ring A_n? Theorem: If A satisfies the standard identity of degree k, then A_n satisfies the standard identity of degree $2kn^2 - n^2 + 1$.

Definition: Suppose that $\{r_1, \cdots, r_q\}$ is a sequence of elements of a ring. *To parenthesize the sequence into j clumps* is to insert j pairs of adjacent, nonoverlapping parentheses. The subsequence within one pair of parentheses constitutes a *clump.* It is odd or even, depending on the number of entries. The *value* of the clump is the product of the entries. If the value is zero, the clump *vanishes.*

In the following let Z represent the integers.

LEMMA 1. *Let k, m, and n be positive integers. Let $\{u_1, \cdots, u_m\}$ be a nonvanishing sequence of matrix units e_{ij} in Z_n.*

(i) *If $m = kn$, there exists i such that the sequnce can be parenthesized into k clumps, each of value e_{ii}.*

(ii) *If $m = (kn - n + 1)n$, there exist i and j such that the sequence can be parenthesized into k clumps, each of value e_{ii}, and each beginning with e_{ij}.*

Proof of (i). Case 1. Suppose there exists i such that at least $k + 1$ of the entries in the sequence have i as initial subscript. Call the first $k + 1$ such entries $y_1, y_2, \cdots, y_{k+1}$. Then parenthesize the sequence as follows: start with y_1. Enclose it in parentheses, together with all entries to the right, if any, up to y_2. Next parenthesize y_2 with all entries up to y_3, etc. We form k clumps, each beginning with a y. Since each clump has to the right an entry with i as initial subscript, and the sequence is nonvanishing, each clump has value e_{ii}.

Case 2. Suppose that for all i, at most k of the entries have i as initial subscript. Since the sequence has kn entries, every i from l through n occurs exactly k times as an initial subscript.

Case 2a. The last entry is an idempotent e_{ii}. There are previous entries $y_1, \cdots, y_{k-1}$, each with i as initial subscript. Start with y_1 and enclose it in parentheses with all entries to the right, up to y_2. Continue, forming $k-1$ clumps, each of value e_{ii}. Then form a final clump consisting of the single e_{ii} at the end.

Case 2b. The last entry is e_{ij}, with $i \neq j$. Then there are k previous entries $y_1, \cdots, y_k$ with j as initial subscript. Parenthesize, forming $k-1$ clumps, beginning with $y_1, y_2, \cdots, y_{k-1}$, respectively. Then form a final clump, beginning with y_k and ending with the last e_{ij}. The result is k clumps, wach of value e_{jj}.

Proof of (ii). Let $m = (kn - n + 1)n$. Let $\{u_1, \cdots, u_m\}$ be a nonvanishing sequence of matrix units. Let $t = kn - n + 1$. By (i) there exists i such that the sequence can be parenthesized into t clumps, each of value e_{ii}. Let $y_1, \cdots, y_t$ be the first entries in these clumps. Each y has i as initial subscript. The second subscript can be any integer from 1 through n. Now

$$t = kn - n + 1 = (k-1)n + 1.$$

Thus for some j, at least k of the y's have j as second subscript. Suppose that $y_{f(1)}, \cdots, y_{f(k)}$ are all e_{ij}. Make new clumps as follows: start with $y_{f(1)}$ and enclose it in parentheses together with all entries to the right, up to $y_{f(2)}$. Continue, forming $k-1$ clumps. In the old parenthesizing $y_{f(k)}$ was the initial entry in a clump of value e_{ii}. Let this old clump be the kth clump in the new parenthesizing. The result is k clumps, each of value e_{ii}, and each beginning with e_{ij}.

Theorem 3.2 of **[2]** established that if A is an algebra satisfying a standard identity, so is A_n. The following theorem improves this result in three ways: (1) the degree of the identity satisfied by A_n is much lower. (2) The theorem holds for rings, not just algebras over fields. (3) The proof is simpler.

THEOREM 1. *If A is a ring satisfying the standard identity of degree k, then A_n satisfies thestandard identity of degree $2kn^2 - n^2 + 1$.*

Proof. Let

$$t = 2kn^2 - n^2 + 1 = (2k-1)n^2 + 1.$$

Choose t simple tensors in $A \otimes Z_n$ of form $a \otimes e_{ij}$, where $a \in A$, and e_{ij} is a matrix unit. Evaluate on these simple tensors the standard polynomial of degree t. Consider only nonvanishing terms.

Case 1. Suppose that for some i, at least k simple tensors have form

$$a_1 \otimes e_{ii}, \cdots, a_k \otimes e_{ii}.$$

Let $y = e_{ii}$. Call the remaining elements

$$b_1 \otimes z_1, b_2 \otimes z_2, \cdots.$$

Insert parentheses on the right side of each term: start with the first y and enclose it with all z's to the right, if any. Similarly parenthesize the next y with its z's, etc. The last y forms a singleton clump. Thus k clumps are created, each beginning with e_{ii}, and each of value e_{ii}. If there are any z's in the clump, call them the *z sub-clump*. It also has value e_{ii}.

Let V be the number of even clumps, and let D be the number of odd clumps. Then $V + D = k$. Each even clump yields two new odd clumps: the initial y and the z sub-clump. The result is $2V + D$ adjacent odd clumps, each of value e_{ii}. Note that $2V + D \geqq V + D = k$.

In each term find the first set of k adjacent odd clumps of value e_{ii}. Create a corresponding set of clumps on the left side. Call two terms equivalent if the following conditions hold on their left sides:

1. The elements to the left of the clumps are the same elements in the same order.

2. The k clumps are the same, but in any order.

3. The elements to the right of the clumps are the same elements in the same order.

Consider a fixed equivalence class. The sum of the terms in the class is a simple tensor whose right side has the common value for the class. The left side is the product of the following:

1. The product of all elements left of the clumps.

2. The standard polynomial of degree k, evaluated on the values of the k clumps, in some order.

3. The product of all the elements right of the clumps. (Because all these clumps are odd, Corollary to Lemma 4 of **[4]** ensures correctness of signs of terms.) Since the second factor vanishes, the conclusion follows.

Case 2. Suppose that Case 1 does not hold. Since there are $(2k - 1)n^2 + 1$ simple tensors, by Lemma 1 (ii) there exist i and j such that at least $2k$ simple tensors have form $a \otimes e_{ij}$. Evidently, $i \neq j$. Let $w_{ii} = e_{ii} + e_{ij}$. Then w_{ii} is idempotent, and

$$e_{ij} = e_{ii} + e_{ij} - e_{ii} = w_{ii} - e_{ii}.$$

In each term replace e_{ij} by $w_{ii} - e_{ii}$. Let N be the original number of e_{ij}'s. Each old term, upon expansion, yields 2^N new terms. Every new term has on the right a monomial in w's and e's. If there are at least k of the e_{ii}'s in the term, it is suitable for Case 1. Otherwise there are at least k of the w_{ii}'s. In this case, define new elements as follows:

$$w_{jj} = -e_{ij} + e_{jj}$$

$$w_{ji} = -e_{ii} - e_{ij} + e_{ji} + e_{jj}.$$

If $i \neq p \neq j$, let

$$w_{pi} = e_{pi} + e_{pj}$$

$$w_{jp} = -e_{ip} + e_{jp}.$$

For the remaining integers from 1 through n, let $w_{pq} = e_{pq}$.

The w's constitute another set of matrix units in Z_n. Each old matrix unit e_{pq} is a linear combination of the w's with integral coefficients. Replace all the e's by w's. The conclusion follows by the linearity of the standard polynomial and by Case 1.

DEFINITION. The *unitary identity of degree k is*

$$\sum_{\pi} x_{\pi(1)} \cdots x_{\pi(k)} = 0,$$

where the sum is over all permutations π of the integers 1 through k.

THEOREM 2. *If A is a ring satisfying the unitary identity of degree k, then A_n satisfies the unitary identity of degree kn.*

Proof. The proof uses Lemma 1 (i) and is similar to Theorem 1 of **[4]**.

THEOREM 3. *If A is an algebra over a field with at least k elements, and A satisfies $x^k = 0$, then A_n satisfies $x^{kn} = 0$.*

Proof. The proof uses Lemma 1(i) and is similar to Theorem 1.2 of **[3]**. Note: That paper uses without definition the term "homogeneous component" of a polynomial. If $f(x_1, \cdots, x_j)$ is a polynomial, the homogeneous component of degree n_1 in x_1, degree n_2 in x_2, etc., is the sum of all terms with degree n_1 *in* x_1, etc.

REFERENCES

1. S. A. Amitsur and J. Levitzki, *Minimal identities for algebras*, Proc. Amer. Math. Soc., **1** (1950), 449–463.
2. Elizabeth Berman, *Group rings, matrix rings, and polynomial identities*, Trans. Amer. Math. Soc., **172** (1972), 241–248.
3. ———, *Matrix rings over polynomial identity rings*, Trans. Amer. Math. Soc., **172** (1972), 231–239.
4. ———, *Tensor products of polynomial identity algebras*, Trans. Amer. Math. Soc., **156** (1971), 259–271. MR 43 # 278.
5. Nathan Jacobson, *Structure of rings*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R. I., 1964. MR 36 # 5158.
6. Thomas P. Kezlan, *Rings in which certain subsets satisfy polynomial identities*, Trans. Amer. Math. Soc., **125** (1966), 414–421.
7. Uri Leron and Amitai Vapne, *Polynomial identities of related rings*, Israel J. Math., **8** (1970), 127–137.
8. D. S. Passman, *Infinite group rings*, Marcel Dekker Inc., New York, 1971.
9. C. P. Procesi and L. Small, *Endomorphism rings of modules over PI-algebras*, Math. Z., **106** (1968), 178–180.

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RINGS OVER WHICH CERTAIN MODULES ARE INJECTIVE

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This paper is concerned with rings for which all modules in one of the following classes are injective: simple modules, quasi-injective modules, or proper cyclic modules. Such rings are known as *V*-rings, *QI*-rings, and *PCI*-rings, respectively. First, some conditions are developed under which the properties of being a *V*-ring, *QI*-ring, or *PCI*-ring are left-right symmetric. In the next section, it is shown that a semiprime Goldie ring is a *QI*-ring if and only if all singular quasi-injective modules are injective. An example is constructed to show that the class of *QI*-rings is properly contained in the class of noetherian *V*-rings. Also, it is shown that the global homological dimension of a *QI*-ring cannot be any larger than its Krull dimension. In the final section, it is shown that a *V*-ring is noetherian if and only if it has a Krull dimension. Examples are put forward to show that a noetherian *V*-ring may have arbitrary finite Krull dimension.

1. Introduction and definitions. A ring R is said to be a *right V-ring* provided all simple right R-modules are injective. According to Villamayor [**16**, Theorem 2.1], this is equivalent to the condition that every right ideal of R is an intersection of maximal right ideals. In particular, the Jacobson radical of any right V-ring is zero, and consequently all right V-rings are semiprime. For further properties and examples of V-rings, we refer the reader to Boyle [**1**], Cozzens [**5**], Cozzens-Johnson [**6**], Farkas-Snider [**9**], Michler-Villamayor [**16**], and Osofsky [**17**].

We recall that a module A is *quasi-injective* provided every homomorphism from a submodule of A into A extends to an endomorphism of A. According to Johnson-Wong [**13**, Theorem 1.1], this is equivalent to the condition that A be a fully invariant submodule of its injective hull, which we denote by $E(A)$. For example, any *semisimple* module (i.e., a module which is a sum of simple submodules) is quasi-injective. A ring R is a *right QI-ring* provided all quasi-injective right R-modules are injective. Inasmuch as all simple right R-modules are quasi-injective, it follows that R must also be a right V-ring. In addition, since all semisimple right R-modules are quasi-injective and thus injective, we see from Kurshan [**15**, Theorem 2.4] that R is right noetherian. Therefore: every right QI-ring is a right noetherian, right V-ring. For further properties and examples of QI-rings, we refer the reader to Boyle **[1]** and Byrd **[2, 3]**.

The *proper cyclic* right modules over a ring R are those cyclic right R-modules R/I for which $R/I \not\cong R$. We say that R is a *right PCI-ring* provided all proper cyclic right R-modules are injective. If R is not a division ring, then all simple R-modules are proper cyclic, from which we conclude that every right PCI-ring is also a right V-ring. According to Faith [**7**, Theorems 14, 17], a right PCI-ring is either semisimple or else is a simple, right semihereditary, right Ore domain. In view of this result, we shall only consider right *PCI-domains* in this paper.

The reader is assumed to be familiar with the notions of singular and nonsingular modules, as in **[10]**, for example. Also we shall need the notions of Krull dimension and critical modules as developed in **[11]**. We remind the reader that a *uniform* module is one in which the intersection of any two nonzero submodules is nonzero. Finally, we use the notation $\mathrm{soc}(A)$ for the socle of a module, and $\text{r.gl.dim.}(R)$ for the right global dimension of a ring R.

2. Left-right symmetry. In this section we consider conditions under which a right V-ring (QI-ring, PCI-domain) is also a left V-ring (QI-ring, PCI-domain). These results are consequences of the existence of a duality (i.e., a contravariant category equivalence) between the categories of finitely generated singular right modules and finitely generated singular left modules. We note that in general a right V-ring need not be a left V-ring, as shown by an example of Michler and Villamayor [**16**, Remark 4.5]. For QI-rings and PCI-domains, however, the question of left-right symmetry in general remains open.

LEMMA 1. *Let R be a right and left noetherian semiprime ring with classical quotient ring Q, and let $\mathcal{F}_R$ $({}_R\mathcal{F})$ denote the category of all finitely generated singular right* (left) *R-modules. If $(Q/R)_R$ and ${}_R(Q/R)$ are both injective, then there exists a duality*

$$\mathrm{Ext}^1_R(-,R)\colon \mathcal{F}_R \to {}_R\mathcal{F}.$$

Proof. Inasmuch as Q_R is nonsingular and injective, we observe that $\mathrm{Hom}_R(-,(Q/R)_R)$ and $\mathrm{Ext}^1_R(-,R_R)$ are naturally equivalent on $\mathcal{F}_R$. Using this, it follows readily that $\mathrm{Ext}^1_R(-,R_R)$ and $\mathrm{Ext}^1_R(-,{}_RR)$ define contravariant functors $\mathcal{F}_R \to {}_R\mathcal{F}$ and ${}_R\mathcal{F} \to \mathcal{F}_R$ respectively, and all that remains is to show that both compositions of these two functors are naturally equivalent to the appropriate identity functors.

In view of the above remarks, $\mathrm{Ext}^1_R(\mathrm{Ext}^1_R(-, R_R), {}_RR)$ is naturally equivalent to $\mathrm{Hom}_R(\mathrm{Ext}^1_R(-, R_R), {}_R(Q/R))$ on $\mathscr{F}_R$. Since ${}_R(Q/R)$ is injective by hypothesis, it follows from Cartan-Eilenberg [**4,** Proposition 5.3, p. 120] that $\mathrm{Hom}_R(\mathrm{Ext}^1_R(-, R_R), {}_R(Q/R))$ is naturally equivalent to $\mathrm{Tor}^R_1(-, \mathrm{Hom}_R({}_RR, {}_R(Q/R)))$ on $\mathscr{F}_R$. Finally, we see by Sandomierski [**18,** Corollary, p. 119] that $\mathrm{Tor}^R_1(-, Q/R)$ is naturally equivalent to the identity functor on $\mathscr{F}_R$.

PROPOSITION 2. *Let R be a right and left noetherian ring with maximal right quotient ring Q, and assume that R is a right V-ring. If $(Q/R)_R$ and ${}_R(Q/R)$ are both injective, then all singular right and left R-modules are semisimple and injective.*

Proof. Since R is a semiprime ring, Q is the classical quotient ring of R. We thus obtain the duality between $\mathscr{F}_R$ and ${}_R\mathscr{F}$ as in Lemma 1. Now $\mathscr{F}_R$ and ${}_R\mathscr{F}$ are noetherian categories, hence it follows from the duality that they are also artinian categories. Thus every object in each of these categories has a composition series. Since R is a right V-ring, we infer from this that every object in $\mathscr{F}_R$ is semisimple, and then it follows from the duality that the same is true in ${}_R\mathscr{F}$. In particular, R/I must be semisimple for any essential right or left ideal I of R. The proposition now follows from Goodearl [**10,** Proposition 3.1].

THEOREM 3. *Suppose R is a right and left noetherian ring with maximal right quotient ring Q, and assume that R is a right V-ring. Then R is right and left hereditary if and only if $(Q/R)_R$ and ${}_R(Q/R)$ are both injective.*

Proof. As in Proposition 2, we see that R is semiprime and that Q is the classical quotient ring of R. Thus Q_R and ${}_RQ$ are both injective, hence if R is right and left hereditary we automatically obtain $(Q/R)_R$ and ${}_R(Q/R)$ injective. Conversely, if Q/R is injective on both sides, Proposition 2 shows that all singular right and left R-modules are injective. According to Goodearl [**10,** Proposition 3.3], R is thus right and left hereditary.

COROLLARY 4. *Suppose that R is a right and left QI-ring with maximal right quotient ring Q. Then R is right and left hereditary if and only if $(Q/R)_R$ and ${}_R(Q/R)$ are both injective.*

Proof. Inasmuch as R is a right and left noetherian, right and left V-ring, this is a direct consequence of Theorem 3.

THEOREM 5. *Let R be a right and left noetherian ring with maximal right quotient ring Q, and assume that $(Q/R)_R$ and ${}_R(Q/R)$ are both injective. Then the following conditions are equivalent*:

(a) *R is a right V-ring.*

(b) *R is a left V-ring.*

(c) *R is a right QI-ring.*

(d) *R is a left QI-ring.*

Proof. Under any of the assumptions a, b, c, d, R is a V-ring on one side or the other and hence is semiprime. Thus we may as well assume that R is semiprime to begin with. In this case Q is also the maximal left quotient ring of R, and so our hypotheses are now left-right symmetric.

a $\Rightarrow$ b: According to Faith [**8,** Theorem 31], R is a finite direct product of simple rings, hence we need only consider the case when R itself is simple. If $\mathrm{soc}({}_RR) \neq 0$, then $\mathrm{soc}({}_RR) = R$ and R is a semisimple ring, whence (b) is automatic. On the other hand, if $\mathrm{soc}({}_RR) = 0$, then all simple left R-modules are singular, hence it follows from Proposition 2 that R is a left V-ring.

b $\Rightarrow$ a: By symmetry.

a $\Rightarrow$ c: Inasmuch as R is right and left hereditary by Theorem 3, this follows from Boyle [**1,** Theorem 5].

c $\Rightarrow$ a: is automatic, and then b $\Leftrightarrow$ d by symmetry.

COROLLARY 6. *Let R be a right and left noetherian domain. Then R is a right PCI-domain if and only if R is a left PCI-domain.*

Proof. If R is a right noetherian PCI-domain, then according to Boyle [**1,** Theorem 7] R is right hereditary. By Small [**19,** Corollary 3], R is also left semihereditary and thus left hereditary. Letting Q denote the classical quotient ring of R, we thus see that $(Q/R)_R$ and ${}_R(Q/R)$ are both injective. Since R is in particular a right V-ring, Theorem 5 now says that R is also a left V-ring. Thus Boyle [**1,** Corollary 10] shows that R is a left PCI-domain.

3. QI-rings. This section is concerned with several aspects of the structure of QI-rings. We begin by looking at semiprime Goldie rings, in which case we show that R is a QI-ring provided only that its singular quasi-injective modules are injective. Second, we give an example to show that not all noetherian V-rings need be QI-rings. Finally, we prove several results about the structure of modules over a QI-ring R which lead to the inequality r.gl.dim.$R \leqq$ K.dim.R.

LEMMA 7. *Let R be any right nonsingular ring. If R_R is finite-dimensional, then all faithful nonsingular quasi-injective right R-modules are injective.*

Proof. Let A be any faithful nonsingular quasi-injective right R-module. Since A is nonsingular, the annihilator of any subset of A is an $\mathcal{S}$-closed right ideal of R in the sense of [**10**, p. 14]. According to Goddearl [**10**, Theorem 1.24], the finite-dimensionality of R_R implies that the $\mathcal{S}$-closed right ideals of R satisfy the descending chain condition. Thus A must have a finite subset $\{a_1, \cdots, a_n\}$ whose annihilator is minimal among the annihilators of finite subsets of A, and then we infer from the faithfulness of A that the annihilator of $\{a_1, \cdots, a_n\}$ is 0. Consequently the element $(a_1, \cdots, a_n) \in A^n$ has zero annihilator, whence A^n contains an isomorphic copy of R_R. Since A^n is quasi-injective by Harada [**12**, Proposition 2.4], it follows easily from Baer's criterion that A^n must be injective. Therefore A is injective.

Unforunately, the hypothesis of faithfulness cannot be omitted from Lemma 7. For if R is the ring of all lower triangular 2×2 matrices over a field F, then its radical J is a minimal right ideal of R and thus is a nonsingular quasi-injective right R-module. However, since J contains no nonzero idempotents it cannot be injective.

Over a commutative noetherian nonsingular ring, faithfulness can be dropped, as shown by Harada [**12**, Propositions 2.5, 2.6]. This result carries over to semiprime Goldie rings, as the following theorem shows.

THEOREM 8. *If R is a semiprime right Goldie ring, then all nonsingular quasi-injective right R-modules are injective.*

Proof. Let A be any nonsingular quasi-injective right R-module, and let Q denote the classical right quotient ring of R, which is a semisimple ring. Now $E(A)$ is nonsingular and so is a right Q-module, hence there exists a ring decomposition $Q = Q_1 \oplus Q_2$ such that $E(A)Q_1 = 0$ and $E(A)$ is faithful over Q_2. If $I = R \cap Q_1$, then I is a two-sided ideal of R such that $E(A)I = 0$. Since it suffices to show that A is a direct summand of $E(A)$, it follows that we need only prove that A is injective as an (R/I)-module. Inasmuch as R/I is a semiprime right Goldie ring with classical right quotient ring Q_2, we may thus assume, without loss of generality, that $E(A)$ is a faithful right Q-module.

According to Lemma 7, it is enough to show that the annihilator $H = \{r \in R \mid Ar = 0\}$ is 0. Since $E(A)$ is a faithful right module over the semisimple ring Q, it must contain a finite subset $\{x_1, \cdots, x_n\}$ whose annihilator in Q is zero. There must be an essential right ideal J of R such that $x_iJ \leqq A$ for all i, whence $x_iJH = 0$ for all i, and thus $JH = 0$. Now HJ is a nilpotent two-sided ideal of R, hence our semiprime hypothesis implies that $HJ = 0$. Inasmuch as J is essential in R, we conclude that $H = 0$, and so A is indeed faithful.

COROLLARY 9. *Let R be a semiprime right Goldie ring. Then R is a right QI-ring if and only if all singular quasi-injective right R-modules are injective.*

Proof. Assume that all singular quasi-injective right R-modules are injective, and consider an arbitrary quasi-injective right R-module A. The singular submodule $Z(A)$ is a fully invariant submodule of A and thus is quasi-injective, hence we obtain that $Z(A)$ is injective. Now $A \cong Z(A) \oplus [A/Z(A)]$ and so $A/Z(A)$ is quasi-injective, whence Theorem 8 says that $A/Z(A)$ must be injective. Therefore A is injective.

As we have remarked above, every QI-ring is a noetherian V-ring, and Byrd **[2]** has raised the converse question of whether every noetherian V-ring must be a QI-ring. The answer is no, as we now show.

EXAMPLE. There exists a right and left noetherian, right and left V-ring R which is not a right QI-ring.

Proof. Let F be a universal differential field of characteristic 0 with respect to two commuting derivations δ_1 and δ_2 (Kolchin **[14**, Theorem, p. 771]), and let $R = F[\theta_1, \theta_2]$ be the ring of linear differential operators over F. We recall that the elements of R are noncommutative polynomials in the indeterminates θ_1, θ_2, subject to the relations $\theta_1\theta_2 = \theta_2\theta_1$ and $\theta_i\alpha = \alpha\theta_i + \delta_i\alpha$ for all $\alpha \in F$. It is easily seen that R is a right and left noetherian ring. Cozzens and Johnson have shown in [**6**, Theorem 1] that R is a left V-ring, and the same argument shows that R is also a right V-ring.

In view of the relation $\theta_i\alpha = \alpha\theta_i + \delta_i\alpha$, we can extend δ_i to a derivation of R by setting $\delta_i r = \theta_i r - r\theta_i$. Note that $R = F[\theta_1][\theta_2]$, i.e., R is equal to the ring of linear differential operators over the differential ring $(F[\theta_1], \delta_2)$. Likewise, $R = F[\theta_2][\theta_1]$.

Inasmuch as $R = F[\theta_1][\theta_2] = F[\theta_1] + \theta_2 R$, we have $R/\theta_2 R \cong F[\theta_1]$ as right $F[\theta_1]$-modules. We compute that the right R-module action on $F[\theta_1]$ is given by its right $F[\theta_1]$-module action together with the rule $x * \theta_2 = -\delta_2 x$. Now $F[\theta_1]$ is a right and left noetherian domain, hence it has a classical quotient division ring Q. We extend δ_2 to a derivation of Q according to the rule $\delta_2(ab^{-1}) = (\delta_2 a)b^{-1} - ab^{-1}(\delta_2 b)b^{-1}$, and then we make Q into a right R-module by using its right R-module action together with the rule $x * \theta_2 = -\delta_2 x$.

Now Q is uniform as a right $F[\theta_1]$-module and thus also as a right R-module. Therefore $E(Q_R)$ is an indecomposable right R-module. Considering $E(Q_R)$ just as a module over the noetherian domain $F[\theta_2]$ (which is a subring of R), it has a torsion submodule which we shall denote by T. We proceed by showing that T is a right R-module which is quasi-injective but not injective. Since T is an $F[\theta_2]$-submodule of $E(Q_R)$, and since $R = F[\theta_2][\theta_1]$, T will be an R-submodule of $E(Q_R)$ provided $T\theta_1 \subseteq T$. Given any $t \in T$, we have $ta = 0$ for some nonzero $a \in F[\theta_2]$. Observing that $\delta_1 a \in F[\theta_2]$, we compute that $t\theta_1 a = t(\delta_1 a) \in T$, and thus $t\theta_1 \in T$. Therefore T is indeed an R-submodule of $E(Q_R)$.

Inasmuch as T is a fully invariant submodule of $E(Q_R)$, T is a quasi-injective right R-module. Observing that $1 * \theta_2 = -\delta_2 1 = 0$, we see that $1 \in T$ and thus $T \neq 0$. Since F is a universal differential field, there must be an element $\alpha \in F$ such that $\delta_2\alpha = 1$. Then $\delta_2(\alpha + \theta_1) = 1$ also, from which we compute that $\delta_2^n(\alpha + \theta_1)^{-1} = (-1)^n n!(\alpha + \theta_1)^{-n-1}$ for all $n > 0$. We now infer that the elements $(\alpha + \theta_1)^{-1} * \theta_2^n$ in Q are right linearly independent over F, i.e., $(\alpha + \theta_1)^{-1}$ is not annihilated by any nonzero elements of $F[\theta_2]$. Thus $(\alpha + \theta_1)^{-1} \notin T$, and so $T \neq E(Q_R)$.

Now $E(Q_R)$ is indecomposable and T is a nontrivial submodule of $E(Q_R)$, hence T cannot be injective. Since T is quasi-injective, R cannot be a right QI-ring.

PROPOSITION 10. [**8**, Proposition 32]. *Let R be a right QI-ring. If E is any nonzero indecomposable injective right R-module, then* $\Lambda = \mathrm{Hom}_R(E, E)$ *is a division ring.*

THEOREM 11. *Let R be a right QI-ring. If A is any nonzero finitely generated right R-module with Krull dimension* α, *then all finitely generated submodules of* $E(A)/A$ *have Krull dimension strictly less than* α.

Proof. First consider the case where A is critical, and suppose $E(A)/A$ has a finitely generated submodule B/A with K.dim.$(B/A) \geqq \alpha$. Since R is a right noetherian, there exist submodules $B_0 = A < B_1 < \cdots < B_n = B$ such that each B_k/B_{k-1} is critical. We must have K.dim.$(B_k/B_{k-1}) \geqq \alpha$ for some k. Thus, replacing B by B_k and setting $C = B_{k-1}$, we have modules $A \leqq C < B$ such that B/C is β-critical for some $\beta \geqq \alpha$.

Now let Q be the sum of all submodules P of $E(A)$ for which $\mathrm{Hom}_R(P, E(B/C)) = 0$. We observe that Q is a fully invariant submodule of $E(A)$, i.e., Q is quasi-injective. Since R is a right QI-ring, Q must be injective. Since A is critical it must be uniform, and thus $E(A)$ is indecomposable, so the only choices for Q are $E(A)$ or 0. Inasmuch as the identity map on B/C extends to a nonzero map $E(A) \to E(A)/C \to E(B/C)$, we obtain $Q \neq E(A)$ and thus $Q = 0$.

Since $Q = 0$, there exists a nonzero map $f: A \to E(B/C)$, which induces a nonzero map from $A_0 = f^{-1}(B/C)$ into B/C. According to Gordon-Robson [**11,** Proposition 2.3], A_0 is α-critical and all nonzero submodules of B/C are β-critical. Inasmuch as $\beta \geqq \alpha$, it follows that f must be a monomorphism, whence A_0 is isomorphic to a nonzero submodule B_0/C of B/C. The map $B_0 \to B_0/C \xrightarrow{\cong} A_0$ extends to a nonzero endomorphism g of $E(A)$, and since $gA = 0$ we see that g cannot be an isomorphism. But then the endomorphism ring of $E(A)$ is not a division ring, which contradicts Proposition 10.

Thus the theorem holds for critical modules. In general, A must have an essential submodule $K = K_1 \oplus \cdots \oplus K_n$, where each K_i is α_i-critical for some $\alpha_i \leqq \alpha$. In view of the results above, every finitely generated submodule of $E(K_i)/K_i$ has Krull dimension strictly less than α_i. It follows that every finitely generated submodule of $E(K)/K$ has Krull dimension strictly less than α, from which the theorem follows.

COROLLARY 12. *If R is any right QI-ring, then* r.gl.dim.$R \leqq$ K.dim.R.

Proof. We need only consider the case when K.dim.$R = N < \infty$. For any right R-module A, let $\phi(A)$ denote the supremum of the Krull dimensions of all finitely generated submodules of A. Note that $\phi(A) \leqq N$, and that $\phi(A) =$ K.dim.A when A is finitely generated. It suffices to show that $id_R(A) \leqq N$ for all nonzero right R-modules A, where $id_R(A)$ denotes the injective dimension of A. We proceed by induction to show that $id_R(A) \leqq \phi(A)$ for all nonzero A.

If $\phi(A) = 0$, then since R is right noetherian all finitely generated submodules of A must have composition series. Inasmuch as R is a right V-ring, all such submodules of R must be semisimple, and thus A itself is semisimple. Now A is quasi-injective and therefore injective, whence $id_R(A) = \phi(A)$.

Now let $\phi(A) = n > 0$ and assume that $id_R(B) \leqq \phi(B)$ for all nonzero modules B with $\phi(B) < n$. Choose an essential submodule of A of the form $K = \bigoplus K_\alpha$, where each K_α is finitely generated. According to Theorem 11, all finitely generated submodules of

$E(K_\alpha)/K_\alpha$ have Krull dimension strictly less than K.dim.K_α, from which we infer that $\phi(E(K_\alpha)/K_\alpha) < n$. It now follows that $\phi(E(A)/A) < n$, and then $id_R(E(A)/A) < n$ by the induction hypothesis. Therefore $id_R(A) \leqq n$, and the induction works.

4. V-rings. In this section we consider the Krull dimension of V-rings. According to Michler-Villamayor [**16,** Theorem 4.2], a right V-ring with right Krull dimension at most one is right noetherian, right hereditary, and Morita-equivalent to a finite direct sum of simple V-domains. In general, all positive integers are possible as Krull dimensions of V-rings, and we exhibit examples of this. We also show that a V-ring has a Krull dimension if and only if it is already noetherian.

PROPOSITION 13. *Let R be a right V-ring. Then R has right Krull dimension if and only if R is right noetherian.*

Proof. All right noetherian rings have right Krull dimension: Gordon-Robson [**11,** Proposition 1.3]. On the other hand, if R is not right noetherian then by using an argument of Faith in [**7,** Corollary 15B] we infer that there exists a cyclic right R-module E whose socle is an infinite direct sum of simple modules. But then E is not finite-dimensional and so does not have Krull dimension, by Gordon-Robson [**11,** Proposition 1.4]. Since E is cyclic, it follows that R does not have Krull dimension.

COROLLARY 14. *If R is a right PCI-domain, then the following conditions are equivalent*:
(a) *R is right noetherian.*
(b) *R has Krull dimension.*
(c) *R has Krull dimension at most* 1.

Proof. Since R is a right V-ring, a $\Leftrightarrow$ b by Proposition 13. If (a) holds, then R is a simple, right hereditary ring by Faith [**7,** Theorem 14]. Inasmuch as R is a right *PCI*-domain, we see that all cyclic submodules of any singular right R-module A are direct summands of A. In case A is finitely generated as well, then it must be finite-dimensional since R is right noetherian, and Goodearl [**10,** Proposition 1.22] shows that A is semisimple. In particular, R/I is semisimple for all essential right ideals I of R, whence Goodearl [**10,** Proposition 3.1] shows that all singular right R-modules are semisimple and injective. It now follows from Michler-Villamayor [**16,** Theorem 4.2] that K.dim.$R \leqq 1$.

It is an open question whether all right *PCI*-rings are right noetherian. Corollary 14 might provide a means for attacking this question.

Cozzens and Johnson **[6]** have constructed examples of noetherian V-rings with arbitrary finite global dimension, and we shall show that these examples also have arbitrary finite Krull dimension. Since these examples are differential operator rings, we begin with two general results on the Krull dimension of differential operator rings.

LEMMA 15. *Let R be any right noetherian ring with a derivation δ. Then the ring of linear differential operators $R[\theta]$ is right noetherian, and*

$$\text{K.dim.}R[\theta] \leqq \text{K.dim.}R + 1$$

Proof. Given any right ideal J of $R[\theta]$ and any nonnegative integer n, let J_n be the set of all leading coefficients of elements of J of degree n, together with 0. Then the collection $\{J_0, J_1, \cdots\}$ is an ascending sequence of right ideals of R, hence we can define a right ideal in the ordinary polynomial ring $R[x]$ by setting $J_0 + J_1x + J_2x^2 + \cdots$. We thus obtain a monotone map ϕ from the right ideal lattice of $R[\theta]$ into the right ideal lattice of $R[x]$, and as in the Hilbert Basis Theorem an easy induction on degrees shows that ϕ is a strictly monotone map. Now $R[x]$ is certainly right noetherian, and $\text{K.dim.}R[x] = \text{K.dim.}R + 1$ by Gordon-Robson **[11**, Theorem 9.2], from which the lemma follows.

PROPOSITION 16. *If F is a field with a finite collection $\delta_1, \cdots, \delta_n$ of commuting derivations, then the ring of linear differential operators $R = F[\theta_1, \cdots, \theta_n]$ has Krull dimension n.*

Proof. By induction on Lemma 15, we see that R is a right noetherian ring with $\text{K.dim.}R \leqq n$. Now set $J_k = \theta_kR + \cdots + \theta_nR$ for $k = 1, \cdots, n$, and set $J_{n+1} = 0$. We prove by induction on k that $\text{K.dim.}R/J_k \geqq k - 1$. Since $R/J_1 \neq 0$, we automatically have $\text{K.dim.}R/J_1 \geqq 0$. Now let $1 \leqq k \leqq n$ and assume that $\text{K.dim.}R/J_k \geqq k - 1$. Inasmuch as the θ_i all commute, we see that $\theta_k^mJ_{k+1} \leqq J_{k+1}$ for all $m > 0$. In fact, we compute that $\{r \in R \mid \theta_k^m r \in J_{k+1}\} = J_{k+1}$, from which it follows that

$$\{r \in R \mid \theta_k^m r \in \theta_k^{m+1}R + J_{k+1}\} = \theta_kR + J_{k+1} = J_k,$$

i.e.,

$$(\theta_k^m R + J_{k+1})/(\theta_k^{m+1}R + J_{k+1}) \cong R/J_k.$$

Since K.dim.$R/J_k \geqq k-1$, it follows that K.dim.$R/J_{k+1} \geqq k$. Thus the induction works, hence we obtain K.dim.$R \geqq n$.

EXAMPLE. Given any positive integer n, there exists a right and left noetherian, right and left V-ring R_n such that K.dim.$R_n = n$.

Proof. Let F be a universal differential field with respect to n commuting derivations $\delta_1, \cdots, \delta_n$ (Kolchin [**14**, Theorem, p. 771]), and let R_n be the ring of linear differential operators $F[\theta_1, \cdots, \theta_n]$. Then R_n is a noetherian V-ring by Cozzens-Johnson [**6**, Theorem 1], and K.dim.$R_n = n$ by Proposition 15.

REFERENCES

1. Anne K. Boyle, *Hereditary QI-rings*, Trans. Amer. Math Soc., (to appear).
2. K. A. Byrd, *When are quasi-injectives injective?*, Canad. Math Bull., **15** (1972), 599–600.
3. ———, *Rings whose quasi-injective modules are injective*, Proc. Amer. Math Soc., **33** (1972), 235–240.
4. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton (1956), Princeton University Press.
5. J. H. Cozzens, *Homological properties of the ring of differential polynomials*, Bull. Amer. Math. Soc., **76** (1970), 75–79.
6. J. H. Cozzens and J. Johnson, *Some applications of differential algebra to ring theory*, Proc. Amer. Math. Soc., **31** (1972), 354–356.
7. Carl Faith, *When are proper cyclics injective?*, Pacific J. Math., **45** (1973), 97–112.
8. ———, *Modules finite over endomorphism ring, in Lectures on Rings and Modules*, Springer-Verlav Lecture Notes No. 246 (1972), 145–189.
9. D. R. Farkas and R. L. Snider, *Group algebras whose simple modules are injective*, Trans. Amer. Math. Soc., (to appear).
10. K. R. Goodearl, *Singular torsion and the splitting properties*, Amer. Math Soc. Memoirs No. 124, (1972).
11. R. Gordon and J. C. Robson, *Krull dimension*, Amer. Math. Soc. Memoirs No. 133, (1973).
12. M. Harada, *Note on quasi-injective modules*, Osaka J. Math., **2** (1965), 351–356.
13. R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc., **36** (1961), 260–268.
14. E. R. Kolchin, *Galois theory of differential fields*, Amer. J. Math., **75** (1953), 753–824.
15. R. P. Kurshan, *Rings whose cyclic modules have finitely generated socle*, J. Algebra, **15** (1970), 376–386.
16. G. Michler and O. E. Villamayor, *On rings whose simple modules are injective*, J. Algebra, **25** (1973), 185–201.
17. B. L. Osofsky, *On twisted polynomial rings*, J. Algebra, **18** (1971), 597–607.
18. F. L. Sandomierski, *Semisimple maximal quotient rings*, Trans. Amer. Math. Soc., **128** (1967), 112–120.
19. L. W. Small, *Semihereditary rings*, Bull. Amer. Math. Soc., **73** (1967), 656–658.

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COVERING THEOREMS FOR FINITE NONABELIAN SIMPLE GROUPS. V.

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In the alternating group A_n, $n = 4k + 1 > 5$, the class C of the cycle $(12 \cdots n)$ has the property that CC covers the group. For $n = 16k$ there is a class C of period $n/4$ in A_n such that CC covers A_n; C is the class of type $(4k)^4$.

1. Introduction. It was shown by E. Bertram [1] that for $n \geqq 5$ every permutation in A_n is the product of two l-cycles, for any l satisfying $[3n/4] \leqq l \leqq n$. Hence A_n can be covered by products of two n-cycles and also by products of two $(n-1)$-cycles. But if n is odd the n-cycles in A_n fall into two conjugate classes C, C', and similarly for the $(n-1)$-cycles if n is even, so that the quoted result does not decide whether

$$CC = A_n. \tag{1}$$

The question was decided affirmatively for $n = 4k + 2$ and negatively for $n = 4k, 4k - 1$ in [2]. The question is now decided affirmatively in the remaining case $n = 4k + 1$, $n \neq 5$.

THEOREM 1. *For $n = 4k + 1 > 5$, the class C of the cycle $(12 \cdots n)$ has property* (1).

The proof is in §§2–4.

Regarding the product CC', it was shown in [2] that CC' covers A_n ($n \geqq 5$) if $n = 4k, 4k - 1$, while if $n = 4k + 1, 4k + 2$, CC' contains all of A_n but the identity.

By an argument quite similar to the proof of Theorem 1, we have proved

THEOREM 2. *For $n = 16k$, the class C of type $(4k)^4$ in A_n has property* (1).

The proof and some related matters are discussed in §5. Note that the class in Theorem 2 has period $n/4$.

2. The case $n = 9$. Let $a = (123456789)$. For every class in A_9, a conjugate b of a can be found such that ab represents (lies in) that class. This assertion is the substance of the table below.

b	ab
a^{-1}	1
(193248765)	(14) (38)
(176235894)	(13) (25) (48) (79)
(132987654)	(193)
(134765289)	(18) (24) (379)
(132798465)	(174) (369)
(184523796)	(135) (274) (698)
(137259486)	(15) (276) (3849)
(123794865)	(1384) (2769)
(132798654)	(17693)
(189623574)	(13) (25) (47986)
(132869745)	(18764) (359)
(132845697)	(18746) (359)
(159348726)	(162495) (38)
(186974532)	(3598764)
a	(135792468) $\sim a$
(125678934)	(315792468)

3. A lemma. In §3 and §4, C will denote the class of the cycle $a = (12 \cdots n)$ in A_n.

LEMMA. *If $n = 4k + 1 > 5$, then CC contains the type $2^{2k}1^1$.*

Proof. If $n \equiv 1 \pmod 8$, then $x =$

$$(n\ n-3\ n-2\ n-1,\ n-4\ n-7\ n-6\ n-5; \cdots; 9678,\ 5234;\ 1)$$

is conjugate to a and

$$ax = (1\ 3)(2\ 4)(5\ 7)(6\ 8) \cdots (n-4\ n-2)(n-3\ n-1).$$

If $n \equiv 5 \pmod 8$, $n > 13$, then $y =$

$$(n\ n-3\ n-2\ n-1,\ n-4\ n-7\ n-6\ n-5; \cdots; 21\ 18\ 19\ 20,$$
$$17\ 14\ 15\ 16;\ 13\ 96\ 10,\ 12\ 78\ 11;\ 5234,\ 1)$$

is conjugate to a and

$$ay = (1\ 3)(2\ 4)(5\ 10)(68)(7\ 11)(9\ 12)(13\ 15)(14\ 16) \cdots$$
$$(n-4\ n-2)(n-3\ n-1).$$

If $n = 13$ use the last 13 letters of the above y. (The pattern of y differs from that of x only in the last block of 8 letters between semi-colons, $13\ 9 \cdots 11$, in which the number of reversals is odd, whereas in every other such block of 8 letters in either x or y, the number of reversals is even.)

4. The induction. The induction proceeds from $n-4$ to $n = 4k+1$. The induction hypothesis is: For every permutation T in A_{n-4}, there are two $(n-4)$-cycles d_1 and d_2, both in the class of the $(n-4)$-cycle $(1\,2 \cdots n-6\ n-5\ n-4)$, and also two other $(n-4)$-cycles d_1' and d_2', both in the class of $(1\,2 \cdots n-6\ n-4\ n-5)$, such that $T = d_1d_2 = d_1'd_2'$.

Let S ($\neq 1$) be a permutation in A_n. To show that CC contains S we consider several cases. In each case we find a conjugate S_1 of S, and a certain permutation g in A_n, such that $T = S_1g^{-1}$ fixes the letters $n, n-1, n-2, n-3$ and thus its restriction to $1, 2, \cdots, n-4$ lies in A_{n-4}.

Case 1. S contains a cycle with 5 or more letters: take

$$g = (n\ n-1\ n-2\ n-3\ n-4).$$

Case 2. S contains no cycle with 5 or more letters, but S contains at least one cycle with 4 letters: take

$$g = (n\ n-1\ n-2\ n-3)(n-4\ n-5).$$

Case 3. S contains no cycle with more than 3 letters, but S does contain two 3-cycles: take

$$g = (n\ n-1\ n-2)(n-3\ n-4\ n-5).$$

Case 4. S is of type $3^1\,2^{2k-2}\,1^2$: take

$$g = (n\ n-1\ n-2).$$

Now, if S contains no cycle longer than a transposition, either S is of type $2^{2k}\,1^1$, whence CC contains S by the lemma, or we have

Case 5. S fixes 5 or more letters: take $g = 1$.

The argument in Case 5 is quite simple. Since S fixes 5 or more letters, S has a conjugate S_1 that fixes $n, n-1, n-2, n-3$. Hence by the induction hypothesis $S_1 = d_1d_2$, where d_1 and d_2 both fix $n, n-1, n-2, n-3$, and can be expressed

$$d_1 = (a_1\, a_2 \cdots a_{n-5}\; n-4), \qquad d_2 = (b_1\, b_2 \cdots b_{n-5}\; n-4),$$

where the permutation $a_i \to b_i$ is an even permutation of the letters $1, 2, \cdots, n-5$. Then $S_1 = d_3 d_4$, with

$$d_3 = (a_1\, a_2 \cdots a_{n-5}\; n\;\; n-1\;\; n-2\;\; n-3\;\; n-4),$$

$$d_4 = (b_1\, b_2 \cdots b_{n-5}\; n-4\;\; n-3\;\; n-2\;\; n-1\;\; n),$$

and d_3, d_4 belong to the same class, be it C or C'. If the other part of the induction hypothesis is used in a similar fashion, the assertion that CC contains S follows.

The details for Case 1 are as follows. Since $T = S_1 g^{-1}$ moves at most the first $n-4$ letters, we have by the induction hypothesis $T = d_1 d_2 = d'_1 d'_2$ where d_1, d_2 $[d'_1, d'_2]$ are from the same class in A_{n-4}. Writing

$$d_1 = (a_1\, a_2 \cdots a_{n-5}\; n-4), \qquad d_2 = (b_1\, b_2 \cdots b_{n-5}\; n-4),$$

the permutation $a_i \to b_i$ is an even permutation of $1, 2, \cdots, n-5$. Now $S_1 = Tg = d_3 d_4$, with $g = (n\;\; n-1\;\; n-2\;\; n-3\;\; n-4)$ and

$$d_3 = (a_1 \cdots a_{n-5}\; n-2\;\; n\;\; n-3\;\; n-1\;\; n-4),$$

$$d_4 = (b_1 \cdots b_{n-5}\; n\;\; n-3\;\; n-1\;\; n-4\;\; n-2).$$

Note that d_3 and d_4 are in the same class, be it C or C', in A_n. By again using d'_1 and d'_2 in place of d_1 and d_2, the proof is completed in this case.

In Case 2, S has a conjugate S_1 such that $T = S_1 g^{-1}$ fixes at least 5 letters. Hence without loss of generality the factors d_1, d_2 $[d'_1, d'_2]$ can be chosen so that $T = d_1 d_2 = d'_1 d'_2$ with

$$d_1 = (a_1 \cdots a_{n-6}\; n-5\;\; n-4), \qquad d'_1 = (a'_1 \cdots a'_{n-6}\; n-5\;\; n-4)$$

$$d_2 = (b_1 \cdots b_{n-6}\; n-4\;\; n-5), \qquad d'_2 = (b'_1 \cdots b'_{n-6}\; n-4\;\; n-5)$$

and where $a_i \to b_i$ $[a'_i \to b'_i]$ is an *odd* permutation of the letters $1, 2, \cdots, n-6$. Now $S_1 = Tg = d_3 d_4$, where

$$d_3 = (a_1 \cdots a_{n-6}\; n-1\;\; n-5\;\; n-3\;\; n-2\;\; n\;\; n-4),$$

$$d_4 = (b_1 \cdots b_{n-6}\; n-5\;\; n-2\;\; n\;\; n-3\;\; n-4\;\; n-1).$$

The permutations d_3 and d_4 belong to the same class in A_n. Priming the a_i and b_i completes the proof in this case.

In Case 3, S has at least two 3-cycles, and has a conjugate S_1 such that $T = S_1g^{-1}$ fixes the letters $n, n-1, n-2, n-3, n-4, n-5$. By the induction hypothesis permutations d_1 and d_2 exist such that $T = d_1d_2$ with

$$d_1 = (n-4\ a_1 \cdots a_k\ n-5\ a_{k+1} \cdots a_{n-6}),$$
$$d_2 = (n-4\ b_1 \cdots b_l\ n-5\ b_{l+1} \cdots b_{n-6}),$$

and where d_1 and d_2 are in the same class in A_n. (We cannot assume that $n-4$ and $n-5$, which are fixed by T, are neighbors in d_1 and d_2, but it is possible that $k=0$ and $l = n-6$ or that $k = n-6$ and $l = 0$.) Now $S_1 = Tg = d_3d_4$, where

$$d_3 = d_1 h, \quad d_4 = h^{-1} d_2 g,$$

with $h = (n-5\ n-3\ n-2)(n-4\ n-1\ n)$. Then d_3 and d_4 are both n-cycles. It has only to be checked that they are in the same class in A_n; to do this is tedious, but straightforward. To complete the proof in this case we observe that since S contains two 3-cycles and $S_1 = d_3d_4$, the decomposition $S_1 = d_3'd_4'$ can be obtained by applying a certain outer automorphism of A_n.

In the only remaining case, S fixes 2 letters, and therefore has a conjugate S_1 such that $T = S_1g^{-1}$ fixes

$$n,\ n-1,\ n-2,\ n-3,\ n-4.$$

Again we have $T = d_1d_2$, where we can write

$$d_1 = (a_1 \cdots a_{n-6}\ n-4\ n-5), \qquad d_2 = (b_1 \cdots b_{n-6}\ n-5\ n-4),$$

and where the permutation $a_i \to b_i$ is an odd permutation of the letters $1, 2, \cdots, n-6$. Then $S_1 = Tg = d_3d_4$, with

$$d_3 = (a_1 \cdots a_{n-6}\ n-1\ n\ n-3\ n-2\ n-4\ n-5),$$
$$d_4 = (b_1 \cdots b_{n-6}\ n-5\ n-4\ n\ n-2\ n-3\ n-1),$$

and these belong to the same class. By priming we again conclude CC contains S, and the proof is complete in all cases. Hence Theorem 1.

5. Covering A_{16k}. By means of an almost identical argument we have shown that the class C of type $4l_1\ 4l_2\ 4l_3\ 4l_4$ ($l_i \geqq 1$) in A_n ($n = 4\Sigma l_i$) has the covering property (1). The lemma required is simpler: Let $m = 4l$, $b = (1\,2 \cdots m)$. Taking $x =$

$$(m\ m-3\ m-2\ m-1,\ m-4\ m-7\ m-6\ m-5, \cdots, 8\,5\,6\,7, 4\,1\,2\,3)$$

gives

$$bx = (1\,3)(2\,m)(4\,6)(5\,7)\cdots(m-4\ m-2)(m-3\ m-1).$$

Hence if D is the class of type $4l_1\ 4l_2 \cdots 4l_r$ (r even) in A_n, then DD contains the type $2^{n/2}$.

In order to start the induction we had to prove that the class C of type 4^4 has the property $CC = A_{16}$. The calculations are too lengthy to be included. (A copy can be had from any of the authors.) This yields Theorem 2.

One can ask how small a period is possible for a class C with property (1). The first result in this direction was that of *Xu* **[4]** who found such a class with period $n-3$ if n is odd and period $n-2$ if n is even. From the result of Bertram quoted in the introduction, it follows that the smallest period of such C is $\leqq 3n/4$. While Theorem 2 does not give covering for all n, it nevertheless yields, among classes C in A_n satisfying (1),

$$\liminf_{n\to\infty} \frac{\text{period of } C}{n} \leqq \frac{1}{4}$$

as opposed to Bertram's 3/4.

From the other direction we have shown **[3]** that for $n > 6$ there is no class C in A_n having property (1) and period 2, and if $n = 12k + 10$ there is no such class of period 3. There may be such a class of period 4, however. More precisely, we conjecture that for $n = 8k$, the class $C = 4^{2k}$ has the covering property (1).

REFERENCES

1. E. A. Bertram, *Even permutations as a product of two conjugate cycles*, J. Combinatorial Theory (A) **12** (1972), 368–380.

2. J. L. Brenner, *Covering theorems for nonabelian simple groups. II*, J. Combinatorial Theory (A), **14** (1973), 264–269.

3. J. L. Brenner, M. Randall, J. Riddell, *Covering theorems for finite nonabelian simple groups. I*, Colloq. Math. **XXXII**.1, 1974 (to appear).

4. Cheng-Hao Xu, *The commutators of the alternating group*, Sci. Sinica **14** (1965), 339–342.

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THE SINGLE VALUED EXTENSION PROPERTY ON A BANACH SPACE

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An operator T which maps a Banach space X into itself has the single valued extension property if the only analytic function f which satisfies $(\lambda I - T)f(\lambda) = 0$ is $f = 0$. Clearly the point spectrum of any operator which does not have the single valued extension property must have nonempty interior. The converse does not hold. However, it is shown below that if $\lambda_0 I - T$ is semi-Fredholm and λ_0 is an interior point of the point spectrum of T, then T does not have the single valued extension property.

It will be convenient to use the following definition.

1. DEFINITION. Let $T: X \to X$ be a closed linear operator mapping a Banach space X into itself, and let λ_0 be a complex number. The operator T has the *single valued extension property at* λ_0 if $f = 0$ is the only solution to $(\lambda I - T)f(\lambda) = 0$ that is analytic in a neighborhood of λ_0. Also, T has the *single valued extension property* if it has this property at every point λ_0 in the complex plane.

2. THEOREM. *Let T be a closed linear operator mapping the Banach space X into itself. If T is onto but not one-one, then T does not have the single valued extension property at $\lambda = 0$.*

Proof. First we produce a candidate for f. Choose any x_0 in X with $\|x_0\| = 1$ and $Tx_0 = 0$, which is possible since T is not one-one. Since T is a closed operator and is onto, it is an open mapping. The open mapping theorem implies there exists a $k > 0$ such that for any element $x \in X$ there is a $y \in X$ with $Ty = x$ and $\|y\| \leqq k\|x\|$. Now choose x_n inductively so that $Tx_n = x_{n-1}$ and $\|x_n\| \leqq k\|x_{n-1}\|$. Define $f(\lambda) = \Sigma x_n \lambda^n$. Since $\|x_n\| \leqq k^n$, the sum converges for $|\lambda| < k^{-1}$, and f is analytic in this neighborhood of zero.

Now to show that $(\lambda I - T)f(\lambda) = 0$. Since T is a closed linear operator, so is $\lambda I - T$. Each of the partial sums $\Sigma_{n=0}^{N} x_n \lambda^n$ is in the domain of $\lambda I - T$, since each x_n was chosen from the domain of T. Furthermore

$$(\lambda I - T) \sum_{n=0}^{N} x_n \lambda^n = x_N \lambda^{N+1}$$

and

$$\|x_N \lambda^{N+1}\| \leqq k^N |\lambda|^{N+1}.$$

But as N goes to infinity, $k^N|\lambda|^{N+1}$ converges to zero for $|\lambda| < k^{-1}$. Since $\lambda I - T$ is a closed map, $f(\lambda) = \lim_N \Sigma_{n=0}^N x_n\lambda^n$ is in the domain of $\lambda I - T$ and $(\lambda I - T)f(\lambda) = \lim_N x_N\lambda^{N+1} = 0$.

The function $f(\lambda)$ obtained in the proof of Theorem 2 is certainly not unique. However, it is typical of any function g satisfying $(\lambda I - T)g(\lambda) = 0$ in the following sense: Suppose T is any closed operator not having the single valued extension property at $\lambda = 0$, and that g is any analytic function defined near $\lambda = 0$ satisfying $(\lambda I - T)g(\lambda) = 0$. Expand g in a Taylor series around $0: g(\lambda) = \Sigma x_n\lambda^n$. It can be shown that: (i) each x_n is in the domain of T; (ii) $Tx_{n+1} = x_n$ for $n = 0, 1, 2, \cdots$; and (iii) $Tx_0 = 0$.

The above discussion holds also at points $\lambda_0 \neq 0$ if we replace every T by $T - \lambda_0 I$, and every λ by $\lambda - \lambda_0$.

There are more interesting ways to express Theorem 2: If T has the single valued extension property, then T is invertible whenever it is onto. Or again, if T has the single valued extension property, then λ_0 is in the spectrum of T if and only if $\lambda_0 - T$ is not onto. In particular this is true for normal operators, spectral operators, etc.

3. COROLLARY. *Let T be a closed linear operator on a Banach space X and suppose Y is a cb sed invariant subspace. If $TY = Y$ but T is not one-one on Y, then T does not have the single valued extension property at* 0.

Actually, in Corollary 3, Y could be a linear manifold that is not closed, *provided* that it can be given a new norm, larger than the original, for which Y is complete (and hence becomes a Banach space).

4. COROLLARY. *Let Y be the domain of a closed linear operator $S: X \to Z$, where Z is a Banach space. If $TY = Y$ but T is not one-one on Y then T does not have the single valued extension property.*

5. COROLLARY. *If there is a bounded linear operator on X which is onto but not one-one, then the set of bounded operators that do not have the single valued extension property at* 0 *has nonempty interior* (in the norm topology). *And thus the set of operators without the single valued extension property has nonempty interior.*

A special case of the following result appears in Colojoară and Foiaş, Chapter 1.

6. COROLLARY. *Let T be a closed linear operator mapping the Banach space X into itself, and assume that the domain of T is dense in X so that the adjoint T^* exists. If T is bounded below but is not onto, then T^* does not have the single valued extension property. Or alternately, if the range of T is closed and T is one-one but not onto, then T^* does not have the single valued extension property.*

Proof. If T is bounded below, its range is closed, and T is one-one. Thus the range of T^* is the orthogonal set to $\{0\}$, which is all of X^*; that is, T^* is onto. Since T is not onto and the range of T is closed, the null space of T^* is not just $\{0\}$. Thus T^* is onto but not one-one, and so does not have the single valued extension property by Theorem 2.

A point λ is in the limit spectrum of T if and only if there is a sequence x_n with $\|x_n\| = 1$ and $(\lambda I - T)x_n$ converging to 0.

7. COROLLARY. *If the closed linear operator T has the single valued extension property, then the limit spectrum of T^* is the entire spectrum of T^*. Similarly, if T^* has the single valued extension property, then the limit spectrum of T is the entire spectrum of T.*

A closed linear operator is semi-Fredholm if the range is closed, and the dimension $n(T)$ of the null space or the codimension $d(T)$ of the range is finite (or both). First we investigate the case where the null space is finite dimensional, after a preliminary lemma needed in both proofs.

8. LEMMA. *Let T be a closed linear operator with closed range mapping a Banach space X into itself, and let N be its null space. For an arbitrary linear manifold M in X, if $M + N$ is closed, then the image $T(M)$ of M is closed.*

9. THEOREM. *Let T be a semi-Fredholm operator mapping a Banach space X into itself with $n(T)$ finite. If the point spectrum of T contains a neighborhood of zero, then T does not have the single valued extension property at $\lambda = 0$.*

Proof. Let Y be the subset of X given by the intersection of the ranges of T^n for all n, that is,

$$Y = \bigcap_{n=1}^{\infty} T^n X.$$

The proof consists of showing that Y is a closed invariant subspace of X, that T maps Y onto itself, and that T is not one-one on Y. Then, applying Corollary 3, we see that T does not have the single valued extension property.

It is obvious that Y is a linear manifold in X and that it is invariant under T. To show that Y is closed, we need only show that T^nX is closed for all n. Applying Lemma 9 to $M = T^nX$, it follows that $T^{n+1}(X) = T(T^nX)$ is closed if and only if $T^nX + N$ is closed. Now by hypothesis N is a finite dimensional closed subspace of X, and $TX$is closed. Since the sum of a closed subspace with a finite dimensional subspace is always closed, T^2X is closed. By induction T^nX is closed, and hence $Y = \bigcap (T^nX)$ is also closed.

The next step is to show that T maps Y onto itself. Let $R_n = T^nX$. For any y in Y there is an x_n in R_n with $Tx_n = y$. Also, $x_n - x_m$ is in N. Now $R_n \bigcap N$ is a decreasing sequence of subspaces. Since N is finite dimensional, this sequence is eventually constant. That is, for some $m, R_m \bigcap N = R_k \bigcap N$ for all $k \geqq m$. Thus $x_m - x_k$, which is in $R_m \bigcap N$, is also in R_k. Since x_k was chosen in R_k, it follows that $x_m = x_k + (x_m - x_k)$ is in R_k as well. That is, x_m is in R_k for all $k \geqq m$; therefore x_m is in $Y = \bigcap R_k$. Thus T does map Y onto itself.

It remains to show that the restriction of T to Y is not one-one. It is clear that if $Tx = \lambda x$ for some $\lambda \neq 0$, then x is in Y. Thus any $\lambda \neq 0$ that is in the point spectrum of T is also in the point spectrum of $T | Y$. Hence by our hypothesis, for some $r > 0$, every λ with $0 < |\lambda| < r$ is in the spectrum of $T | Y$. Since the spectrum of $T | Y$ is closed, it also contains 0. But then since $T | Y$ is onto, it cannot be one-one.

In summary, the resriction of T to Y maps Y onto itself but is not one-one. Hence T does not have the single valued extension property.

Taylor (1966) has shown the following: Let T be an arbitrary linear operator on a vector space (no topological properties are necessary). If $n(T)$ is finite, then $Y = \bigcap T^nX$ satisfies $TY = Y$. This result can fail if $n(T)$ is infinite.

A slight extension of this theorem is possible. We may replace the assumption that the range of T is closed by: the range of T^k is closed for some k. Since $n(T^k) \leqq kn(T)$ (see Taylor, 1966), we have that $n(T^k)$ is also finite. Then (using the notation of the proof of Theorem 9) $Y = \bigcap T^nX$ is also equal to $\bigcap (T^k)^nX$. The argument of the proof applied to T^k shows that Y is closed and that $T^kY = Y$, but T^k is not one-one on Y. But of course this means that $TY = Y$ and T is not one-one on Y. Hence, by Corollary 3, T does not have the single valued extension property.

10. THEOREM. *Let T be a closed linear operator mapping a Banach space X into itself, and suppose that the codimension of the range $d(T)$ is finite. If the point spectrum of T contains a neighborhood of zero, then T does not have the single valued extension property at $\lambda = 0$.*

Note that if T is a closed operator and the codimension (the dimension of X/R) of the range is finite, then the range is closed and so T is automatically a semi-Fredholm operator. (See Kato, 1966, p. 233, problem 5.7.)

Proof. The idea of this proof is similar to that of Theorem 9. Let $Y = \bigcap T^n X$; it will be shown that Y is a closed subspace invariant under T, and that T maps Y onto itself but is not one-one.

First we show that Y is closed; it is obviously a linear manifold that is invariant under T. Let $R_n = T^n X$. Since $d(T)$ is finite, then $d(T^n) \leqq nd(T)$ is also finite. It follows that $R_n + N = R_n \bigoplus F_n$ where F_n is a finite dimensional space. We now show that R_n is closed by an induction argument: $R_{n+1} = T(R_n) = T(R_n + N)$. From Lemma 8, R_{n+1} is closed if $R_n + N$ is. But $R_n + N = R_n \bigoplus F_n$ is closed since R_n is closed and F_n is finite dimensional. Thus $Y = \bigcap R_n$ is closed.

Next it will be shown that T maps Y onto itself. It will be sufficient to show that for some $m, R_k \cap N = R_m \cap N$ for $k \geqq m$; for then we complete the proof as in Theorem 9. For a proof of the contrapositive, suppose that for an infinite number of n there is a z_n in $R_n \cap N$ but not in $R_{n+1} \cap N$. Then $z_n = T^n u_n$, where u_n is not in R_1. But the u_n are linearly independent. For if $\Sigma_{k=1}^{K} a_k u_k = 0$, then taking T^K and recalling that z_k is in N, we get $a_K = 0$. Then recusively we get $a_k = 0$ for $k = K-1, \cdots, 1$. Thus the u_n form an infinite linearly independent set not in R_1. This contradicts the assumption that the condimension of R_1 is finite. Thus T maps Y onto itself as in the proof of Theorem 9.

Finally, we show that T is not one-one on Y exactly as in the proof of Theorem 9.

In conclusion, Y is a closed subspace of X which is invariant under T, and T maps Y onto itself but is not one-one on Y. Thus T does not have the single valued extension property.

The two theorems above can be summarized to say that: *if T is a semi-Fredholm operator and the point spectrum of T contains a neighborhood of zero, then T does not have the single valued extension property at $\lambda = 0$.*

The requirement that T be semi-Fredholm in the above theorems seems to be crucial. Even though the range is closed and the point spectrum contains a neighborhood of zero, if $n(T) = d(T) = \infty$, T may still have the single valued extension property. (Certain normal operators on a nonseparable Hilbert space will work). An attempt to extend the proofs of Theorems 9 and 10 to operators which are not semi-Fredholm must encounter the following two difficulties: The subspace $Y = \bigcap T^n X$ may fail to be closed, or T may not map Y onto itself.

In Theorem 2 it was shown that an operator T which is onto but not one-one does not have the single valued extension property. Such operators are semi-Fredholm operators with $n(T) \geqq 1$ and $d(T) = 0$. A rather natural extension of this theorem is now possible.

11. COROLLARY. *Let T be a closed linear operator with closed range mapping a Banach space X into itself. If the dimension $n(T)$ of the null space is strictly greater than the codimension $d(T)$ of the range, then T does not have the single valued extension property.*

Proof. From the theory of semi-Fredholm operators, for sufficiently small perturbations S, $n(T+S) - d(T+S) = n(T) - d(T)$ (see Kato, Theorem 5.22). Thus for small λ,

$$\begin{aligned} n(\lambda I - T) &= n(T) - d(T) + d(\lambda I - T) \\ &> d(\lambda I - t) \geqq 0. \end{aligned}$$

That is, $\lambda I - T$ is not one-one for λ sufficiently small. Thus T is a semi-Fredholm operator with point spectrum containing a neighborhood of zero. By Theorem 10, T does not have the single valued extension property.

One would like to show that if $n(T) < d(T)$, then T does have the single valued extension property (at least at 0). Unfortunately, this is not true. For, if S is the right shift on Hilbert space, then $n(S) = 0$ and $d(S) = 1$; if T is any operator, then $n(T \oplus S) = n(T)$ and $d(T \oplus S) = d(T) + 1$. In this way we may extend T to a new operator $T \oplus S \oplus \cdots \oplus S$ with d arbitrarily large and n fixed, without affecting the single valued extendibility (or lack of it).

For a closed linear operator T having a dense domain on a Banach space X there is a unique adjoint operator T^* defined on a total subset of the dual space X^*.

12. Corollary. *Let T be a semi-Fredholm operator on X with domain dense in X. If $n(T) < d(T) \leqq \infty$, then T^* does not have the single valued extension property at $\lambda = 0$.*

Proof. For semi-Fredholm operators,

$$n(T^*) = d(T), \quad \text{and} \quad d(T^*) = n(T).$$

Hence $d(T^*) < n(T^*)$, and by Theorem 11, T^* does not have the single valued extension property.

13. Corollary. *If T is a closed linear operator on a Banach space with dense domain and closed range, and if both T and T^* have the single valued extension property, then $n(T) = d(T)$.*

14. Corollary. *Let T be a closed linear operator on a Banach space with dense domain and with $n(T) = d(T)$ finite. Then T has the single valued extension property near $\lambda = 0$ if and only if T^* does.*

Proof. Since $d(T)$ is finite and T is a closed operator, it follows that the range is closed and hence T is a semi-Fredholm operator. If T does not have the single valued extension property near 0, then $n(\lambda I - T) > 0$ for λ in a neighborhood of zero. Then

$$\begin{aligned} d(\lambda I - T) &= d(T) - n(T) + n(\lambda I - T) \\ &= n(\lambda I - T) > 0. \end{aligned}$$

Thus $n(\lambda I^* - T^*) = d(\lambda I - T)$ is strictly positive in a neighborhood of zero. But then T^* is a Fredholm operator whose point spectrum contains an open set, and so by Theorem 10, T^* does not have the single valued extension property.

Conversely, suppose T does have the single valued extension property. Then from Theorem 10, $n(\lambda I - T) = 0$ in a deleted neighborhood of zero (using the fact that $n(\lambda I - T)$ is constant in a deleted neighborhood of a point where $\lambda I - T$ is semi-Fredholm). Hence $d(\lambda I - T) = 0$ in this neighborhood. This implies that T^* has the single valued extension property near $\lambda = 0$.

Two more concepts are useful at this point. Consider the iterates $T^k, k = 0, 1, 2, \cdots$, of the operator T. The null space $N(T^{k+1})$ always contains $N(T^k)$ and may be strictly larger. But if for some

$p, N(T^{p+1}) = N(T^p)$, then for all $k > p$, $N(T^{k+1}) = N(T^k)$. The smallest p satisfying the above is the *ascent* of T. It may happen that the equation is not satisfied for any p; in this case the ascent is infinite. The *descent* is defined in a similar way with the ranges of T instead of the null spaces. It is the smallest q with $T^{q+1}X = T^qX$, and is infinite if no such q exists.

15. THEOREM. *Let T be a semi-Fredholm operator on X. Then T has the single valued extension property near 0 if and only if the ascent of T is finite.*

Proof. If the ascent is finite, then certainly T has the single valued extension property near 0. For if $(\lambda I - T)f(\lambda) = 0$ and $f(\lambda) = \Sigma x_n \lambda^n$, then x_n is in $N(T^{n+1})$ but not in $N(T^n)$. Hence $N(T^{n+1}) \neq N(T^n)$ for any n.

Suppose that the ascent is infinite. Since T is semi-Fredholm, if the nullity $n(T)$ is infinite, then the deficiency $d(T)$ is finite; and, by Theorem 11, T does not have the single valued extension property. Thus assume that $n(T)$ is finite, and let

$$Y = \bigcap_{n=1}^{\infty} T^n X.$$

As was shown in the proof of Theorem 9, Y is a closed, invariant subspace, and T maps Y onto Y. Since $n(T) < \infty$, the null space N in finite dimensional, and so $(T^kX) \cap N$ is eventually constant. Since the ascent is infinite, $(T^kX) \cap N \neq (0)$, for all k. It then follows that $Y \cap N \neq (0)$; that is, T is not one-one on Y. From Corollary 3, T does not have the single valued extension property.

16. COROLLARY. *If T is a semi-Fredholm operator with domain dense in X, then T^* has the single valued extension property if and only if the descent of T is finite.*

Proof. Since the range of T^k is closed for all k (as was shown in the proofs of Theorems 9 and 10), the null space of T^{*k} is the set of x^* orthogonal to the range of T^k. Hence the ascent of T^* is the descent of T, and the conclusion follows by Theorem 15.

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REFERENCES

1. Errett Bishop, *A duality theory for arbitrary operators*, Pacific J. Math., **9** (1959), 379–397.
2. Ion Colojoară and Ciprian Foias, *Theory of Generalized Spectral Operators*, New York: Gordon and Breach, 1968.
3. Nelson Dunford, *Spectral operators*, Pacific J. Math., **4** (1954), 321–354.
4. Nelson Dunford and Jacob Schwartz, *Linear Operators*, New York: Interscience 1957.
5. Seymour Goldberg, *Unbounded Linear Operators; Theory and Applications*, New York: McGraw Hill, 1966.
6. Tosio Kato, *Perturbation Theory for Linear Operators*, New York: Springer-Verlag, 1966.
7. G. M. Keselman, *On the single-value analytic continubility of the resolvent of a bounded linear operator*, (Russian), Uspekhi Matematicheskikh Nauk, **17** (1962), 135–139.
8. D. G. Lay, *Spectral analysis using ascent, descent, nullity, and defect*, Mathematische Annalen, **184** (1967), 197–214.
9. Angus E. Taylor, *Theorems on ascent, descent, nullity, and defect of linear operators*, Mathematische Annalen, **163** (1966), 18–49.
10. F. H. Vasilescu, *Residual properties for closed operators on Fréchet spaces*, Illinois J. Math., **15** (1971), 377–386.

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A GOLDIE THEOREM FOR DIFFERENTIABLY PRIME RINGS

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The main goal of this paper is to prove analogues of the Goldie theorems for associative rings with derivations. It is shown that a differentiably prime ring, with suitable chain conditions, has a differentiably simple Artinian total ring of quotients, and, conversely, that a differential subring which is an order in a differentiably simple Artinian ring is a differentiably prime ring which has the chain conditions referred to above. A similar theorem concerning differentiably semiprime rings which are orders in differentiably semi-simple rings is also given.

Suppose that A is an associative ring and that D is a set of derivations of A, a *derivation* of A being any function d on A such that $d(a+b)=da+db$, and $d(ab)=(da)b+a(db)$ for all $a,b\in A$. An ideal I of A is a *D-ideal* provided $d(I)\subseteq I$ for all $d\in D$. A is *D-prime* provided $HI=0$ implies $H=0$ or $I=0$ for any D-ideals H and I of A. A is *D-semiprime* provided A has no nonzero nilpotent D-ideals. A is *D-semisimple* provided there are no nonzero D-ideals contained in the Jacobson radical of A. A is *D-simple* provided $A^2\neq 0$ and A has no proper D-ideals. Finally, A is said to be *differentiably prime* (resp. semiprime, semisimple, simple) provided A is D-prime (resp. semiprime, semisimple, simple) for some set of derivations D, hence for the set $D=\operatorname{der}A$ of all derivations of A.

The ring A is D-prime if and only if the left (right) annihilator of a nonzero left (right) D-ideal is zero. Another equivalent is that if $(\delta a)b(\delta' c)=0$ for all products δ,δ' of derivations in D, and for every $b\in A$, then $a=0$ or $c=0$. Other easy consequences of the definitions are that differentiably prime implies differentiably semiprime, and that differentiably simple implies differentiably prime.

It is straightforward to show, using arguments suitable for prime rings, that every differentiably prime ring has characteristic zero or a prime number. If A is a differentiably prime ring of characteristic zero, and satisfies a suitable chain condition (Corollary to Lemma 2 in §2 below), then A must be a prime ring. This is analogous to the fact that a differentiably simple ring of characteristic zero with a minimal ideal must be simple [3, Corollary to Thm. 4].

The ring Q is a *total ring of left quotients* for the ring A provided $A\subseteq Q$, every nonzero-divisor (regular element) in A is invertible in Q, and each $q \in Q$ can be written $q = b^{-1}a$ for some $a, b \in A$, with b a regular element in A. It will also be said that A is a subring and *left order* in Q. A has a total ring of left quotients if and only if A satisfies the (left) Öre condition: $a, b \in A$ with b regular implies there exist $c, e \in A$ with e regular such that $ea = cb$.

The following fact is crucial to the main theorems of this paper: Suppose that A has a total ring of left quotients Q, and that d is any derivation of A. Then d extends in a unique fashion to give a derivation $\hat{d}$ of Q. If $q \in Q$ is written $q = b^{-1}a$, then $\hat{d}q = -b^{-1}(db)b^{-1}a + b^{-1}(da)$. To prove this, it must be shown that $\hat{d}$ is well-defined, additive, and obeys the product rule on Q. This is tedious, but can be done using the Öre condition and the definition of $\hat{d}$. Note that $\hat{d}b^{-1} = -b^{-1}(db)b^{-1}$, which will be found to be necessary if $\hat{d}$ is applied to both sides of the equation $b^{-1}b = 1$, and one solves for $\hat{d}b^{-1}$. If A has a total ring of left quotients Q and D is a set of derivations of A, let $\hat{D} = \{\hat{d} \mid d \in D\}$ where $\hat{d}$ is the extension to Q as above.

A is a *left Goldie ring* provided A has no infinite ascending chain of left annihilators, and no infinite direct sum of left ideals. The two main theorems of this paper can now be stated. The proofs are in §2.

THEOREM 1. *Assume that A is a D-prime ring, that the nil radical N of A is nilpotent, that A satisfies the ascending chain condition on right annihilators, and that both A and A/N are left Goldie rings. Then A has a total ring of left quotients Q which is a $\hat{D}$-simple, left Artinian ring. Conversely, assume that A is a $\hat{D}$-closed subring and left order in the $\hat{D}$-simple, left Artinian ring Q. Then A is D-prime, where $D = \{\hat{d}|_A \mid \hat{d} \in \hat{D}\}$, the nil radical N of A is nilpotent, A satisfies the ascending chain condition on right annihilators, and both A and A/N are left Goldie rings.*

THEOREM 2. *Assume that A is a D-semiprime, left Noetherian ring. Then A has a total ring of left quotients Q which is a $\hat{D}$-semisimple, left Artinian ring. As a partial converse, if A is a $\hat{D}$-closed subring and left order in the $\hat{D}$-semisimple, left Artinian ring Q, then A is D-semiprime, where $D = \{\hat{d}|_A \mid \hat{d} \in \hat{D}\}$.*

If A has characteristic zero then the statements of Theorems 1 and 2 reduce to versions of the Goldie theorems. This observation uses the Corollary to Lemma 2 proved in §2. Of course, if D is the empty set, then the theorems again reduce to the Goldie theorems. It should be noted, however, that the Goldie theorems (and proofs contributed by various authors) are used in the proofs of Theorems 1 and 2.

A differentiably simple ring Q with a minimal ideal is either simple or there exists a simple ring S of prime characteristic, and a positive integer n, such that $Q \cong S \otimes B_n$ where the tensor product is over Z_p (p the characteristic of S) and B_n denotes the (commutative associative with unit) truncated polynomial algebra $Z_p[X_1, \cdots, X_n]/(X_1^p, \cdots, X_n^p)$, by [**1**, Main Theorem]. Hence Q is Artinian if and only if S is. An example of a differentiably prime, but not prime, ring would be one of the form $P \otimes B_n$ where P is a subring and left order in S.

A differentiably semisimple, left Artinian ring Q is a direct sum $Q = Q_1 \oplus \cdots \oplus Q_k$ where the Q_i are differential ideals of Q and each Q_i is a differentiably simple, left Artinian ring [**1**, Th. 8.2, Cor. 8.3]. Using this expression for Q and the type of example used for a differentiably prime ring in the previous paragraph, one can construct examples of differentiably semiprime rings which are not differentiably prime.

The left orders in a simple Artinian ring are characterized in the Faith-Utumi Theorem. In analogy, it could be asked whether a differential subring and left order in $S \otimes B_n$ *need* be of the form $P \otimes B_n$ where P is a subring and left order in S. The following is an example, for any prime characteristic p, of a commutative differential subring and order A in a differentiably simple Artinian ring of the form $B_1(E)$, where E is some field of characteristic p, such that A is not of the form $B_1(I)$ for any subring I of A which is an integral domain consisting of differential constants of A.

Let $B = Z_p[u, v]$, the polynomial ring in commuting indeterminants u and v, and let $A = B[(u/v)+x]$ considered as a subring of $B_1(E)$ where E is the field of quotients of B. Consider $B_1(E)$ as $E[x]$ where $x^p = 0$ and let $d = d/dx$ denote differentiation by x. Then A is a d-subring and order in $B_1(E)$, but straightforward calculation shows that A cannot be written $B_1(I)$ for any suitable I described at the end of the discussion in the preceding paragraph.

2. Proofs of the theorems.

LEMMA 1. *Assume that A is a D-semiprime ring and that the nil radical N of A is nilpotent. Then $a + N$ is regular in A/N implies that a is regular in A.*

LEMMA 2. *Assume that A is a D-prime ring, that the nil radical N of A is nilpotent, and that A has the ascending chain condition on left annihilators and on right annihilators. Then the ideal divisors of zero of A are nilpotent, A/N is a prime ring, and a regular in A implies that $a + N$ is regular in A/N.*

COROLLARY. *If, in addition to the hypotheses in Lemma 2, A is assumed to have characteristic zero, then A must be a prime ring.*

Proof of Lemma 1. Suppose that $a + N$ is regular in A/N but that $ab = 0$ for some b in A. Then $b \in N$, and, for any derivation d of A, $0 = d(ab) = (da)b + a(db)$, or $a(db) = -(da)b \in N$, and so $db \in N$, since again $a + N$ is regular in A/N. This proves the first step of the following induction. Suppose $\delta b \in N$ for every string of derivations $\delta = d_1 \cdots d_k$ of length k. If a string of derivations of length $k+1$ is written in the form $d\delta$ where d is a derivation and δ is a string of length k, then $0 = d\delta(ab) = \Sigma(\delta_1 a)(\delta_2 b) + a(d\delta b)$ where δ_1, δ_2 are strings of derivations of length at most k, hence each $\delta_2 b \in N$, and so $a(d\delta b) \in N$ and so $d\delta b \in N$. Thus, if δ is any string of derivations of A, then $\delta b \in N$. This implies that the differential ideal I of A generated by b is contained in N, hence that I is a nilpotent differential ideal of A. Since A is D-semiprime for some set of derivations D, it must be the case that $I = 0$, hence $b = 0$. A similar argument shows that $ba = 0$ implies that $b = 0$.

Proof of Lemma 2. Suppose that $aB = 0$, where a is a nonzero element of A, and B is an ideal of A. Then $B \supseteq B^2 \supseteq B^3 \supseteq \cdots$ is a descending chain of ideals of A and $\ell(B) \subseteq \ell(B^2) \subseteq \ell(B^3) \subseteq \cdots$ where $\ell(B)$ denotes the left annihilator of B in A. If U is the union of all the $\ell(B^i)$, then U is a differential ideal of A, since if $x \in \ell(B^i)$ then $dx \in \ell(B^{i+1})$ for any derivation d of A. Since A has the ascending chain condition on left annihilators, $U = \ell(B^k)$ for some k, and since $a \in \ell(B^k)$, $\ell(B^k) \neq 0$. But $\ell(B^k)B^k = 0$, and since A is D-prime for some set of derivations D, it must be the case that $B^k = 0$. A similar argument using the ascending chain condition on right annihilators shows that $Ba = 0$ implies B is nilpotent.

The proof that A/N is a prime ring is straightforward.

Now suppose that a is regular in A but that $ab \in N$ for some b in A. If $N = 0$, then $ab = 0$, so $b = 0$. If $N \neq 0$, then $abN^{k-1} = 0$ where $N^{k-1} \neq 0$, $N^k = 0$. Thus $bN^{k-1} = 0$ since a is regular, so b is contained in an ideal divisor of zero, hence $b \in N$. Similarly, $ba \in N$ implies $b \in N$.

Proof of Corollary. By [3, Proof of Thm. 4], if A is a primary ring (ideal divisors of zero are nilpotent) whose additive group is torsion free, then da lies in a nilpotent ideal if a does, for every derivation d of A. This would imply that N is a nilpotent differential ideal, hence $N = 0$, since A is D-prime for some set of derivations D. Thus $A \cong A/N$ is prime by Lemma 2.

Proof of Theorem 1. Assume the hypotheses on A in the statement of the theorem. The proof of the first part of the theorem is separated into parts.

(i) *The existence of Q.* For this it is shown that A satisfies the Öre condition. By Lemmas 1 and 2, it follows that $\bar{a} = a + N$ is regular in $\bar{A} = A/N$ if and only if a is regular in A, and that $\bar{A}$ is a prime Goldie ring. First we show that if $a, b \in A$ are *both* regular then there exist $c, e \in A$ with c regular such that $ea = cb$. Let $M = \{x \in A \mid xb \in Aa\}$. We claim that $\bar{M}$, the image of M in $\bar{A}$, is an essential left ideal of $\bar{A}$. Note that $\bar{M}$ is essential in $\bar{A}$ if and only if for every left ideal L of A, $M \cap L \subseteq N$ implies $L \subseteq N$. So suppose that $M \cap L \subseteq N$. Then $Aa \cap Lb \subseteq N$ since if $y \in Aa \cap Lb$, then $y = ua = vb$ or $v \in M \cap L \subseteq N$, thus $y = vb \in N$. But $\overline{Aa} = \bar{A}\,\bar{a}$ is essential in $\bar{A}$ by [2, Lemma 7.2.3, p. 174], since $\bar{a}$ is regular in $\bar{A}$, and $\bar{A}$ is a prime Goldie ring. Hence $Lb \subseteq N$. Since $\bar{b}$ is regular in A, $L \subseteq N$. Therefore $\bar{M}$ is an essential left ideal in $\bar{A}$, hence there exists a $\bar{c} \in \bar{M}$ with $\bar{c}$ regular in $\bar{A}$ by [2, Lemma 7.2.5, p. 175]. Thus there exists a c regular in A such that $cb = ea$ for some $e \in A$.

Next, the above is used to show that the full Öre condition holds for A. Suppose $a, b \in A$ with b regular in A. Since $\bar{A}$ is a prime Goldie ring, and hence satisfies the Öre condition, there exist $\bar{e}, \bar{c}$ in $\bar{A}$ with $\bar{e}$ regular in $\bar{A}$, so e is regular in A, such that $\bar{e}\,\bar{a} = \bar{c}\,\bar{b}$ or $ea = cb + n$ for some $n \in N$. Let $u = c - e$, and write $ea = ub + (eb + n)$. Now $(eb + n) + N = eb + N$ is regular in $\bar{A}$ so $eb + n$ is regular in A. Using the preceding paragraph, there are r, s in A with r regular in A such that $r(eb + n) = sb$. Hence $rea = rub + r(eb + n) = rub + sb = (ru + s)b$ or $(re)a = (ru + s)b$ which gives the Öre condition since re is regular in A.

(ii) *If N_1 is the nil radical of Q, then $N_1 = QN$ and $N = N_1 \cap A$.* $N_1 \cap A$ is a nil ideal of A so $N_1 \cap A \subseteq N$. Since $(QN)^k \subseteq QN^k$ and N is nilpotent, QN is nilpotent, thus $QN \subseteq N_1$, and so $N \subseteq QN \subseteq N_1$. Hence $N = N_1 \cap A$. Now $N_1 \subseteq QN$ since if $b^{-1}m \in N_1$, then $b(b^{-1}m) = m \in N_1 \cap A = N$, and so $b^{-1}m \in QN$. Hence $N_1 = QN$.

(iii) *A/N is a prime Goldie ring whose total ring of left quotients is isomorphic to Q/N_1. Hence Q/N_1 is a simple, left Artinian ring.* Let $Q(A/N)$ denote the total ring of left quotients of A/N, and let ϕ be the map from Q to $Q(A/N)$ defined by $\phi(b^{-1}a) = (b + N)^{-1}(a + N)$. It is straightforward to show that ϕ is well-defined and a homomorphism onto $Q(A/N)$ with kernel N_1.

(iv) *Q is $\hat{D}$-simple.* Suppose that I is a nonzero $\hat{D}$-ideal of Q. Then $I \cap A$ is a nonzero D-ideal of A. $I \cap A \not\subseteq N$ since A is

D-prime. Hence $I \not\subseteq N_1$. Thus $(I + N_1)/N_1$ is a nonzero ideal of Q/N_1. Since Q/N_1 is simple, $I + N_1 = Q$. If $I \neq Q$ then I contains only non-invertible elements of Q. From the statement $Q = I + N_1$ it will be shown that no element of Q is invertible, which is not possible, hence $I = Q$. Now suppose that $x \in I + N_1$ is invertible in Q, $I \neq Q$. Write $x = a^{-1}b + n$ where $n \in N_1$ and $a^{-1}b \in I$. Since $a^{-1}b$ is not invertible, b is not regular in A. So there exists a nonzero z in A such that $bz = 0$ or $zb = 0$. The element $ax = b + an$ must also be invertible. Using $bz = 0$, $axz = anz \in N_1$. Since ax is invertible, $z \in N_1$. Let z^k be the smallest nonzero power of z such that $anz^k = 0$. Then $axz^k = bz^k + anz^k = 0$, which is impossible since ax is invertible and $z^k \neq 0$. A similar argument can be given for the case $zb = 0$. Hence no element of $I + N_1$ can be invertible if $I \neq Q$.

(v) *Q is left Artinian.* This will be proved by showing that Q has a minimal ideal. For in that case Q is a differentiably simple ring with a minimal ideal. Thus by [**1**, Main Theorem], Q must either be simple, or there exists a simple ring S of prime characteristic and a positive integer n such that $Q \cong S \otimes B_n$. If Q is simple, then $N = N_1 = 0$, so $Q \cong Q/0 \cong Q(A/0)$ is left Artinian since in this case A is a prime ring. On the other hand, if $Q = S \otimes B_n$, then $S \cong Q/N_1$, so S is Artinian by (iii), and so $Q = S \otimes B_n$ is also left Artinian.

It can be assumed that Q is not simple. Hence $H = \{q \in Q \mid N_1 q = 0\}$ contains a nonzero element m. Then QmQ is a nonzero left Q/N_1-module using the action $\bar{q}y = qy$ for all $q \in Q, y \in QmQ$. This action is well-defined since $N_1(QmQ) = 0$. Moreover, the Q/N_1-submodules of QmQ are just those left ideals of Q contained in QmQ. Q/N_1 is a simple, left Artinian ring, so QmQ is a completely reducible left Q/N_1-module. Hence QmQ is a direct sum of minimal left ideals of Q. But Q contains no infinite direct sum of left ideals since A does not. Hence QmQ is a finite direct sum of minimal left ideals of Q, so QmQ is an Artinian left Q/N_1-module. Therefore, QmQ, being an ideal of Q itself, must contain a minimal ideal of Q.

Now assume the hypotheses on Q and A stated in the converse of the theorem. Then A is D-prime by an argument similar to one that can be given to show that a subring and left order in a simple Artinian ring must be prime, as in [**2**, Thm. 7.2.3, p. 177]. That the nil radical N of A is nilpotent, and that both A and A/N are Goldie rings follows from [**4**, Part II, §2]. Since Q is left Artinian, Q satisfies the descending chain condition on left annihilators, hence so does A since A is a left order in Q. Thus A satisfies the ascending chain condition on right annihilators. This shows that A must satisfy all of the chain conditions used in the first part of the theorem.

Proof of Theorem 2. If A is a D-semiprime, left Noetherian ring then, using Lemma 1 and [**4**, Part II, §1], A has a left Artinian total ring of left quotients Q.

To show that Q is $\hat{D}$-semisimple, suppose that H is a $\hat{D}$-ideal of Q contained in the Jacobson radical of Q. Since Q is left Artinian, the Jacobson radical of Q is nilpotent. Thus H is a nilpotent $\hat{D}$-ideal of Q. Hence $H \cap A$ is a nilpotent D-ideal of A, and so $H \cap A = 0$. Consequently, $H = 0$, and thus Q is $\hat{D}$-semisimple.

For the partial converse, assume that A is a $\hat{D}$-closed subring and left order in the $\hat{D}$-semisimple, left Artinian ring Q, and let $D = \{d|_A \,|\, d \in \hat{D}\}$. Then $Q = Q_1 \oplus \cdots \oplus Q_k$ where the Q_i are $\hat{D}$-ideals of Q and $\hat{D}$-simple rings, by [**1**, Thm. 8.2, Cor. 8.3], so that any $\hat{D}$-ideal of Q must be a sum of some of the Q_i. To show now that A must be D-semiprime, use an argument just like one that can be given to show that a subring and left order in a semisimple, left Artinian ring must be semiprime [**2**, Thm. 7.2, p. 177], only using differential ideals instead of ordinary ideals.

REFERENCES

1. R. E. Block, *Determination of the differentiably simple rings with a minimal ideal*, Annals of Math., **90** (1969), 433–459.

2. I. N. Herstein, *Noncommutative Rings*, MAA Carus Monographs, # 15, 1968.

3. E. C. Posner, *Differentiably simple rings*, Proc. Amer. Math. Soc., **11** (1960), 337–343.

4. L. W. Small, *Orders in Artinian rings*, J. Algebra, **4** (1966), 13–41.

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ON ONE-SIDED PRIME IDEALS

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This paper contains some results on prime right ideals in weakly regular rings, especially V-rings, and in rings with restricted minimum condition. Theorem 1 gives information about the structure of V-rings: A V-ring with maximum condition for annihilating left ideals is a finite direct sum of simple V-rings. A characterization of rings with restricted minimum condition is given in Theorem 2: A nonprimitive right Noetherian ring satisfies the restricted minimum condition iff every critical prime right ideal $\neq(0)$ is maximal. The proof depends on the simple observation that in a nonprimitive ring with restricted minimum condition all prime right ideals $\neq(0)$ contain a (two-sided) prime ideal $\neq(0)$. An example shows that Theorem 2 is not valid for right Noetherian primitive rings. The same observation on nonprimitive rings leads to a sufficient condition for rings with restricted minimum condition to be right Noetherian.

It remains an open problem whether there exist nonnoetherian rings with restricted minimum condition (clearly in the commutative case they are Noetherian).

Theorem 1 is a generalization of the well known: A right Goldie V-ring is a finite direct sum of simple V-rings (e.g., **[2]**, p. 357). Theorem 2 is a noncommutative version of a result due to Cohen **[1**, p. 29]. Ornstein has established a weak form in the noncommutative case **[11**, p. 1145].

In §§1, 2, 3 the unity in rings is assumed (except in Proposition 2.1), but most of the results are valid for rings without unity, as shown in §4.

Of basic importance for the following is Lambek's and Michler's work **[8]** from which also most notions have been taken.

1. Basic concepts. A right ideal L of the ring R is called *prime* (*semiprime*), if $xRy\,(xRx)\subseteq L$ implies $x\in L$ or $y\in L\,(x\in L)$ for all $x,y\in R$. As in the two-sided case an equivalent definition is: $AB\subseteq L\,(A^2\subseteq L)$ implies A or $B\subseteq L\,(A\subseteq L)$ for all right ideals $A,B\subseteq R$.

A right ideal M of R is called *critical*, if R/M is a "supporting module" [**3**, p. 35] for a hereditary torsion theory, i.e., if there exists a hereditary torsion theory (T,F) such that $R/M\in F$, but $R/N\in T$ for all $N \supsetneqq M$. An important among under the critical right ideals are the 1-*critical* ones. A right ideal L is 1-critical, if R/L is not Artinian, but R/N for every $N \supsetneqq L$. It is easy to see that then R/L is a supporting module for the hereditary torsion theory, generated by the class of simple R-modules.

A ring R satisfies the *proper restricted minimum condition*, if (0) is a 1-critical right ideal of R. If additionally R is allowed to be right Artinian, R satisfies the *restricted minimum condition.*

A right ideal L of a ring R is *irreducible* (*indecomposable*), if every submodule $\neq (0)$ of R/L is essential (if R/L is indecomposable). Every critical right ideal is irreducible. The following facts are needed:

(a) Every prime (semiprime) right ideal L contains a two-sided prime (semiprime) ideal, namely $R^{-1}L$. The proof is not hard.

(b) If L is an irreducible right ideal of the right Noetherian ring R, there is a critical prime right ideal of the form $s^{-1}L$, $s \in R \setminus L$ [**8**, p. 370].

(c) Every irreducible semiprime right ideal is prime [**8**, p. 370].

(d) Any ring R with proper restricted minimum condition is a right Öre domain [**11**, p. 1149].

For a subset T of a ring R and a right or left ideal $L \subseteq R$ one defines $T^{-1}L := \{x \in R;\ Tx \subseteq L\}$, $LT^{-1} := \{x \in R;\ xT \subseteq L\}$.

2. Weakly regular rings and V-rings.

LEMMA 2.1. *The following conditions for a ring R are equivalent:*

(a) *$L = L^2$ for every right ideal L of R.*

(b) *Every right ideal is semiprime.*

(c) *Every right ideal is the intersection of prime right ideals.*

A ring satisfying these equivalent conditions is called weakly regular.

Proof. (a) $\Rightarrow$ (b). Let N be any right ideal. $xRx \subseteq N$ implies $xRxR \subseteq N$, so $xR \subseteq N$, i.e. $x \in N$.

(b) $\Rightarrow$ (c). Every right ideal is the intersection of irreducible semiprime right ideals. In view of §1, (c) these are prime.

(c) $\Rightarrow$ (a). The intersection of prime right ideals is semiprime. L^2 semiprime implies $L^2 = L$.

Besides the regular rings the best known weakly regular ones are the simple rings and the *V-rings.* A ring R is called V-ring, if every homomorphic image of R_R has zero radical. Equivalently all simple R-modules are injective (see (10)).

PROPOSITION 2.1. *A ring R with maximum condition for annihilating left ideals and $x \in xR$ for all $x \in R$ has a right unity.*

Proof. Let be $L_e = \{x - ex\,;\, x \in R\}$, $e \in R$; $0L_e^{-1}$ is maximal for some e. Assume $0L_e^{-1} \subsetneq R$, i.e. $yL_e \neq (0)$ for some $y \in R$. Then $(y - ye)R \neq (0)$ and so $y - ye \neq 0$. There exists an $e' \in R$, such that $y - ye = (y - ye)e'$; $y = yf$ with $f = e + e' - ee'$, so $y \in 0L_f^{-1}$; $z \in 0L_e^{-1}$ implies $z = ze$, thus $z = zf$, i.e. $z \in 0L_f^{-1}$. Altogether one has $0L_e^{-1} \subsetneq 0L_f^{-1}$, but this is impossible. Because of $0L_e^{-1} = R$, $(r - re)R = (0)$ for all $r \in R$, that means e is a right unity.

The proof is essentially the same as the one of A. Kertész [**5**, p. 237] for: A left Noetherian ring has a right unity, iff $x \in xR$ for all $x \in R$.

PROPOSITION 2.2. *A ring R with maximum condition for annihilating left ideals is weakly regular, iff it is a finite direct sum of simple rings.*

Proof. Let I be an ideal of R and L a right ideal of the ring I. Then $LR = LRLR \subseteq LI \subseteq L$, i.e., every right ideal of I is a right ideal of R. Let be $0 \neq x \in I, Z$ be the ring of integers. $Zx + xI$ is a right ideal of I, therefore of R. Now $x \in Zx + xI = (Zx + xI)^2 = Zx^2 + x^2I + xIx + xIxI = xI$. Because of Proposition 2.1 there is an idempotent $e \in I$ with $Ie = Re = I$; $ex = xe$ for all $x \in R$, otherwise there were an $y \in R$ with $I \ni z = ey - ye \neq 0$, thus $eze = ez = 0$. Therefore $(0) = RezR = IzR \supseteq (zR)^2 = zR$. This is a contradiction. It follows that the second summand in $R = I \oplus R(1 - e)$ is an ideal, and the assertion is true because of a basic argument.

COROLLARY 2.1. *A weakly regular ring with maximum condition for annihilating left ideals is a simple ring, iff it is prime.*

According to (7) the simple rings are just these ones, in which every right ideal is prime.

A consequence of Proposition 2.2 is

THEOREM 1. *A V-ring R with maximum condition for annihilating left ideals is a finite direct sum of simple V-rings.*

COROLLARY 2.2 [**2**, p. 357]. *A right Goldie V-ring R is a finite direct sum of simple rings.*

Proof. R satisfies the maximum condition for annihilating left ideals [e.g. **4**, p. 173].

The same argument holds for left Goldie (right) V-rings.

3. Rings with restricted minimum condition.

LEMMA 3.1. *If R is a ring with restricted minimum condition, R is primitive or every indecomposable prime right ideal $\neq (0)$ is maximal.*

Proof. Let L be any prime right ideal $\neq (0)$. As R/L is Artinian, it contains a minimal submodule E/L. If R is not primitive, $Er \subseteq L$ holds for some $0 \neq r \in R$, i.e. $E^{-1}L \neq (0)$; $xR + L = E$ for some $x \in E \setminus L$; $E^{-1}L \subseteq L$, for $y \in E^{-1}L$ implies $(xR + L)y = xRy + Ly \subseteq L$, thus $y \in L$; $L \supseteq R^{-1}L = E^{-1}L \neq (0)$, because every ideal contained in L is contained in $R^{-1}L$; the other inclusion is obvious. By §1, (a) $E^{-1}L$ is a prime ideal, so maximal, as $R/E^{-1}L$ is an Artinian ring. As L is indecomposable, R/L is an indecomposable $R/E^{-1}L$-module, thus simple. This means, L is a maximal right ideal of R.

LEMMA 3.2. *If R is a right Noetherian ring and every critical prime right ideal $\neq (0)$ maximal, R satisfies the restricted minimum condition.*

Proof. Suppose the assertion is false. Then there is a right ideal $L \neq (0)$, maximal with respect to R/L is nonartinian. Therefore L is 1-critical. Because of §1, (b) a critical prime right ideal N of the form $N = s^{-1}L$, $s \notin L$, exists. $R/s^{-1}L \cong (sR + L)/L$ is nonartinian. This is a contradiction, as in a right Noetherian ring every right ideal of the form $s^{-1}L$, $L \neq (0)$, $s \notin L$, is different from zero.

THEOREM 2. *In a nonprimitive right Noetherian ring R the following conditions are equivalent:*

(a) *Every critical prime right ideal $\neq (0)$ is maximal.*
(b) *Every irreducible prime right ideal $\neq (0)$ is maximal.*
(c) *Every indecomposable prime right ideal $\neq (0)$ is maximal.*
(d) *R satisfies the restricted minimum condition.*

REMARK. Compare it with Theorem 3.6 (a noncommutative version of the Krull-Akizuki Theorem) in [**8**, p. 373].

The following example shows that there are primitive right Noetherian rings which satisfy the restricted minimum, but contain a critical nonmaximal prime right ideal $\neq (0)$: Let $K(z)$ be the ring of rational functions over a field K with $\mathrm{Char}(K) = 0$ and R be the ring of differential polynomials in one indeterminate x over $K(z)$, i.e. $xf = fx + f', f \in K(z)$; R is known to be a simple principal left and right ideal domain. Hence it is a ring with proper restricted minimum condition [see **11**, p. 1149]. Since every right ideal is prime, it is enough to find a critical nonmaximal right ideal. $L = (z + x)xR$ is not maximal; it is properly contained only in R and $(z + x)R$; for $(g_0 + g_1x)(h_0 + h_1x) = (z)x + x^2, g_0, g_1 \neq 0, h_0, h_1 \in K(z)$, leads to $hk + k' = 0$, $h + k = z$ with $h = (g_0 - g_1')g_1^{-1}$, $k = g_1h_0$. The only solution in $K(z)$ is $h = z$, $k = 0$. It follows $(g_0 + g_1x)R = (z + x)R$. To show that R/L is a supporting module for the torsion theory generated by the simple module $E = R/(z + x)R$, $E \not\cong F = R/xR$ must be proved: There is an element $e \neq 0$ in F, e.g., $e = 1 \bmod(xR)$, with $ex = 0$, but no such an element in E; it can be checked by similar methods as above.

In (7) and (9) it was stated: Any ring is right Noetherian iff every prime right ideal is finitely generated. An easy consequence of this is:

PROPOSITION 3.1. *A nonprimitive ring R with restricted minimum condition is right Noetherian iff the square of every principal right ideal is finitely generated.*

Proof. If R is Artinian, it is always Noetherian. Alternatively R satisfies the restricted minimum condition. It was already proven (Lemma 3.1) that every prime right ideal $L \neq (0)$ contains an ideal $RxR \neq (0)$. L modulo RxR is finitely generated, as R/xR is right Noetherian. RxR itself is a finitely generated right ideal, because $RxR \cong (xR)^2$ as R-modules; thus L is finitely generated, too. The other direction is trivial.

4. The results for rings without unity. Clearly the definition of a prime (semiprime, irreducible, indecomposable) right ideal is the same as in rings with unity. Torsion theories on Mod-R can be regarded as torsion theories on Mod-R_1, where R_1 denotes the usual unitary overring of R. So the other concepts are defined, too. Setting $R^{-1}L \cap L$ instead of $R^{-1}L$, §1, (a) remains true. (b) only holds for rings with unity, (c) only for rings R with $x \in xR$, $x \in R$. (d) can be generalized in the following way:

PROPOSITION 4.1. *Any ring R with $R^2 \neq (0)$ and proper restricted minimum condition is a right Öre domain and can be imbedded as an ideal into a unitary ring with proper restricted minimum condition.*

Proof. Let be $0 \neq x \in R$ and $(0) \neq \{x\}^{-1}0$. Then $xR \cong R/\{x\}^{-1}0$ is an Artinian R-module, so must be zero (otherwise R/xR Artinian implies R Artinian), i.e. $0R^{-1} \neq (0)$. The intersection of any two nonzero right ideals M, L cannot be zero by a similar argument. It remains to show that $R^2 = (0)$, if $0R^{-1} \neq (0)$: $0R^{-1}$ is a trivial R-module, hence as a group nonartinian with Artinian proper homomorphic images. It is easy to see that $0R^{-1}$ must be some proper subgroup of Q, the additive group of the rational numbers. Every $0 \neq r \in 0R^{-1}$ defines a group homomorphism $R/0R^{-1} \to 0R^{-1}$ by $\bar{x} \to xr$. It obviously is well defined and monic, so the additive group of $0R^{-1}$ is a subgroup of Q. No proper subgroup $\neq (0)$ of Q is the additive group of an Artinian ring [e.g., **5**, p. 225], hence $R/0R^{-1}$ must be zero, i.e., $R^2 = (0)$.

Let K be the (right) quotient field of R and E be the subring generated by 1; $E \cong Z$ or $E \cong Z(p)$; R is a right essential ideal in $S = E + R$, and every right ideal of R is a right ideal of S. If $\mathrm{Char}(K) = 0$, then $xR \cap xE \neq (0)$ for all $0 \neq x \in R$, otherwise the trivial right R-module $(xE + xR)/xR \cong xE/(xR \cap xE)$ were not Artinian. Thus there is an $r \in R$ and an $0 \neq n \in E$ such that $xr = xn$; hence $0 \neq r = n \in R \cap E$, and $S/R \cong E/(R \cap E)$ is finite, especially an Artinian S-module. The same follows immediately, if $\mathrm{Char}(K) = p$. $(R + L)/L \cong R/(L \cap R) \neq R$ is Artinian for all right ideals L of S, likewise $S/(R + L)$. So S/L is an Artinian S-module, and the assertion is proved.

Lemma 2.1, Proposition 2.2 and Corollary remain true for rings without unity, as easily can be checked. Because of Proposition 2.1 in Proposition 2.2 the existence of the unity follows necessarily.

There is a great difference between V-rings with and without unity. The latter are not weakly regular in general (and the simple modules need not be injective), as is shown by the ring R with four elements and exactly two right unities; it also is a counterexample that the next Theorem remains valid for $0R^{-1} = (0)$.

THEOREM 1′. *A V-ring R with $R^{-1}0 = (0)$ and maximum condition for annihilating left ideals is a finite direct sum of simple V-rings with unity.*

Proof. $R^2 \subseteq M$ for any maximal right ideal $M \supseteq R^3$, otherwise $R^2 + M = R$ implies $M \supseteq R^3 + MR = R^2$. So $R^3 = R^2 =: S$. The ring S has a unity: $xS = \bigcap N$, where N denotes the set of maximal R-submodules of S containing xS; $S \supseteq S \cap NS^{-1} \supseteq N$; $S \cap NS^{-1} \neq S$, as $S = S^2 \not\subseteq N$, therefore $S \cap NS^{-1} = N$. Now $x \in S \cap (xS)S^{-1} = \bigcap (S \cap NS^{-1}) = \bigcap N = xS$; this holds for all $x \in S$. Because of Proposition 2.1 S contains a right unity e. If $y = er - re \neq Q$, $0 = Seye = Sey = Sy$. This is impossible, as $S \cap S^{-1}0 \subseteq R^{-1}(R^{-1}0) = 0$. So e is the unity of S. Now $R = S \oplus R(1-e)$, and the ideal $R(1-e)$ is isomorphic to R/R^2 as a ring. $R(1-e) = (0)$, as $R(1-e)R(1-e) = R(R(1-e)) = (0)$ and $R^{-1}0 = (0)$. Theorem 1 completes the proof.

Proposition 3.1 remains true for rings R with $R^2 \neq (0)$. This follows from Proposition 4.1 and from the fact that the unitary overring S of R is nonprimitive, right Noetherian with restricted minimum condition and the square of every principal right ideal finitely generated iff R is.

REFERENCES

1. I. S. Cohen, *Commutative rings with restricted minimum condition*, Duke Math. J., **17** (1950), 27–42.
2. C. Faith, *Algebra: Rings, Modules and Categories I*, Berlin-Heidelberg-New York, 1973.
3. O. Goldman, *Rings and modules of quotients*, J. Algebra, **13** (1969), 10–47.
4. I. N. Herstein, *Noncommutative Rings*, Carus Math. Mon., 1968.
5. A. Kertész, *Vorlesungen über artinsche Ringe*, Budapest, 1968.
6. K. Koh, *Quasisimple modules and other topics in ring theory*, Lectures on rings and modules, 323–426, Berlin-Heidelberg-New York, 1972.
7. K. Koh, *On prime one-sided ideals*, Canad. Math. Bull., **14** (1971), 259–260.
8. J. Lambek-G. Michler, *The torsion theory at a prime ideal of a right Noetherian ring*, J. Algebra, **25** (1973), 364–389.
9. G. Michler, *Prime right ideals and right Noetherian rings*, Ring Theory, 251–255, London 1972.
10. G. Michler-O.E. Villamayor, *On rings whose simple modules are injective*, J. Algebra, **25** (1973), 185–201.
11. A. J. Ornstein, *Rings with restricted minimum condition*, Proc. Amer. Math. Soc., **19** (1968), 1145–1150.
12. V. S. Ramamurthi, *Weakly regular rings*, Canad. Math. Bull., **16** (1973), 317–321.

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PRODUCT INTEGRALS AND THE SOLUTION OF INTEGRAL EQUATIONS

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Functions are from R to N or $R \times R$ to N, where R denotes the set of real numbers and N denotes a normed complete ring. If $\beta > 0$, H and G are functions from $R \times R$ to N, f and h are functions from R to N, each of H, G and dh has bounded variation on $[a, b]$ and $|H| < 1 - \beta$ on $[a, b]$, then the following statements are equivalent:

(1) f is bounded on $[a, b]$, each of $\int_a^b H$, $\int_a^b G$ and $(LR)\int_a^b (fG + fH)$ exists and

$$f(x) = h(x) + (LR)\int_a^x (fG + fH)$$

for $a \leqq x \leqq b$, and

(2) each of ${}_x\Pi^y(1 + \sum_{j=1}^{\infty} H^j)$, ${}_x\Pi^y(1 + G)$ and

$$(R)\int_x^y dh\left(1 + \sum_{j=1}^{\infty} H^j\right) {}_s\Pi^y(1 + G)\left(1 + \sum_{j=1}^{\infty} H^j\right)$$

exists for $a \leqq x < y \leqq b$ and

$$f(x) = h(a)\,{}_a\Pi^x(1 + G)\left(1 + \sum_{j=1}^{\infty} H^j\right) + (R)\int_a^x dh\left(1 + \sum_{j=1}^{\infty} H^j\right) {}_s\Pi^x(1 + G)\left(1 + \sum_{j=1}^{\infty} H^j\right)$$

for $a \leqq x \leqq b$.

This result is obtained without requiring the existence of integrals of the form

$$\int_a^b \left| G - \int G \right| = 0 \quad \text{and} \quad \int_a^b |1 + G - \Pi(1 + G)| = 0.$$

This article is part of a sequence of results on the solution of integral equations initiated by two papers by H. S. Wall **[28][29]** on continuous continued fractions and harmonic matrices. He studied certain techniques for solving integral equations which are associated with product integration and his results have been extended in various directions by J. S. MacNerney **[18][19][20][21][22]**, J. W. Neuberger

[24][25][26], T. H. Hildebrandt [13], J. R. Dorroh [4], B. W. Helton [5][6][7], D. B. Hinton [14], J. V. Herod [11], C. W. Bitzer [2][3], D. L. Lovelady [16][17] and J. A. Reneke [27]. The results here connect closely with those of B. W. Helton [5, §5, pp. 307–315].

B. W. Helton [5, Theorem 5.1, p. 310] solved the integral equation

$$\text{(a)} \qquad f(x) = h(x) + (LR)\int_a^x (fG + fH)$$

by using product integral techniques. In his development, the existence of integrals of the form

$$\text{(b)} \qquad \int_a^b \left|G - \int G\right| = 0 \quad \text{and} \quad \int_a^b |1 + G - \Pi(1+G)| = 0$$

plays an important part. For real valued functions, A. Kolmogoroff [15, p. 669] has shown that if $\int_a^b G$ exists, then $\int_a^b \left|G - \int G\right|$ exists and is zero. Further, W. D. L. Appling [1, Theorem 2, p. 155] and B. W. Helton [5, Theorem 4.1, p. 304] have shown that there exist other classes of functions such that the existence of $\int_a^b G$ is sufficient to assure that $\int_a^b \left|G - \int G\right|$ exists and is zero. Also, B. W. Helton [5, Theorem 4.2, p. 305] has shown that for some settings the existence of ${}_x\Pi^y(1+G)$ for $a \leqq x < y \leqq b$ is sufficient to assure that $\int_a^b |1 + G - \Pi(1+G)|$ exists and is zero. However, it has been shown by W. D. L. Appling [1, Theorem 2, p. 155] and the author [8, pp. 153–154] that the existence of $\int_a^b G$ and ${}_x\Pi^y(1+G)$ for $a \leqq x < y \leqq b$ is not sufficient to imply the existence of the integrals in (b). In the following, we solve the integral equation in (a) without requiring the existence of the integrals in (b).

All integrals and definitions are of the subdivision-refinement type, and functions are from either R to N or $R \times R$ to N, where R denotes the set of real numbers and N denotes a ring which has a multiplicative identity element represented by 1 and a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. Lower case letters are used to denote functions from R to N, and capital letters are used to denote functions from $R \times R$ to N. Unless noted otherwise, functions on $R \times R$ are assumed to be defined only for elements $\{a, b\}$ of $R \times R$ such that $a < b$. If $D = \{x_q\}_{q=0}^n$ is a subdivision of $[a, b]$, then $D(I) = \{[x_{q-1}, x_q]\}_{q=1}^n$, $f_q = f(x_q)$ and $G_q = G(x_{q-1}, x_q)$. Further, $\{x_{qr}\}_{r=0}^{n(q)}$ represents a subdivision of $[x_{q-i}, x_q]$ and $G_{qr} = G(x_{q,r-l}, x_{qr})$.

The statement that $\int_a^b G$ exists means there exists an element L of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a,b]$ such that if J is a refinement of D, then

$$\left| L - \sum_{J(I)} G \right| < \epsilon.$$

The statement that ${}_a\Pi^b(1+G)$ exists means there exists an element L of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a,b]$ such that if J is a refinement of D, then

$$\left| L - \prod_{J(I)} (1+G) \right| < \epsilon.$$

The statement $(LR)\int_a^b (fG + fH)$ exists means $\int_a^b C$ exists, where

$$C(r,s) = f(r)G(r,s) + f(s)H(r,s).$$

We adopt the conventions that

$$\int_a^a G = 0 \quad \text{and} \quad {}_a\Pi^a(1+G) = 1.$$

Further,

$$\sum_{q=i}^{j} G_q = 0 \quad \text{and} \quad \prod_{q=i}^{j} (1+G_q) = 1,$$

where $i > j$.

The statements that G is bounded on $[a,b]$, $G \in OP^\circ$ on $[a,b]$ and $G \in OB^\circ$ on $[a,b]$ mean there exist a subdivision D of $[a,b]$ and a number B such that if $\{x_q\}_{q=0}^n$ is a refinement of D, then

(1) $|G_q| < B$ for $q = 1, 2, \cdots, n$,

(2) $|\Pi_{q=i}^j (1+G_q)| < B$ for $1 \leqq i \leqq j \leqq n$, and

(3) $\Sigma_{q=1}^n |G_q| < B$,

respectively. Similarly, statements of the form $|G| < \beta$ are to be interpreted in terms of subdivisions and refinements. Observe that every function in OB° is also in OP°.

The statement that $G \in OM^*$ on $[a,b]$ means ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$ and if $\epsilon > 0$ then there exists a subdivision D of $[a,b]$ such that if $\{x_q\}_{q=0}^n$ is a refinement of D and $0 \leqq p < q \leqq n$, then

$$\left| {}_{x_p}\Pi^{x_q}(1+G) - \Pi_{i=p+1}^{q}(1+G_i) \right| < \epsilon.$$

Also, $G \in OL^\circ$ on $[a,b]$ only if $\lim_{x\to p^+} G(p,x)$, $\lim_{x\to p^-} G(x,p)$, $\lim_{x,y\to p^+} G(x,y)$ and $\lim_{x,y\to p^-} G(x,y)$ exist for $a \leqq p \leqq b$, and $G \in OA^\circ$ on $[a,b]$ only if $\int_a^b G$ exists and $\int_a^b \left| G - \int G \right|$ exists and is zero. For additional background with respect to this paper, see work by B. W. Helton **[5]** **[6]** and J. S. MacNerney **[20]**. Further, additional background on product integration is given by P. R. Masani **[23]**.

LEMMA 1. *If G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a,b]$, then $\int_a^b G$ exists if and only if ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$* [**10**, Theorem 4].

LEMMA 2. *If H and G are functions from $R \times R$ to N, $H \in OL^\circ$ on $[a,b]$, $G \in OB^\circ$ on $[a,b]$ and either $\int_a^b G$ exists or ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $\int_a^b HG$ and $\int_a^b GH$ exist and ${}_x\Pi^y(1+HG)$ and ${}_x\Pi^y(1+GH)$ exist for $a \leqq x < y \leqq b$* [**10**, Theorem 5].

LEMMA 3. *If G is a function from $R \times R$ to N, $G \in OB^\circ$ on $[a,b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $G \in OM^*$ on $[a,b]$* [**10**, Theorem 1].

LEMMA 4. *If $\epsilon > 0$, H is a function from $R \times R$ to N and $H \in OL^\circ$ on $[a,b]$, then there exist a subdivision $\{t_i\}_{i=0}^p$ of $[a,b]$ and a sequence $\{k_j\}_{j=1}^p$ such that if $1 \leqq j \leqq p$ and $t_{j-1} < x < y < t_j$, then*

$$|H(x,y) - k_j| < \epsilon$$

[**6**, Lemma, p. 498].

LEMMA 5. *If H and G are functions from $R \times R$ to N, $H \in OL^\circ$ on $[a,b]$ and $G \in OA^\circ$ and OB° on $[a,b]$, then $HG \in OA^\circ$ on $[a,b]$* [**6**, Theorem 2, p. 494].

LEMMA 6. *If F and U are functions from $R \times R$ to N, F and U are in OB° on $[a,b]$, $F \in OA^\circ$ on $[a,b]$, ${}_x\Pi^y(1+U)$ exists for $a \leqq x < y \leqq b$ and*

$$(R)\int_x^y F_s\Pi^y(1+U)$$

exists for $a \leqq x < y \leqq b$, *then*

$$\int_a^b \left| (R) \int_x^y F_s \Pi^y (1 + U) - F(x, y) \right|$$

exists and is zero [**5**, Lemma, p. 307].

The main result now follows.

THEOREM. *If* $\beta > 0$, H *and* G *are functions from* $R \times R$ *to* N, f *and* h *are functions from* R *to* N, *each of* H, G *and* dh *is in* $OB°$ *on* $[a, b]$ *and* $|H| < 1 - \beta$ *on* $[a, b]$, *then the following statements are equivalent*:

(1) f *is bounded on* $[a, b]$, *each of* $\int_a^b H$, $\int_a^b G$ *and*

$$(LR) \int_a^b (fG + fH)$$

exists and

$$f(x) = h(x) + (LR) \int_a^x (fG + fH)$$

for $a \leqq x \leqq b$, *and*

(2) *each of* ${}_x\Pi^y(1 + \Sigma_{j=1}^{\infty} H^j)$, ${}_x\Pi^y(1 + G)$ *and*

$$(R) \int_x^y dh \left(1 + \sum_{j=1}^{\infty} H^j\right) {}_s\Pi^y(1 + G) \left(1 + \sum_{j=1}^{\infty} H^j\right)$$

exists for $a \leqq x < y \leqq b$ *and*

$$f(x) = h(a)\, {}_a\Pi^x (1 + G)\left(1 + \sum_{j=1}^{\infty} H^j\right)$$
$$+ (R) \int_a^x dh \left(1 + \sum_{j=1}^{\infty} H^j\right) {}_s\Pi^x(1 + G) \left(1 + \sum_{j=1}^{\infty} H^j\right)$$

for $a \leqq x \leqq b$.

Before proving the theorem, we point out the results of considering left and right integrals, respectively. If $H \equiv 0$, then we have the integral equation

$$\text{(a)} \qquad f(x) = h(x) + (L) \int_a^x fG.$$

This equation involves only a left integral, and its solution is

(b) $$f(x) = h(a)_a\Pi^x(1+G) + (R)\int_a^x dh_s\Pi^x(1+G).$$

On the other hand, if $G \equiv 0$, then we have the integral equation

(c) $$f(x) = h(x) + (R)\int_a^x fG.$$

This equation involves only a right integral, and its solution is

(d) $$f(x) = h(a)_a\Pi^x\left(1+\sum_{j=1}^{\infty} H^j\right) + (L)\int_a^x dh_r\Pi^x\left(1+\sum_{j=1}^{\infty} H^j\right).$$

If z is in N and $|z| < 1$, then $1 + \sum_{j=1}^{\infty} z^j$ exists and is $(1-z)^{-1}$. Thus, in (d) and in the theorem itself, it is possible to substitute $(1-H)^{-1}$ for $1 + \sum_{j=1}^{\infty} H^j$. To obtain some feeling for why invertibility-related conditions are placed on H but not on G, consider the first approximations to equations (a) and (c). For (a), we have that

$$f(x) \doteq h(x) + f(a)\,G\,(a,x);$$

while for (c), we have that

$$f(x) \doteq h(x) + f(x)\,H\,(a,x),$$

and hence that

$$f(x) \doteq h(x)\,[1 - H(a,x)]^{-1}.$$

For additional discussion of product integrals, inverses and integral equations, the reader is referred to papers by J. V. Herod **[12]** and the author **[9]**.

The main result is now established.

Proof. To simplify notation in the following work, we use the interval functions T, U and V to denote

$$(1+G)\left(1+\sum_{j=1}^{\infty} H^j\right),$$

$$G + \sum_{j=1}^{\infty} H^j + G\sum_{j=1}^{\infty} H^j$$

and

$$1+\sum_{j=1}^{\infty} H^j,$$

respectively. Further, we use C to denote the interval function

$$C(r,s)=f(r)G(r,s)+f(s)H(r,s).$$

Proof (1)→(2). Since $\int_a^b H$ exists and $H \in OB^\circ$ on $[a,b]$, it follows that $H \in OL^\circ$ on $[a,b]$, and hence, $1+\Sigma_{j=1}^{\infty} H^j \in OL^\circ$ on $[a,b]$. Thus, the existence of

$$\int_a^b H\left(1+\sum_{j=1}^{\infty} H^j\right)=\int_a^b \sum_{j=1}^{\infty} H^j$$

follows from Lemma 2. Therefore, the existence of ${}_x\Pi^y V$ for $a \leqq x < y \leqq b$ follows from Lemma 1. Also, Lemma 1 implies the existence of ${}_x\Pi^y(1+G)$ for $a \leqq x < y \leqq b$ from the existence of $\int_a^b G$. Lemma 2 can be used to establish the existence of $\int_a^b G\, \Sigma_{j=1}^{\infty} H^j$. Therefore, since each of

$$\int_a^b G, \int_a^b \sum_{j=1}^{\infty} H^j \quad \text{and} \quad \int_a^b G \sum_{j=1}^{\infty} H^j$$

exists, we have that $\int_a^b U$ exists, and thus, the existence of ${}_x\Pi^y T$ for $a \leqq x < y \leqq b$ can be established by applying Lemma 1. Finally, since $V(r,s)_s\Pi^y T$ is in OL° on $[a,b]$, the existence of

$$(R)\int_x^y dh\, V_s\Pi^y T$$

for $a \leqq x < y \leqq b$ can be obtained from the existence of $\int_a^b dh$ through the use of Lemma 2.

Suppose $a \leqq x \leqq b$. We now show that

$$f(x)=h(a)_a\Pi^x T+(R)\int_a^x dh\, V_s\Pi^x T.$$

If $a=x$, the result follows immediately. Therefore, suppose $a<x$.

Let $\epsilon > 0$. Since $|H| < 1 - \beta$ on $[a, x]$, G, H and dh are in OB° on $[a, x]$ and f and V are bounded on $[a, x]$, there exist a subdivision D_1 of $[a, x]$ and a number B such that if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

(1) $|H_i| < 1 - \beta$ for $i = 1, 2, \cdots, n$,

(2) $\sum_{i=1}^n |dh_i V_i| < B$,

(3) $\sum_{i=1}^n |C_i V_i| < B$,

(4) $\sum_{i=1}^n \left| \left[\int_{x_{i-1}}^{x_i} C \right] V_i \right| < B$, and

(5) $|V_i \Pi_{k=i+1}^n T_k| < B$ for $i = 1, 2, \cdots, n$.

Since ${}_r\Pi^s T$ exists for $a \leqq r \leqq s \leqq x$ and $U \in OB^\circ$ on $[a, x]$, it follows from Lemma 3 that there exists a subdivision D_2 of $[a, x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_2 and $0 \leqq p < q \leqq n$, then

(1) $|{}_{x_p}\Pi^{x_q} T - \Pi_{i=p+1}^q T_i| < \epsilon(16B)^{-1}$,

(2) $|\Pi_{i=p+1}^q T_i - {}_{x_p}\Pi^{x_q} T| < \epsilon(16B)^{-1}$, and

(3) $|h(a)_a\Pi^x T - h(a)\Pi_{i=1}^n T_i| < \epsilon/4$.

Since $(R)\int_a^x dh\, V_s \Pi^x T$ exists, there exists a subdivision D_3 of $[a, x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_3, then

$$\left| (R)\int_a^x dh\, V_s \Pi^x T - \sum_{i=1}^n dh_i\, V_{i\, x_i}\Pi^x T \right| < \epsilon/8.$$

Since $V(r, s)_s \Pi^x T$ is in OL° on $[a, x]$, it follows from Lemma 4 that there exist a subdivision $D_4 = \{t_i\}_{i=0}^p$ of $[a, x]$ and a sequence $\{k_j\}_{j=1}^p$ such that if $1 \leqq j \leqq p$ and $t_{j-1} < r < s < t_j$, then

$$|V(r, s)_s \Pi^x T - k_j| < \epsilon(16B)^{-1}.$$

Since $C \in OB^\circ$ on $[a, x]$ and $\int_a^x C$ exists, there exist subdivisions $\{r_j\}_{j=0}^{p+1}$ and $\{s_j\}_{j=0}^{p+1}$ of $[a, x]$ such that

(1) $t_{j-1} < r_j < s_j < t_j$ for $j = 1, 2, \cdots, p$, and

(2) $\sum_{k=1}^{n(j)} \left| C_{jk} - \int_{x_{j,k-1}}^{x_{jk}} C \right| < \epsilon[8B(p+1)]^{-1}$ for $j = 1, 2, \cdots, p+1$ and each refinement $\{x_{jk}\}_{k=0}^{n(j)}$ of $\{s_{j-1}, t_{j-1}, r_j\}$.

Further, for $j = 1, 2, \cdots, p$, there exist subdivisions E_j of $[r_j, s_j]$ such that if F_j is a refinement of E_j, then

$$\sum_{j=1}^p \left| \sum_{F_j(I)} C - \int_{r_j}^{s_j} C \right| |k_j| < \epsilon/8.$$

Let D denote the subdivision

$$\bigcup_{i=1}^4 D_i \cup \{r_j\}_{j=0}^{p+1} \cup \{s_j\}_{j=0}^{p+1} \bigcup_{j=1}^p E_j$$

of $[a, x]$, and suppose $\{x_i\}_{i=0}^n$ is a refinement of D. For $j = 1, 2, \cdots, p$, let K_j be the set such that $i \in K_j$ only if $r_j < x_i \leqq s_j$. Let K and L denote the sets

$$\bigcup_{j=1}^{p} K_j \quad \text{and} \quad \{i\}_{i=1}^n - \bigcup_{j=1}^{p} K_j,$$

respectively.

We now establish two inequalities that are necessary to complete the proof. First,

$$\begin{aligned}
&\left|(R)\int_a^x dh\, V\, {}_s\Pi^x T - \sum_{i=1}^n dh_i\, V_i \prod_{k=i+1}^n T_k\right| \\
&\quad \leqq \left|\sum_{i=1}^n dh_i\, V_i\, {}_{x_i}\Pi^x T - \sum_{i=1}^n dh_i\, V_i \prod_{k=i+1}^n T_k\right| \\
&\quad\quad + \left|(R)\int_a^x dh\, V\, {}_s\Pi^x T - \sum_{i=1}^n dh_i\, V_i\, {}_{x_i}\Pi^x T\right| \\
&\quad < \sum_{i=1}^n |dh_i\, V_i| \left|{}_{x_i}\Pi^x T - \prod_{k=i+1}^n T_k\right| + \epsilon/8 \\
&\quad < B[\epsilon(16B)^{-1}] + \epsilon/8 < \epsilon/4.
\end{aligned}$$

Second,

$$\begin{aligned}
&\left|\sum_{i=1}^n \left[C_i - \int_{x_{i-1}}^{x_i} C\right] V_i \prod_{k=i+1}^n T_k\right| \\
&\quad \leqq \left|\sum_{i\in K} \left[C_i - \int_{x_{i-1}}^{x_i} C\right] V_i \prod_{k=i+1}^n T_k\right| \\
&\quad\quad + \sum_{i\in L} \left|C_i - \int_{x_{i-1}}^{x_i} C\right| \left|V_i \prod_{k=i+1}^n T_k\right| \\
&\quad < \left|\sum_{i\in K} \left[C_i - \int_{x_{i-1}}^{x_i} C\right] V_i \prod_{k=i+1}^n T_k\right| \\
&\quad\quad + (p+1)\{\epsilon[8B(p+1)]^{-1}\}B \\
&\quad \leqq \left|\sum_{i\in K} \left[C_i - \int_{x_{i-1}}^{x_i} C\right] V_i\, {}_{x_i}\Pi^x T\right| \\
&\quad\quad + \sum_{i\in K} \left|\left[C_i - \int_{x_{i-1}}^{x_i} C\right] V_i\right| \left|\prod_{k=i+1}^n T_k - {}_{x_i}\Pi^x T\right| \\
&\quad\quad + \epsilon/8
\end{aligned}$$

$$
\begin{aligned}
&< \left| \sum_{i \in K} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_{i x_i} \Pi^x T \right| \\
&\quad + 2B[\epsilon(16B)^{-1}] + \epsilon/8 \\
&\leqq \sum_{j=1}^{p} \left| \sum_{i \in K_j} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_{i x_i} \Pi^x T \right| + \epsilon/4 \\
&\leqq \sum_{j=1}^{p} \left| \sum_{i \in K_j} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] k_j \right| \\
&\quad + \sum_{j=1}^{p} \sum_{i \in K_j} \left| C_i - \int_{x_{i-1}}^{x_i} C \right| \, | V_{i x_i} \Pi^x T - k_j | + \epsilon/4 \\
&< \sum_{j=1}^{p} \left| \sum_{i \in K_j} C_i - \int_{r_j}^{s_j} C \right| \, | k_j | \\
&\quad + 2B[\epsilon(16B)^{-1}] + \epsilon/4 \\
&< \epsilon/8 + 3\epsilon/8 = \epsilon/2.
\end{aligned}
$$

If we employ the iterative technique used by B. W. Helton [5, p. 311], we have that

$$
\begin{aligned}
f(x) = \sum_{i=1}^{n} \left[\int_{x_{i-1}}^{x_i} C - C_i \right] V_i \prod_{k=i+1}^{n} T_k \\
+ h(a) \prod_{i=1}^{n} T_i + \sum_{i=1}^{n} dh_i \, V_i \prod_{k=i+1}^{n} T_k.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left| h(a)\, {}_a\Pi^x T + (R) \int_a^x dh \, V_s \Pi^x T - f(x) \right| \\
&\quad \leqq \left| h(a)\, {}_a\Pi^x T - h(a) \prod_{i=1}^{n} T_i \right| \\
&\qquad + \left| (R) \int_a^x dh \, V_s \Pi^x T - \sum_{i=1}^{n} dh_i \, V_i \prod_{k=i+1}^{n} T_k \right| \\
&\qquad + \left| \sum_{i=1}^{n} \left[C_i - \int_{x_{i-1}}^{x_i} C \right] V_i \prod_{k=i+1}^{n} T_k \right| \\
&\quad < \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon.
\end{aligned}
$$

Therefore, (1) implies (2).

Proof (2)→(1). It follows from the bounded variation of the various functions involved that f is bounded on $[a,b]$. Since $\Sigma_{j=1}^{\infty} H^j \in OB^\circ$ on $[a,b]$, it follows from Lemma 1 that

$$\int_a^b \sum_{j=1}^{\infty} H^j = \int_a^b H\left(1+\sum_{j=1}^{\infty} H^j\right)$$

exists. Recall that $(1+\Sigma_{j=1}^{\infty} H^j)^{-1}$ exists and is $1-H$. Thus, since $1-H \in OL^\circ$ on $[a,b]$, it follows from Lemma 2 that $\int_a^b H$ exists. Further, it follows from Lemma 1 that $\int_a^b G$ exists. The existence of $\int_a^b C$ now follows from the existence of $\int_a^b G$ and $\int_a^b H$ by applying Lemma 2.

Suppose $a \leqq x \leqq b$. We now show that

$$f(x) = h(x) + (LR)\int_a^x (fG + fH).$$

If $a = x$, the result follows immediately. Therefore, suppose $a < x$.

Let $\epsilon > 0$. There exist a subdivision D_1 of $[a,x]$ and a number B such that if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

(1) $|H_i| < 1-\beta$ for $i = 1,2,\cdots,n$,
(2) $\Sigma_{i=1}^n |G_i| < B$,
(3) $\Sigma_{i=1}^n |H_i| < B$,
(4) $\Sigma_{i=1}^n |dh_i V_i| < B$, and
(5) $|{}_{x_p}\Pi^{x_q} T| < B$ for $0 \leqq p < q \leqq n$.

Since $\int_a^x G$ exists and $\Sigma_{j=1}^{\infty} H^j \in OL^\circ$ on $[a,b]$, it follows from Lemma 2 that $\int_a^x G\,\Sigma_{j=1}^{\infty} H^j$ exists. Thus, the existence of $\int_a^x U$ follows from the existence of

$$\int_a^x G,\ \int_a^x \sum_{j=1}^{\infty} H^j \quad \text{and} \quad \int_a^x G \sum_{j=1}^{\infty} H^j.$$

Therefore,

$${}_r\Pi^t (1+U) = {}_r\Pi^t T$$

exists for $a \leqq r \leqq t \leqq x$ by Lemma 1. Now, it follows from Lemma 3 that $U \in OM^*$ on $[a, x]$. Hence, there exists a subdivision D_2 of $[a, x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_2 and $0 \leqq p < q \leqq n$, then

$$\left| f(a)\,{}_{x_p}\Pi^{x_q} T - f(a) \prod_{i=p+1}^{q} T_i \right| < \epsilon(6B)^{-1}.$$

Since dh is in OA° and OB° on $[a, x]$ and $V \in OL^\circ$ on $[a, x]$, it follows from Lemma 5 that $dh\,V \in OA^\circ$ on $[a, x]$. Thus, since $U \in OB^\circ$ on $[a, x]$ and ${}_s\Pi^t T$ exists for $a \leqq s < t \leqq x$, it follows from Lemma 6 that

$$\int_a^x \left| (R)\int_u^v dh\,V\,{}_s\Pi^v T - dh(v)V(u, v) \right| = 0.$$

From the existence of this integral and the fact that $U \in OM^*$ on $[a, x]$, it follows that there exists a subdivision D_3 of $[a, x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_3 and $0 \leqq p < q \leqq n$, then

(1) $\Sigma_{i=1}^n \left| (R)\int_{x_{i-1}}^{x_i} dh\,V\,{}_s\Pi^{x_i} T - dh_i\,V_i \right| < \epsilon(12B^2)^{-1}$, and

(2) $\left| {}_{x_p}\Pi^{x_q} T - \Pi_{k=p+1}^q T_k \right| < \epsilon(12B^2)^{-1}$.

Thus, if $\{x_i\}_{i=0}^n$ is a refinement of $D_1 \cup D_3$ and $0 < p \leqq n$, then

$$\begin{aligned}
&\left| (R)\int_a^{x_p} dh\,V\,{}_s\Pi^{x_p} T - \sum_{i=1}^{p} dh_i\,V_i \prod_{k=i+1}^{p} T_k \right| \\
&\quad \leqq \left| (R)\int_a^{x_p} dh\,V\,{}_s\Pi^{x_p} T - \sum_{i=1}^{p} dh_i\,V_i\,{}_{x_i}\Pi^{x_p} T \right| \\
&\qquad + \sum_{i=1}^{p} |dh_i\,V_i| \left| {}_{x_i}\Pi^{x_p} T - \prod_{k=i+1}^{p} T_k \right| \\
&\quad < \left| \sum_{i=1}^{p} \left[(R)\int_{x_{i-1}}^{x_i} dh\,V\,{}_s\Pi^{x_i} T - dh_i\,V_i \right] {}_{x_i}\Pi^{x_p} T \right| \\
&\qquad + B[\epsilon(12B^2)^{-1}] \\
&\quad \leqq \sum_{i=1}^{p} \left| (R)\int_{x_{i-1}}^{x_i} dh\,V\,{}_s\Pi^{x_i} T - dh_i\,V_i \right| \left| {}_{x_i}\Pi^{x_p} T \right| + \epsilon(12B)^{-1} \\
&\quad < B[\epsilon(12B^2)^{-1}] + \epsilon(12B)^{-1} = \epsilon(6B)^{-1}.
\end{aligned}$$

It follows from the existence of the integrals involved that there exists a subdivision D_4 of $[a, x]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_4, then

$$\left| f(a) \prod_{i=1}^{n} T_i - f(a)\, {}_a\Pi^x T \right| < \epsilon/6$$

and

$$\left| \sum_{i=1}^{n} dh_i\, V_i \prod_{k=i+1}^{n} T_k - (R) \int_a^x dh\, V_s \Pi^x T \right| < \epsilon/6.$$

Let D denote the subdivision $\cup_{i=1}^4 D_i$ of $[a, x]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Observe that

$$\begin{aligned} \prod_{i=m}^{n} T_i &= 1 + \sum_{i=m}^{n} \left(\prod_{k=m}^{i-1} T_k \right) U_i \\ &= 1 + \sum_{i=m}^{n} \left(\prod_{k=m}^{i-1} T_k \right) G_i + \sum_{i=m}^{n} \left(\prod_{k=m}^{i} T_k \right) H_i. \end{aligned}$$

Further,

$$\begin{aligned} &\sum_{i=1}^{n} dh_i\, V_i \sum_{k=i+1}^{n} \left(\prod_{j=i+1}^{k-1} T_j \right) G_k \\ &\qquad = \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i-1} T_k \right] G_i \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^{n} dh_i\, V_i \sum_{k=i+1}^{n} \left(\prod_{j=i+1}^{k} T_j \right) H_k \\ &\qquad = \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i} T_k \right] H_i. \end{aligned}$$

These identities can be established by induction and are used in subsequent manipulations.

We now work out a further identity to aid in establishing the existence of the desired integral. By employing the previously stated identities, we have that

$$
\begin{aligned}
& f(a)\prod_{i=1}^{n} T_i + \sum_{i=1}^{n} dh_i\, V_i \prod_{k=i+1}^{n} T_k \\
&\quad = f(a)\left[1+\sum_{i=1}^{n}\left(\prod_{k=1}^{i-1} T_k\right) G_i + \sum_{i=1}^{n}\left(\prod_{k=1}^{i} T_k\right) H_i\right] \\
&\qquad + \sum_{i=1}^{n} dh_i\, V_i\left[1+\sum_{k=i+1}^{n}\left(\prod_{j=i+1}^{k-1} T_j\right) G_k + \sum_{k=i+1}^{n}\left(\prod_{j=i+1}^{k} T_j\right) H_k\right] \\
&\quad = f(a)\left[1+\sum_{i=1}^{n}\left(\prod_{k=1}^{i-1} T_k\right) G_i + \sum_{i=1}^{n}\left(\prod_{k=1}^{i} T_k\right) H_i\right] \\
&\qquad + \sum_{i=1}^{n} dh_i\, V_i + \sum_{i=1}^{n} dh_i\, V_i \sum_{k=i+1}^{n}\left(\prod_{j=i+1}^{k-1} T_j\right) G_k \\
&\qquad + \sum_{i=1}^{n} dh_i\, V_i \sum_{k=i+1}^{n}\left(\prod_{j=i+1}^{k} T_j\right) H_k \\
&\quad = f(a)\left[1+\sum_{i=1}^{n}\left(\prod_{k=1}^{i-1} T_k\right) G_i + \sum_{i=1}^{n}\left(\prod_{k=1}^{i} T_k\right) H_i\right] \\
&\qquad + \sum_{i=1}^{n} dh_i + \sum_{i=1}^{n} dh_i\, V_i H_i \\
&\qquad + \sum_{i=2}^{n}\left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i-1} T_k\right] G_i \\
&\qquad + \sum_{i=2}^{n}\left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i} T_k\right] H_i \\
&\quad = f(a)\left[1+\sum_{i=1}^{n}\left(\prod_{k=1}^{i-1} T_k\right) G_i + \sum_{i=1}^{n}\left(\prod_{k=1}^{i} T_k\right) H_i\right] \\
&\qquad + h(x) - h(a) + \sum_{i=2}^{n}\left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i-1} T_k\right] G_i \\
&\qquad + \sum_{i=1}^{n}\left[\sum_{j=1}^{i} dh_j\, V_j \prod_{k=j+1}^{i} T_k\right] H_i \\
&\quad = f(a)\sum_{i=1}^{n}\left(\prod_{k=1}^{i-1} T_k\right) G_i + f(a)\sum_{i=1}^{n}\left(\prod_{k=1}^{i} T_k\right) H_i \\
&\qquad + h(x) + \sum_{i=2}^{n}\left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i-1} T_k\right] G_i \\
&\qquad + \sum_{i=1}^{n}\left[\sum_{j=1}^{i} dh_j\, V_j \prod_{k=j+1}^{i} T_k\right] H_i.
\end{aligned}
$$

Now, by employing the identity developed in the preceding paragraph, we have that

$$
\begin{aligned}
&\left| h(x) + \sum_{i=1}^{n} f(x_{i-1})G_i + \sum_{i=1}^{n} f(x_i)H_i - f(x) \right| \\
&\quad < \left| h(x) + \sum_{i=1}^{n} f(x_{i-1})G_i + \sum_{i=1}^{n} f(x_i)H_i \right. \\
&\qquad \left. - \left[f(a) \prod_{i=1}^{n} T_i + \sum_{i=1}^{n} dh_i\, V_i \prod_{k=i+1}^{n} T_k \right] \right| + \epsilon/6 + \epsilon/6 \\
&\quad = \left| h(x) + \sum_{i=1}^{n} f(x_{i-1})G_i + \sum_{i=1}^{n} f(x_i)H_i \right. \\
&\qquad - \left\{ f(a) \sum_{i=1}^{n} \left(\prod_{k=1}^{i-1} T_k \right) G_i + f(a) \sum_{i=1}^{n} \left(\prod_{k=1}^{i} T_k \right) H_i \right. \\
&\qquad + h(x) + \sum_{i=2}^{n} \left[\sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i-1} T_k \right] G_i \\
&\qquad \left. \left. + \sum_{i=1}^{n} \left[\sum_{j=1}^{i} dh_j\, V_j \prod_{k=j+1}^{i} T_k \right] H_i \right\} \right| + \epsilon/3 \\
&\quad \leqq \sum_{i=2}^{n} \left| f(a)\, {}_a\Pi^{x_{i-1}} T - f(a) \prod_{k=1}^{i-1} T_k \right| |G_i| \\
&\qquad + \sum_{i=1}^{n} \left| f(a)\, {}_a\Pi^{x_i} T - f(a) \prod_{k=1}^{i} T_k \right| |H_i| \\
&\qquad + \sum_{i=2}^{n} \left| (R) \int_a^{x_{i-1}} dh\, V\, {}_v\Pi^{x_{i-1}} T - \sum_{j=1}^{i-1} dh_j\, V_j \prod_{k=j+1}^{i-1} T_k \right| |G_i| \\
&\qquad + \sum_{i=1}^{n} \left| (R) \int_a^{x_i} dh\, V\, {}_v\Pi^{x_i} T - \sum_{j=1}^{i} dh_j\, V_j \prod_{k=j+1}^{i} T_k \right| |H_i| \\
&\qquad + \epsilon/3 \\
&\quad < B[\epsilon(6B)^{-1}] + B[\epsilon(6B)^{-1}] + B[\epsilon(6B)^{-1}] + B[\epsilon(6B)^{-1}] \\
&\qquad + \epsilon/3 \\
&\quad = \epsilon.
\end{aligned}
$$

Therefore, $(LR)\int_a^x (fG + fH)$ exists and is $f(x) - h(x)$. Hence, (2) implies (1).

B. W. Helton states three additional theorems on the solution of integral equations by product integration [**5**, Theorems 5.2, 5.3, 5.4, pp.

313–314]. The techniques used in the present paper to avoid requiring the existence of the integrals

$$\int_a^b \left| G - \int G \right| = 0 \quad \text{and} \quad \int_a^b |1 + G - \Pi(1+G)| = 0$$

can also be applied to these results.

References

1. W. D. L. Appling, *Interval functions and real Hilbert spaces*, Rend. Circl. Mat. Palermo, (2) **11** (1962), 154–156.
2. C. W. Bitzer, *Stieltjes-Volterra integral equations*, Illinois J. Math., **14** (1970), 434–451.
3. ———, *Convolution, fixed point, and approximation in Stieltjes-Volterra integral equations*, J. Australian Math. Soc., **14** (1972), 182–199.
4. J. R. Dorroh, *Integral equations in normed Abelian groups*, Pacific J. Math., **13** (1963), 1143–1158.
5. B. W. Helton, *Integral equations and product integrals*, Pacific J. Math., **16** (1966), 297–322.
6. ———, *A product integral representation for a Gronwall inequality*, Proc. Amer. Math. Soc., **23** (1969), 493–500.
7. ———, *Solutions of* $f(x) = f(a) + (RL)\int_a^x (fH + fG)$ *for rings*, Proc. Amer. Math. Soc., **25** (1970), 735–742.
8. J. C. Helton, *An existence theorem for sum and product integrals*, Proc. Amer. Math. Soc., **39** (1973), 149–154.
9. ———, *Product integrals and inverses in normed rings*, Pacific J. Math., **51** (1974), 155–166.
10. ———, *Mutual existence of sum and product integrals*, Pacific J. Math., **56** (1975), 495–516.
11. J. V. Herod, *Solving integral equations by iteration*, Duke Math. J., **34** (1967), 519–534.
12. ———, *Multiplicative inverses of solutions for Volterra-Stieltjes integral equations*, Proc. Amer. Math. Soc., **22** (1969), 650–656.
13. T. H. Hildebrandt, *On systems of linear differentio-Stieltjes-integral equations*, Illinois J. Math., **3** (1959), 352–373.
14. D. B. Hinton, *A Stieltjes-Volterra integral equation theory*, Canad. J. Math., **18** (1966), 314–331.
15. A. Kolmogoroff, *Untersuchungen über den Integralbegriff*, Math. Ann., **103** (1930), 654–696.
16. D. L. Lovelady, *Perturbations of solutions of Stieltjes integral equations*, Trans. Amer. Math. Soc., **155** (1971), 175–187.
17. ———, *Multiplicative integration of infinite products*, Canad. J. Math., **23** (1971), 692–698.
18. J. S. MacNerney, *Stieltjes integrals in linear spaces*, Ann. of Math., (2) **61** (1955), 354–367.
19. ———, *Continuous products in linear spaces*, J. Elisha Mitchell Sci. Soc., **71** (1955), 185–200.
20. ———, *Integral equations and semigroups*, Illinois J. Math., **7** (1963), 148–173.
21. ———, *A linear initial-value problem*, Bull. Amer. Math. Soc., **69** (1963), 314–329.
22. ———, *A nonlinear integral operation*, Illinois J. Math., **8** (1964), 621–638.
23. P. R. Masani, *Multiplicative Riemann integration in normed rings*, Trans. Amer. Math. Soc., **61** (1947), 147–192.
24. J. W. Neuberger, *Continuous products and nonlinear integral equations*, Pacific J. Math., **8** (1958), 529–549.
25. ———, *Concerning boundary value problems*, Pacific J. Math., **10** (1960), 1385–1392.
26. ———, *A generator for a set of functions*, Illinois J. Math., **9** (1965), 31–39.

27. J. A. Reneke, *A Product integral solution of a Stieltjes-Volterra integral equation*, Proc. Amer. Math. Soc., **24** (1970), 621–626.

28. H. S. Wall, *Concerning continuous continued fractions and certain systems of Stieltjes integral equations*, Rend. Circ. Mat. Palermo, (2) **2** (1953), 73–84.

29. ———, *Concerning harmonic matrices*, Arch. Math., **5** (1954), 160–167.

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THE RANGE OF A NORMAL DERIVATION

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The inner derivation δ_A implemented by an element A of the algebra $\mathscr{B}(\mathscr{H})$ of bounded linear operators on the separable Hilbert space $\mathscr{H}$ is the map $X \to AX - XA$ $(X \in \mathscr{B}(\mathscr{H}))$. The main result of this paper is that when A is normal, range inclusion $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ is equivalent to the condition that $B = f(A)$ where $\Lambda(z,w) = (f(z) - f(w))(z - w)^{-1}$ (taken as 0 when $z = w$) has the property that $\Lambda(z,w)t(z,w)$ is a trace class kernel on $L^2(\mu)$ whenever $t(z,w)$ is such a kernel. Here μ is the dominating scalar valued spectral measure of A constructed in multiplicity theory. In order that a Borel function f satisfy this condition it is necessary that f be equal almost everywhere to a Lipschitz function with derivative in $\sigma(A)$ at each limit point of $\sigma(A)$ and it is sufficient (for A self-adjoint) that $f \in C^{(3)}(\mathbf{R})$.

For such operators B there is also a factorization $\delta_B = \delta_A\tau = \tau\delta_A$ by an ultraweakly continuous linear map τ from $\mathscr{B}(\mathscr{H})$ into itself satisfying $\tau(A_1'XA_2') = A_1'\tau(X)A_2'$ for $X \in \mathscr{B}(\mathscr{H})$ and A_1', A_2' commuting with A.

When $\mathscr{H}$ is finite dimensional $(X, Y) = \operatorname{trace}(XY^*)$ is an inner product and $\mathscr{B}(\mathscr{H}) = \mathscr{R}(\delta_A) \oplus \{A^*\}'$ is the orthogonal direct sum of the range of δ_A and $\{A^*\}' = \{Y \in \mathscr{B}(\mathscr{H}) : YA^* = A^*Y\}$, the commutant of the adjoint of A. This simple fact suggests that $\mathscr{R}(\delta_A)$ is a natural subspace, like the commutant, associated with A. The orthogonal decomposition also shows that range inclusion $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ holds for a pair of operators if and only if $B \in \{A\}''$, or equivalently, if and only if B is a polynomial in A. In this case $\delta_B = \delta_A\tau = \tau\delta_A$ with τ as above.

When $\mathscr{H}$ is infinite dimensional the situation is less clear. We do not know whether $\mathscr{R}(\delta_A) \cap \{A^*\}' = 0$ in general. The sum $\mathscr{R}(\delta_A) + \{A^*\}'$ is always weakly dense in $\mathscr{B}(\mathscr{H})$ but is rarely norm closed; in fact for A normal it is closed if and only if A has a finite specturm **[1]**.

The condition $B \in \{A\}''$ is neither sufficient for $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ (even if A is positive and compact **[19]**), nor necessary [Yang Ho, private communication].

If $A \in \mathscr{B}(\mathscr{H})$ and $B = f(A)$, where f is analytic in a neighborhood of the specturm of A, then $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ but range inclusion does not imply $B = f(A)$ for some analytic f **[20]**. Finally, if $\{A\}'$ contains no nonzero trace class operator then the norm closure of $\mathscr{R}(\delta_A)$ contains the ideal of compact operators **[22]**. In this case there are operators $B \notin \{A\}''$ with $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)^-$. There are normal operators A with this property (multiplication by x in $L^2(0, 1)$ for example) and compact operators [16] but none that is both normal and compact. (See the remark following (2.1) below).

As byproducts of our study of the range of a normal derivation we obtain improvements of the results of [19, 20] and a simpler proof of the theorem of [1] mentioned above. Our results also yield solutions to some asymptotic commutativity problems raised in [3].

1. Auxiliary results. If E, F, G are Banach spaces, $S \in \mathcal{B}(E, G)$ and $T \in \mathcal{B}(F, G)$, then the closed graph theorem implies that $\mathcal{R}(S) \subset \mathcal{R}(T)$ if and only if there is $R \in \mathcal{B}(E, F/\mathrm{Ker}(T))$ with $S = \tilde{T}R$, where $\tilde{T}$ is the element of $\mathcal{B}(F/\mathrm{Ker}(T), G)$ associated with T. In particular, range inclusion $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ of derivations on $\mathcal{B}(\mathcal{H})$ amounts to a factorization $\delta_B = \tilde{\delta}_A\sigma$ with σ a bounded linear operator from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})/\{A\}'$. Our first goal is to show that if A is normal this trivial factorization can be sharpened to: $\delta_B = \delta_A\sigma$ for some ultraweakly continuous linear operator σ from $\mathcal{B}(\mathcal{H})$ into *itself.* For this and later applications we need some simple facts about range inclusion in general.

LEMMA (1.1). *If $S, T \in \mathcal{B}(E, F)$ the following are equivalent:*

(1) *There exists a constant c such that $\|Sx\| \leqq c\|Tx\|$ for all $x \in E$.*

(2) *There exists a constant c such that for each $f \in F^*$ there is a $g \in F^*$ with $\|g\| \leqq c\|f\|$ and $S^*f = T^*g$.*

(3) $\mathcal{R}(S^*) \subset \mathcal{R}(T^*)$.

Proof. Suppose that (1) holds, and fix $f \in F^*$. Then $Tx \to \langle Sx, f\rangle$ extends to a bounded linear functional on F by the Hahn-Banach theorem and therefore there is a vector $g \in F^*$ with $\|g\| \leqq c\|f\|$ such that $\langle Sx, f\rangle = \langle Tx, g\rangle$ for all $x \in E$. Hence $S^*f = T^*g$ so that (2) is satisfied.

Clearly (2) implies (3). Suppose that (3) holds. If $f \in F^*$ then $S^*f = T^*g$ for some $g \in F^*$ and therefore $|\langle Sx, f\rangle| = |\langle x, T^*g\rangle| \leqq \|g\|\|Tx\|$ for each $x \in E$. The uniform boundedness theorem implies that there is a constant c such that $|\langle Sx, f\rangle| \leqq c\|Tx\|\|f\|$ for all $x \in E$ and all $f \in F^*$. Then $\|Sx\| \leqq c\|Tx\|$ so that (1) holds.

COROLLARY (1.2). *If $S, T \in \mathcal{B}(E, F)$ then $\mathcal{R}(S^{**}) \subset \mathcal{R}(T^{**})$ if and only if $\mathcal{R}(S) \subset \mathcal{R}(T^{**})$ where $\mathcal{R}(S)$ is identified with its canonical image in F^{**}.*

Proof. Since S^{**} is an extension of S the necessity is trivial. Suppose $\mathcal{R}(S) \subset \mathcal{R}(T^{**})$. If $x \in E$ then there is $\xi \in E^{**}$ with

$Sx = T^{**}\xi$ and hence $|\langle x, S^*f\rangle| = |\langle T^{**}\xi, f\rangle| \leqq \|\xi\| \|T^*f\|$ for $f \in F^*$. Hence by the uniform boundedness theorem there is a constant c such that $|\langle x, S^*f\rangle| \leqq c\|x\| \|T^*f\|$ for $x \in E$ and $f \in F^*$. Consequently $\|S^*f\| \leqq c\|T^*f\|$ and so $\mathcal{R}(S^{**}) \subset \mathcal{R}(T^{**})$ by Lemma (1.1).

COROLLARY (1.3). *If $S, T \in \mathcal{B}(E, F)$ these are equivalent:*
(1) $\mathcal{R}(S^*) \subset \mathcal{R}(T^*)$
(2) $\mathcal{R}(S^*) \subset \mathcal{R}(T^{***})$
(3) $\mathcal{R}(S^{***}) \subset \mathcal{R}(T^{***})$.

Proof. Conditions (2) and (3) are equivalent by Corollary (1.2). Also (1) trivially implies (2). Suppose (2) holds. If $f \in F^*$ then $S^*f = T^{***}\xi$ for some $\xi \in F^{***}$. Now each such ψ has the form $\xi = \xi_0 + \xi_1$ where $\xi_0 \in F^0$ and ξ_1 F^*. Also T^{***} extends T^* and maps F^0 into E^0 and so $S^*f - T^*\xi_1 = T^{***}\xi_0 \in E^* \cap E^0 = \{0\}$. Thus $S^*f = T^*\xi_1 \in \mathcal{R}(T^*)$. Therefore (2) implies (1).

In the next result and in several subsequent arguments we shall make use of the duality relations between the Banach space $\mathcal{K} = \mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$, equipped with the usual sup norm, and the Banach space $\mathcal{T} = \mathcal{T}(\mathcal{H})$ of trace class operators on $\mathcal{H}$, equipped with the trace norm $\| \ \|_{\mathcal{T}}$. Recall **[4, 14]** that $\mathcal{T}$ may be isometrically identified with the conjugate space of $\mathcal{K}$ and that $\mathcal{B}(\mathcal{H})$ is the conjugate space of $\mathcal{T}$. The canonical bilinear form here is $\langle X, T\rangle = \operatorname{trace}(XT) = \operatorname{trace}(TX)$ for $T \in \mathcal{T}$ and X belonging to either $\mathcal{B}(\mathcal{H})$ or to $\mathcal{K}$. Finally, the ultraweak topology on $\mathcal{B}(\mathcal{H})$ is the weak* topology $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{T})$.

COROLLARY (1.4). *These are equivalent for $A, B \in \mathcal{B}(\mathcal{H})$:*

(1) *There exists a constant c such that $\|\delta_B(X)\| \leqq c\|\delta_A(X)\|$ for all $X \in \mathcal{B}(\mathcal{H})$.*

(2) *There exists a constant c such that for each $T \in \mathcal{T}(\mathcal{H})$ there is $S \in \mathcal{T}(\mathcal{H})$ with $\|S\|_{\mathcal{T}} \leqq c\|T\|_{\mathcal{T}}$ and $\delta_B(T) = \delta_A(S)$.*

(3) $\delta_B^*(\mathcal{B}(\mathcal{H})^*) \subset \delta_A^*(\mathcal{B}(\mathcal{H})^*)$.

(4) $\delta_B(\mathcal{T}(\mathcal{H})) \subset \delta_A(\mathcal{T}(\mathcal{H}))$.

Proof. Conditions (3) and (4) are equivalent by Corollary (1.3) with $S = (\delta_{\mathcal{B}} \mid \mathcal{K})$ and $T = (\delta_A \mid \mathcal{K})$. Also (1) and (3) and (2) and (4) are equivalent by Lemma (1.1).

COROLLARY (1.5). *These are equivalent for $A, B \in \mathcal{B}(\mathcal{H})$:*
(1) $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$.
(2) $\delta_B(\mathcal{K}) \subset \mathcal{R}(\delta_A)$.
(3) *There exists a bounded linear map σ from $\mathcal{B}(\mathcal{H})$ into*

$\mathcal{B}(\mathcal{H})/\{A\}'$ such that $\delta_B = \tilde{\delta}_A\sigma$ where $\tilde{\delta}_A : \mathcal{B}(\mathcal{H})/\{A\}' \to \mathcal{R}(\delta_A)$ is the canonical map associated with δ_A.

(4) *There exists a constant c such that $\|\delta_B(T)\|_{\mathcal{T}} \leqq c\,\|\delta_A(T)\|_{\mathcal{T}}$ for all $T \in \mathcal{T}(\mathcal{H})$.*

Proof. Conditions (1) and (4) are equivalent by Lemma (1.1) applied to $S = (\delta_B \mid \mathcal{T})$ and $T = (\delta_A \mid \mathcal{T})$. Conditions (1) and (2) are equivalent by Corollary (1.2) applied to the restrictions of δ_A and δ_B to $\mathcal{K}$, and (1) and (3) are equivalent by the remark preceding Lemma (1.1).

2. Normal derivations. In this section we show that if A is a normal operator on a Hilbert space $\mathcal{H}$ (assumed to be separable here and in the remainder of the paper) then range inclusion $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ holds only for operators $B \in \{A\}''$. We use the fact that there is a projection P of norm one from $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$ with the property $P(A_1'XA_2') = A_1'P(X)A_2'$ for $X \in \mathcal{B}(\mathcal{H})$ and A_1', A_2' in $\{A\}'$. In fact any projection of norm one onto the commutant has this commutativity property **[17]**. The existence of such projections is a standard fact in the theory of von Neumann algebras [2; Chapter 2]. A simple way to obtain one is to choose a unitary operator V with $\{V\}' = \{A\}'$ and set $P(X) = \operatorname{glim}_{n\to\infty} V^{*n}XV^n$ where glim is a fixed generalized limit on l^∞ and the equality is in the weak (inner product) sense. (See **[23]**)

The commutativity property of P immediately implies $\mathcal{R}(\delta_A) \subset \mathcal{R}(1-P)$ but one does not have equality here in general since for A normal, $\mathcal{R}(\delta_A) + \{A\}'$ is norm closed only in the trivial case in which the spectrum of A is finite **[1]**. The following fact is sufficient for our needs here:

LEMMA (2.1). *$\mathcal{R}(\delta_A)$ and $\mathcal{R}(1-P)$ have same ultraweak closures. Hence if $A\xi = \lambda\xi$ and $A\eta = \lambda\eta$ for $\xi, \eta \in \mathcal{H}$ then $((1-P)(X)\xi, \eta) = 0$.*

Proof. We have $\mathcal{R}(\delta_A) \subset \mathcal{R}(1-P)$ so that by considering annihilators in $\mathcal{T}(\mathcal{H})$ it suffices to show that $\langle(1-P)(X), T\rangle = 0$ for each trace class operator T that commutes with A. Now for such a T we have the polar factorization $T = U(T^*T)^{\frac{1}{2}}$ where U is a partial isometry and both factors belong to $\{A\}'$. Since $P(XU) = P(X)U$ it suffices to consider the case $T \geqq 0$. In fact, by the spectral theorem we need only show that $\langle(1-P)(X), E\rangle = 0$ for $X \in \mathcal{B}(\mathcal{H})$ and $E \in \{A\}'$ a projection of finite rank. Now for the projection P constructed in **[12]** this can easily be verified since $P(X)$ is obtained from the operators V^*XV with V unitary in $\{A\}''$. However we can also prove the assertion assuming only th existence of P as follows: With respect to the decomposition $\mathcal{H} = \mathcal{R}(E) \oplus \mathcal{R}(E)^\perp = \mathcal{H}_0 \oplus \mathcal{H}_1$ simple calculations with two by two operat matrices show that P induces a norm one projection p from

$\mathcal{B}(\mathcal{H}_0)$ onto the commutant of $A_0 = (A \mid \mathcal{H}_0)$ such that $\langle X - P(X), E\rangle =$ trace$(X_0 - p(X_0))$, where $X_0 = E(X \mid \mathcal{H}_0)$, and this last quantity is 0 because the formula $\mathcal{B}(\mathcal{H}_0) = \mathcal{R}(\delta_{A_0}) + \{A_0^*\}'$ shows that $\mathcal{R}(1-p) = \mathcal{R}(\delta_{A_0})$, that is, $X_0 - p(X_0)$ is a commutator.

The second assertion of the Lemma follows from the first by obserivng that the operator $\xi \otimes \eta$ defined by $(\xi \otimes \eta)(\zeta) = (\zeta, \eta)\xi$ is a trace class operator commuting with A so that $0 = \langle (1-P)(X)), \xi \otimes \eta\rangle = ((1-P)(X)\xi, \eta)$.

Remark. A similar duality argument shows that if A is normal and compact and if $B \in \mathcal{B}(\mathcal{H})$, then $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)^-$ if and only if $B - \lambda \in \{A\}'' \cap \mathcal{K}$ for some scalar λ.

The next result appears in [7; Theorem 3.2]:

Lemma (2.2). *If A is a normal operator on $\mathcal{H}$ then*

$$\bigcap_{z \in C} \mathcal{R}(A - zI) = \{0\}.$$

Although we shall make no use of the fact, it is worth observing that (2.2) implies a stronger version of itself.

Corollary (2.3). *Let μ be a* (positive, regular) *Borel measure on* **C** *with compact support and let A be the operator $f(z) \to zf(z)$ on $L^2(\mu)$. If $f \in \mathcal{R}(A - zI)$ for μ almost every $z \in$* **C** *then $f = 0$.*

Proof. Let $\{K_n\}$ be a sequence of compact sets with $\lim \mu(K_n) = \|\mu\|$ and $f \in \mathcal{R}(A - zI)$ for all $z \in K_n$. If P_n is the spectral projection corresponding to K_n then $P_n f \in \mathcal{R}(P_n(A - \lambda I)P_n)$ for all $\lambda \in K_n$ and also for $\lambda \in \mathbf{C} \setminus K_n$ because $\sigma(P_nA \mid P_nH) \subset K_n$. Thus $P_nf = 0$ for all n and consequently $f = 0$.

We shall also need the following result of Korotkov. For a proof see [10, 21]:

Lemma (2.4). *Let μ be a Borel measure on* **C** *of compact support. If $T \in \mathcal{B}(L^2(\mu))$ has $\mathcal{R}(T) \subset L^\infty(\mu)$ then T is a Hilbert-Schmidt operator with kernel $t \in L^2(\mu \times \mu)$ satisfying*

$$\text{ess sup}_z \int |t(z,w)|^2 \, d\mu(w) \leqq K^2$$

where K is the norm of T as an operator from L^2 to L^∞.

We come now to the main result of this section.

THEOREM (2.5). *If A is a normal operator on $\mathscr{H}$ then $\mathscr{R}(\delta_A)$ contains no nonzero left or right ideal of $\mathscr{B}(\mathscr{H})$. Hence if $B \in \mathscr{B}(\mathscr{H})$ and $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ then $B \in \{A\}''$.*

Proof. Observe first that for any $S, T \in \mathscr{B}(\mathscr{H})$ the identity $X\delta_S(T') = \delta_S(XT') - \delta_S(X)T'$ implies that if $\mathscr{R}(\delta_S) \subset \mathscr{R}(\delta_T)$ then $\mathscr{R}(\delta_T)$ must contain the left (and dually, also the right) ideal of $\mathscr{B}(\mathscr{H})$ generated by $\delta_S(T')$ for each T' commuting with T. Hence if $\mathscr{R}(\delta_T)$ is known not to contain any left or right ideal ideals then $S \in \{T\}''$. Thus the second assertion of the theorem is a simple consequence of the first.

For $\xi, \eta \in \mathscr{H}$ let $\xi \otimes \eta$ denote the operator $\zeta \to (\zeta, \eta)\xi$. Every left ideal contains an ideal $\mathscr{H} \otimes \eta$ and so it is enough to show that $\mathscr{H} \otimes \eta \subset \mathscr{R}(\delta_A)$ implies $\eta = 0$. (The assertion for right ideals follows on taking adjoints.)

By restricting to the smallest reducing subspace of A that contains η we can suppose that A is the operator $f(z) \to zf(z)$ in $\mathscr{H} = L^2(\mu)$ for some regular Borel measure μ on $\mathbf{C}$ of compact support. Let P be a projection of norm 1 from $\mathscr{B}(\mathscr{H})$ onto $\{A\}'$.

For $\xi \in L^2(\mu)$ let $\gamma(\xi)$ be the unique element of $\mathscr{B}(\mathscr{H})$ with

$$\delta_A(\gamma(\xi)) = \xi \otimes \eta$$
$$P(\gamma(\xi)) = 0.$$

The operator γ is continuous by the closed graph theorem and if $M_h \in \mathscr{B}(\mathscr{H})$ is the multiplication operator on $\mathscr{H}$ induced by $h \in L^\infty(\mu)$ then $\gamma(M_h\xi) = M_h\gamma(\xi)$ because $\delta_A(M_h\gamma(\xi)) = M_h\delta_A(\gamma(\xi)) = M_h\xi \otimes \eta$ and $P(M_h\gamma(\xi)) = M_hP(\gamma(\xi)) = 0$. In particular, $\gamma(h) = M_h(\gamma(1))$ for $h \in L^\infty(\mu)$.

If $h \in L^\infty(\mu)$, $\xi \in L^2(\mu)$ then $\|M_h\gamma(1)\xi\| = \|\gamma(h)\xi\| \leqq \|\gamma\| \, \|h\|_2 \|\xi\|_2$ so that $h \to M_h\gamma(1)\xi = h \cdot \gamma(1)\xi$ is continuous in the L^2 norm, and therefore $\gamma(1)\xi \in L^\infty(\mu)$ with $\|\gamma(1)\xi\|_\infty \leqq \|\gamma\| \, \|\xi\|_2$. By Lemma (2.4) the operator $\gamma(1)$ is Hilbert-Schmidt with kernel $t(z, w)$ satisfying ess sup $\int |t(z, w)|^2 d\mu(w) = K^2 < \infty$. Fix a vector $\xi \in L^2(\mu)$. Then there is a measurable set $E = E_\xi$ with $\mu(E') = 0$ and

$$\delta_A(\gamma(1))\xi(z) = \int (z - w)t(z, w)\xi(w)d\mu(w)$$

for $z \in E$. Since $\delta_A(\gamma(1))\xi = (1 \otimes \eta)\xi = (\xi, \eta)$, the Cauchy-Schwarz inequality gives

$$|(\xi, \eta)| \leqq \left(\int |(z - w)\xi(w)|^2 d\mu(w)\right)^{\frac{1}{2}} \left(\int |t(z, w)|^2 d\mu(w)\right)^{\frac{1}{2}}$$

for $z \in E$. Therefore $|(\xi, \eta)| \leqq K \|(A - zI)\xi\| = K \|(A - zI)^*\xi\|$ almost everywhere, and consequently, for all $z \in \sigma(A)$ by continuity of the right side of this inequality. It follows from this that $\eta \in \mathcal{R}(A - zI)$ for each $z \in \mathbf{C}$ (see the proof of Lemma (1.1)) and therefore $\eta = 0$ by Lemma (2.2).

COROLLARY (2.6). *Let A be a normal operator on $\mathcal{H}$. If $B \in \mathcal{B}(\mathcal{H})$ and $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ then $\delta_B = \delta_A \sigma = \sigma \delta_A$ for some ultraweakly continuous linear operator σ from $\mathcal{B}(\mathcal{H})$ into itself with the property $\sigma(A_1'XA_2') = A_1'\sigma(X)A_2'$ for $X \in \mathcal{B}(\mathcal{H})$ and A_1', A_2' commuting with A.*

Proof. Suppose $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$. Then by (1.5) we can factor $\delta_B = \tilde{\delta}_A \tau_0$ for some $\tau_0 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\{A\}'$. Making use of a projection P of norm 1 from $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$ we can replace τ_0 by $\tau \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ to get $\delta_B = \delta_A \tau$. Thus for $X \in \mathcal{B}(\mathcal{H})$, $\tau(X)$ is the unique operator satisying $P(\tau(X)) = 0$, $\delta_A(\tau(X)) = \delta_B(X)$. Since $B \in \{A\}''$ it is easy to check that τ inherits the commutativity properties of P, that is $\tau(A_1'XA_2') = A_1' \cdot (X)A_2'$.

We now replace τ by an ultraweakly continuous σ with the desired properties by the following device: the map $(\tau \mid \mathcal{K})$ from $\mathcal{K}$ into $\mathcal{B}(\mathcal{H}) = \mathcal{K}^{**}$ has adjoint $(\tau \mid \mathcal{K})^*$ from $\mathcal{K}^{***}$ into $\mathcal{K}^*$. For $T \in \mathcal{T} = \mathcal{K}^* \subset \mathcal{K}^{***}$ we therefore obtain an operator $\alpha(T) = (\tau \mid \mathcal{K})^*(T)$ in $\mathcal{T}$. It is clear that $\alpha \in \mathcal{B}(\mathcal{T})$, that $(\delta_A \mid \mathcal{T})\alpha = \alpha(\delta_A \mid \mathcal{T}) = (\delta_B \mid \mathcal{T})$ and that $\delta_B = \alpha^* \delta_A = \delta_A \alpha^*$. Since $\alpha^* \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ agrees with τ on $\mathcal{K}$ we also have $\alpha^*(A_1'XA_2') = A_1'\alpha^*(X)A_2'$, first for $X \in \mathcal{K}$, then by ultraweak continuity of α^* and of multiplication by a fixed element of $\mathcal{B}(\mathcal{H})$, for all X in the ultraweak closure of $\mathcal{K}$ which is $\mathcal{B}(\mathcal{H})$. Thus $\sigma = \alpha^*$ satisfies the requirements of the Corollary.

3. On a separable space an operator B in the second commutant of a normal operator A must be a bounded Borel function of A. In this section we determine which such functions are admissible for range inclusion of the corresponding derivations. For this we need to develop some background information about Hadamard multipliers.

DEFINITION (3.1). A matrix (γ_{ij}) is a Hadamard multiplier of $\mathcal{B}(\mathcal{H})$ if there is an orthonormal basis $\{\xi_i\}$ of $\mathcal{H}$ such that for each $X \in \mathcal{B}(\mathcal{H})$ there is an operator $\Gamma(X) \in \mathcal{B}(\mathcal{H})$ with $(\Gamma(X)\xi_j, \xi_i) = \gamma_{ij}(X\xi_j, \xi_i)$ for all i, j.

Thus (γ_{ij}) is a H-multiplier if the Hadamard product of (γ_{ij}) and the matrix of any bounded operator is again the matrix of a bounded operator. I. Schur **[15]** gives several sufficient conditions that (γ_{ij}) be an H-multiplier, among them being that (γ_{ij}) is itself the matrix of an operator.

LEMMA (3.2). (1) *The condition that (γ_{ij}) be an H-multiplier is independent of the choice of orthonormal basis of $\mathcal{H}$.*

(2) *If* (γ_{ij}) *is an H-multiplier then* $X \to \Gamma(X)$ *defines a bounded linear operator on* $\mathcal{B}(\mathcal{H})$. *Moreover* Γ *is ultraweakly continuous and is the adjoint of the H-multiplier on* $\mathcal{T}(\mathcal{H})$ *induced by the transpose of* (γ_{ij}).

(3) *Let* $\{\Gamma_n\}$ *be a sequence of H-multipliers. If* $\sup_n \|\Gamma_n\| < \infty$ *and if* $(\Gamma_n(X)\xi_j, \xi_i) \to \alpha_{ij}(X\xi_j, \xi_i)$ *for each* $X \in \mathcal{B}(\mathcal{H})$ *and each* i, j *then* (α_{ij}) *is an H-multiplier.*

(4) *Let* $\alpha_{ij} = \begin{cases} 1 & \text{if} \quad i \leqq j \\ 0 & \text{if} \quad i > j \end{cases}$. *Then* (α_{ij}) *is not an H-multiplier on* $\mathcal{B}(\mathcal{H})$ *for* $\mathcal{H} = l^2(\mathbf{Z})$ *or for* $\mathcal{H} = l^2(\mathbf{Z}^+)$.

Proof. (1) If $\{\eta_i\}$ is another orthonormal basis of $\mathcal{H}$ and if $(\Gamma'(X)\eta_j, \eta_i) = \gamma_{ij}(X\eta_j, \eta_i)$ then $\Gamma'(X) = U^*\Gamma(UXU^*)U$ where U is the unitary operator defined by $U\eta_i = \xi_i$.

(2) That Γ is bounded on $\mathcal{B}(\mathcal{H})$ is a simple consequence of the closed graph theorem. The other two assertions of (2) are easy to verify.

(3) The hypotheses imply that for each $X \in \mathcal{B}(\mathcal{H})$ the map $T \to \lim_n \langle \Gamma_n(X), T\rangle$ is a bounded linear functional on the subspace of $\mathcal{T}(\mathcal{H})$ consisting of finite linear combinations of the operators $\xi_i \otimes \xi_j$. Since the dual of $\mathcal{T}$ is $\mathcal{B}(\mathcal{H})$ there is an operator $Z \in \mathcal{B}(\mathcal{H})$ such that $\langle Z, T\rangle = \lim_n \langle \Gamma_n(X), T\rangle$ for any T of the form $\xi_i \otimes \xi_j$. It follows that $(\alpha_{ij}(X\xi_j, \xi_i))$ is the matrix of Z and thus that (α_{ij}) is an H-multiplier.

(4) Consider first the case in which (α_{ij}) is a doubly infinite matrix $(i, j \in \mathbf{Z})$ and let $\xi_j(e^{i\theta}) = e^{ij\theta}$ be the usual basis of $L^2(0, 2\pi)$. If M_φ denotes the operator $f \to \varphi \cdot f$ on L^2 for a given $\varphi \in L^\infty$, then the Hadamard product of (α_{ij}) and the matrix of M_φ is the matrix associated with $M_{\tilde{\varphi}}$ where $\tilde{\varphi}$ is the function whose Fourier coefficients $(\tilde{\varphi}, \xi_n)$ vanish for $n < 0$ and agree with those of φ for $n \geqq 0$. Since there are $\varphi \in L^\infty$ for which $\tilde{\varphi} \notin L^\infty$ it follows that (α_{ij}) cannot be an H-multiplier of $L^2(0, 2\pi)$ and consequently cannot be an H-multiplier of $l^2(\mathbf{Z})$ either.

Consider now the matrix $\beta_{ij} = 1$ or 0 depending on whether or not $i \leqq j$ for $i, j \in \mathbf{Z}^+$. Then (β_{ij}) cannot be an H-multiplier of $l^2(\mathbf{Z}^+)$ because the doubly infinite matrix (α_{ij}) just mentioned is the sum of the direct sum $(\beta_{ij}) \oplus (\beta_{-j,-i})$ and a matrix which is an obvious H-multiplier of $l^2(\mathbf{Z})$.

There is another, perhaps more natural, way to see that the doubly infinite matrix (α_{ij}) of (4) is not a Hadamard multiplier of $l^2(\mathbf{Z})$ that we now sketch. It is enough to show that (α_{ij}) does not induce an operator α on the trace class matrices $\mathcal{T}$ on $l^2(\mathbf{Z})$. Now $\mathcal{T}$ is isometric with $l^2(\mathbf{Z}) \hat{\otimes} l^2(\mathbf{Z})$ so the convolution product gives rise to the map $(\rho S)_k = \Sigma_{i-j=k}\, s_{ij}$ of $\mathcal{T}$ into $A(\mathbf{Z})$ and in fact $A(\mathbf{Z})$ is isometric with $\mathcal{T}/\mathrm{Ker}(\rho)$. Clearly $\alpha(\mathrm{Ker}(\rho)) \subset \mathrm{Ker}(\rho)$ so α lifts to $\alpha' \in \mathcal{B}(A(\mathbf{Z}))$. Here α' is the operation of multiplication by the characteristic function of $\mathbf{Z}^+$ and it is well known that $A(\mathbf{Z})$ is *not* closed under this operation. ($L^1(\mathbf{T})$ is not closed under the Hilbert transform.) The harmonic analysis used here appears in [**13**; pp. 80–81] and [**9**; p. 64].

Hadamard multipliers are important for studying the range of a derivation mainly because of the following simple fact: (Recall that a *diagonal operator* is an operator for which there is an orthonormal basis of eigenvectors.)

LEMMA (3.3). *Suppose $A \in \mathcal{B}(\mathcal{H})$ is a diagonal operator with matrix* $\operatorname{diag}(\lambda_1, \lambda_2, \cdots)$ *with respect to the orthonormal basis $\{\xi_i\}$. If $B \in \mathcal{B}(\mathcal{H})$ then $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ if and only if $B = \operatorname{diag}(\mu_1, \mu_2, \cdots)$ with respect to $\{\xi_i\}$ and (γ_{ij}) is an H-multiplier of $\mathcal{B}(\mathcal{H})$ where*

$$\gamma_{ij} = \begin{cases} (\mu_i - \mu_j)(\lambda_i - \lambda_j)^{-1} & \text{if} \quad \lambda_i \neq \lambda_j \\ 0 & \text{if} \quad \lambda_i = \lambda_j \end{cases}.$$

Proof. Suppose $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$. Then $B \in \{A\}''$ by (2.5) and hence B has a diagonal matrix with respect to the given basis. If $X \in \mathcal{B}(\mathcal{H})$ there is $Z \in \mathcal{B}(\mathcal{H})$ with $\delta_B(X) = \delta_A(Z)$. Computing (i, j) entries yields $(\mu_i - \mu_j)(X\xi_j, \xi_i) = (\lambda_i - \lambda_j)(Z\xi_j, \xi_i)$. Now we can choose Z so that $P(Z) = 0$ where P is a norm one projection from $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$ For such a choice Lemma (2.1) shows that $(Z\xi_j, \xi_i) = 0$ whenever $\lambda_i = \lambda_j$ and therefore $(Z\xi_j, \xi_i) = \gamma_{ij}(X\xi_j, \xi_i)$ with γ_{ij} defined as in the statement of the Lemma. This proves that the multiplier condition is necessary for $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ and it is clear that it is also sufficient.

LEMMA (3.4). *Let μ be a finite measure on $\mathbf{C}$ and let A be the operator $h(z) \to zh(z)$ on $L^2(\mu)$. Suppose that $f \in C(\sigma(A))$ and that $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$. If B is a diagonal operator with distinct eigenvalues $\lambda_1, \lambda_2, \cdots$ in $\sigma(A)$ then $\mathcal{R}(\delta_{f(B)}) \subset \mathcal{R}(\delta_B)$.*

Proof. Let $\{\xi_i\}$ be an orthonormal basis of $\mathcal{H}$ such that $B\xi_i = \lambda_i\xi_i$ for $i = 1, 2, \cdots$. Fix an integer $n > 0$ and let $\varphi_1, \varphi_2, \cdots, \varphi_n$ be the normalized characteristic functions in $L^2(\mu)$ of disjoint neighborhoods $N_1, N_2, \cdots, N_n$ of the λ_i, each having diameter at most n^{-1}. Finally let $U = U_n$ be a partial isometry from $\mathcal{H}$ into $L^2(\mu)$ with

$$U\xi_i = \begin{cases} \varphi_i & 1 \leq i \leq n \\ 0 & i > n \end{cases}.$$

By (2.6) there is a map α from $\mathcal{T} = \mathcal{T}(L^2(\mu))$ into itself such that $(\delta_{f(A)} | \mathcal{T}) = \alpha(\delta_A | \mathcal{T}) = \delta_A\alpha$, and this induces a map β from $\mathcal{T}(\mathcal{H})$ into itself, namely $\beta(T) = U^*\alpha(UTU^*)U$. An easy calculation gives $(\beta(\xi_i \otimes \xi_j)\xi_l, \xi_k) = \beta_{ij}^{(n)}((\xi_i \otimes \xi_j)\xi_l, \xi_k)$

$$\beta_{ij}^{(n)} = \begin{cases} \mu(N_i)^{-1}\mu(N_j)^{-1} \int_{N_i} \int_{N_j} \lambda(z,w)\,d\mu(z)\,d\mu(w) & 1 \leqq i, j \leqq n \\ 0 & i > n \text{ or } j > n \end{cases}$$

and $\lambda(z,w) = (f(z) - f(w))(z-w)^{-1}(1-\delta_{z,w})$. Now if Γ_n is the Hadamard multiplication on $\mathcal{T}(\mathcal{H})$ associated with the matrix $(\beta_{ij}^{(n)})$ it follows that Γ_n and β agree on the operators $\xi_i \otimes \xi_j$ and since both operators are ultraweakly continuous we have $\|\Gamma_n\| = \|\beta\| \leqq \|\alpha\|$. Hence the matrix $(\gamma_{ij}^{(n)})$ defined by $\gamma_{ij}^{(n)} = \beta_{ij}^{(n)} - \delta_{ij}\beta_{ij}^{(n)}$ has multiplier norm at most $2\|\alpha\|$. Since $\gamma_{ij}^{(n)} \to \gamma_{ij}$ where

$$\gamma_{ij} = \begin{cases} (f(\lambda_j) - f(\lambda_i))(\lambda_j - \lambda_i)^{-1} & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$$

it follows from (3.2) that (γ_{ij}) is an H-multiplier on $\mathcal{T}(\mathcal{H})$ with multiplier norm at most $2\|\alpha\|$. Thus $(\delta_{f(B)} | \mathcal{T}) = \alpha_B(\delta_B \mid \mathcal{T}) = \delta_B\alpha_B$ for an operator α_B on $\mathcal{T}(\mathcal{H})$ with norm at most $2\|\alpha\|$ so that $\mathcal{R}(\delta_{f(B)}) \subset \mathcal{R}(\delta_B)$.

Our next result shows that the question whether $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$ depends only on the values of f on $\sigma(A)$ and not on the normal operator A itself:

COROLLARY (3.5). *Let A and f be as in the statement of the Lemma. If B is any normal operator with $\sigma(B) \subset \sigma(A)$ then $\mathcal{R}(\delta_{f(B)}) \subset \mathcal{R}(\delta_B)$.*

Proof. Any normal operator B on $\mathcal{H}$ is a countable direct sum of operators $h(z) \to zh(z)$ on L^2 spaces and each of these is uniformly approximable by diagonal operators (approximate the identity function by a simple function.) Hence B itself is the uniform limit of a sequence $\{B_n\}$ of diagonal operators on $\mathcal{H}$ and clearly we may also choose the B_n to have distinct diagonal entries for each n. Now for each n there is an operator α_n on $\mathcal{T}(\mathcal{H})$ with $(\delta_{f(B_n)} | \mathcal{T}) = \alpha_n(\delta_{B_n} \mid \mathcal{T}) = \delta_{B_n}\alpha_n$ and $\sup_n \|\alpha_n\| < \infty$ by the proof of the Lemma. Fix $X \in \mathcal{B}(\mathcal{H})$ and let $Z_n = \alpha_n^*(X)$. Since the Z_n are bounded we can pass to a subsequence if necessary to insure that the sequence $\{Z_n\}$ converges ultraweakly to some $Z \in \mathcal{B}(\mathcal{H})$. Then

$$\delta_{f(B)}(X) = (f(B) - f(B_n))X + (B_n - B)Z_n + (BZ_n - Z_nB) + Z_n(B - B_n) + X(f(B_n) - f(B))$$

so that, taking ultraweak limits, $\delta_{f(B)}(X) = \delta_B(Z) \in \mathcal{R}(\delta_B)$ as required.

THEOREM (3.6). *Let A be a normal operator on $\mathcal{H}$ with dominating scalar spectral measure μ* (see [5; Theorem 10, p. 916].). *If $B \in \mathcal{B}(\mathcal{H})$*

then $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ *if and only if* $B = f(A)$ *where* $f \in C(\sigma(A))$ *and* $(\alpha t)(z, w) = (f(z) - f(w))(z - w)^{-1} t(z, w)$ (taken as 0 when $z = w$) *is a trace class kernel with respect to* μ *whenever* $t(z, w)$ *is such a kernel.*

Proof. Suppose first that A is the operator $h(z) \to zh(z)$ on $L^2(\mu)$. If $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ then $B \in \{A\}''$ by Theorem (2.5) so that $B = f(A)$ for some $f \in L^\infty(\mu)$. Also, by (2.5) there is an operator α from the trace class operators $\mathcal{T}$ on $L^2(\mu)$ into itself such that $(\delta_B \mid \mathcal{T}) = \alpha(\delta_A \mid \mathcal{T}) = (\delta_A \mid \mathcal{T})\alpha$. If $t(z, w)$ is the kernel of a trace class operator on $L^2(\mu)$ then $(z - w)(\alpha t)(z, w) = (f(z) - f(w))t(z, w)$ almost everywhere $\mu \times \mu$. Now if μ has mass at z_0 and ξ_0 is the corresponding normalized characteristic function, then $(\alpha t)(z_0, z_0) = \langle \xi_0 \otimes \xi_0, \alpha t \rangle = \langle \alpha^*(\xi_0 \otimes \xi_0), t \rangle = 0$ because the range of α^* is contained in the range of $1 - P$, where P is the projection of $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$ used to construct α, and therefore $0 = P\alpha^*(\xi_0 \otimes \xi_0) = \alpha^*(P(\xi_0 \otimes \xi_0)) = \alpha^*(\xi_0 \otimes \xi_0)$. Thus $(\alpha t)(z, w) = (f(z) - f(w))(z - w)^{-1} t(z, w)$ (taken as 0 when $z = w$) and this holds $\mu \times \mu$ almost everywhere. To complete the proof we appeal to Theorem (4.1) below which implies that the multiplier condition on f just established forces f to be equal μ, a.e. to a continuous function on $\sigma(A)$.

Conversely, if αt is a trace class kernel for each trace class kernel t then $t \to \alpha t$ defines a bounded operator on the trace class by the closed graph theorem and $\alpha(\delta_A)t(z, w) = (f(z) - f(w))t(z, w) = \delta_{f(A)} t(z, w)$ for $z \neq w$ by definition of α and for $z = w$ since both sides of the equation are 0. Thus $\delta_B = \delta_A \alpha^*$ and $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$.

Consider now the general case. If $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ then there is a reducing subspace $\mathcal{H}_0$ of $\mathcal{H}$ on which A is unitarily equivalent to the operator $h(z) \to zh(z)$ on $L^2(\mu)$. The subspace $\mathcal{H}_0$ reduces B and since $\delta_B(\mathcal{H}_0) \subset \delta_A(\mathcal{H}_0)$ the first part of the argument shows that $B = f(A)$ where f has the asserted properties.

Conversely if $B = f(A)$ with f of the given form then $\mathcal{R}(\delta_{B_0}) \subset \mathcal{R}(\delta_{A_0})$ where A_0, B_0 are the restrictions of A, B to $\mathcal{H}_0$. Corollary (3.5) then implies that $\mathcal{R}(\delta_{\tilde{B}}) \subset \mathcal{R}(\delta_{\tilde{A}})$ where $\tilde{A}, \tilde{B}$ are the direct sum of countably many copies of A_0 and B_0 respectively. Since A is the restriction of $\tilde{A}$ to a reducing subspace it follows that $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$.

COROLLARY (3.7). *Let* A *be a normal operator on* $\mathcal{H}$ *and let* $B \in \mathcal{B}(\mathcal{H})$. *Then* $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ *if and only if there is a constant* c *such that* $\|\delta_B(X)\| \leqq c \|\delta_A(X)\|$ *for all* $X \in \mathcal{B}(\mathcal{H})$.

Proof. Suppose $\|\delta_B(X)\| \leqq c \|\delta_A(X)\|$ for all X. Then $B \in \{A\}''$ so that $B = f(A)$ for some bounded Borel function f. Also if $T \in \mathcal{T}$ then $\delta_B(T) = \delta_A(S)$ for some $S \in \mathcal{T}$ by (1.4) and, in the notation of (3.6), αt is the kernel of the trace class operator S_1 satisfying

$$\langle (1-P)(K), S\rangle = \langle K, S_1\rangle$$

for all $K \in \mathcal{K}$, i.e., $S_1 = S - (P^*(S)\,|\,\mathcal{K})$. Thus $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$. The converse is clear from (2.6).

4. We now consider the problem of determining the elements of $C(\sigma(A))$ that satisfy the multiplier condition of Theorem (3.6). It seems that, just as there is no satisfactory way of recognizing which periodic functions have absolutely convergent Fourier series, so no easy description of the class of functions in Theorem (3.6) exists.

THEOREM (4.1). *Let μ be a finite measure on* **C** *of compact support and let $f \in L^\infty(\mu)$ satisfy the criterion of Theorem* (3.6). *Then f is equal* a.e. *μ to a continuous function which satisfies a Lipschitz condition and is differentiable relative to $\sigma(A)$ at each nonisolated point of $\sigma(A)$.*

Proof. Let $\Lambda(z, w) = (f(z) - f(w))(z-w)^{-1}$ for $z \neq w$ and $\Lambda(z, z) = 0$. The function Λ is μ-measurable and if ξ, η are unit vectors in $L^2(\mu)$ then the trace class norm of the operator with kernel $\Lambda(z, w)\xi(z)\eta(w)$ is no less than its Hilbert-Schmidt norm

$$\{\textstyle\int\int |\Lambda(z, w)|^2\, |\xi(z)|^2\, |\eta(w)|^2\, d\mu(z)d\mu(w)\}^{\frac{1}{2}}$$

for which the upper bound as ξ, η vary is $K = \operatorname{ess\,sup} |\Lambda(z, w)|$. Thus $\Lambda \in L^\infty(\mu \times \mu)$ and $|f(z) - f(w)| \leqq K\,|z-w|$ for almost all (z, w). Put

$$E = \{z \in \operatorname{Supp}(\mu) \colon |f(z) - f(w)| \leqq K\,|z-w| \text{ for almost all } w \in \operatorname{Supp}(\mu)\}.$$

The complement of E is of measure 0. If $z_1, z_2 \in E$ we have

$$\begin{aligned} |f(z_1) - f(z_2)| &\leqq |f(z_1) - f(w)| + |f(w) - f(z_2)| \\ &\leqq K(|z_1 - w| + |z_2 - w|) \end{aligned}$$

except for values of w in a set of measure 0. If $\mu(\{z_1\}) = 0$ then we can find a sequence of values of w outside this exceptional set converging to z_1 so $|f(z_1) - f(z_2)| \leqq K\,|z_1 - z_2|$. If $\mu(\{z_1\}) \neq 0$ we get the same inequality from the fact that z_2 belongs to E. As E is dense in $\operatorname{Supp}(\mu)$ and f is uniformly continuous on E there is a continuous function g on $\operatorname{Supp}(\mu)$ equal to f on E, that is, equal to f a.e. By continuity $|g(z) - g(w)| \leqq K\,|z-w|$ for z, w in $\sigma(A)$.

Suppose now that λ is a nonisolated point of $\sigma(A)$ and that g is not differentiable at λ. Then, as g is a Lipschitz function, we can find a sequence $\{\lambda_i\}$ $(-\infty < i < \infty)$ of distinct points in $\sigma(A) - \{\lambda\}$ with $\lambda_i \to \lambda$ as $i \to \pm\infty$ and

$$\lim_{i \to -\infty} (g(\lambda_i) - g(\lambda))(\lambda_i - \lambda)^{-1} = p$$

$$\lim_{i \to +\infty} (g(\lambda_i) - g(\lambda))(\lambda_i - \lambda)^{-1} = q \neq p.$$

Replacing g by $h(z) = (g(z) - pz)(q - p)^{-1}$ if necessary we can assume that $p = 0$, $q = 1$ since g satisfies the criterion of Theorem (3.6) if and only if h does.

If

$$\gamma_{ij} = \begin{cases} (g(\lambda_i) - g(\lambda_j))(\lambda_i - \lambda_j)^{-1} & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$$

then (γ_{ij}) is an Hadamard multiplier of $\mathcal{B}(l^2(\mathbf{Z}))$ by (3.3) and (3.4). This implies that the matrix (β_{ij}) where $\beta_{ij} = \gamma_{-i,j}$ for $i, j \geqq 1$ is an H-multiplier of $\mathcal{B}(l^2(\mathbf{Z}^+))$. Indeed if $\{\varphi_i : -\infty < i < \infty\}$ is an orthonormal basis of $l^2(\mathbf{Z})$ and $\{\xi_i : i \geqq 1\}$ is an orthonormal basis of $l^2(\mathbf{Z}^+)$ then the operator Γ' on $\mathcal{B}(l^2(\mathbf{Z}^+))$ corresponding to H-multiplication by (β_{ij}) is given by $\Gamma'(X) = V^*\Gamma(VXU^*)U$ where Γ is the operator on $\mathcal{B}(l^2(\mathbf{Z}^+))$ associated with (γ_{ij}) and U, V are the isometries from $l^2(\mathbf{Z}^+)$ into $l^2(\mathbf{Z})$ defined by $U\xi_i = \varphi_i$, $V\xi_i = \varphi_{-i}$ for $i \geqq 1$.

Fix a positive integer n. Since

$$\lim_{i\to\infty} \lim_{j\to\infty} \beta_{ij} = 0, \qquad \lim_{j\to\infty} \lim_{i\to\infty} \beta_{ij} = 1$$

we can find one-to-one maps π, σ from $\mathbf{Z}^+$ into itself such that

$$|\beta_{\pi(i),\sigma(j)} - 1| < n^{-1} \qquad \text{for} \qquad i > j \geqq 1$$

$$|\beta_{\pi(i),\sigma(j)}| < n^{-1} \qquad \text{for} \qquad 1 \leqq i \leqq j.$$

(See [**8**; p. 694].) Since $(\beta_{\pi(i),\sigma(j)})$ is clearly an H-multiplier with multiplier norm at most equal to the multiplier norm of the matrix (β_{ij}), and since n can be taken arbitrarily large here, it follows from (3) of Lemma (3.2) that (α_{ij}) is a Hadamard multiplier where

$$\alpha_{ij} = \begin{cases} 1 & \text{for } i > j \\ 0 & \text{for } i \leqq j \end{cases}.$$

This contradicts (4) of Lemma (3.2) and completes the proof.

COROLLARY (4.2). *Suppose that f is continuous on the closed unit disk D and that $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$ for some normal operator A with $\sigma(A) = D$. Then f is analytic at each point of D. In fact, f and its derivative f' belong to H^∞, f satisfies the Lipschitz condition $|f(e^{i\alpha}) - f(e^{i\beta})| \leqq K|\alpha - \beta|$ on the unit circle, and the Taylor series of f is absolutely convergent on D.*

Proof. That f and f' belong to H^∞ is clear from the theorem. The other assertions about f are consequence of these, a theorem of Hardy and Littlewood, and Hardy's inequality. See [**6**; pp. 48, 78].

There is a natural anti-involution $\tau \to \tau^*$ on $\mathcal{B}(\mathcal{B}(\mathcal{H}))$ defined by $\tau^*(X) = \tau(X^*)^*$ for $X \in \mathcal{B}(\mathcal{H})$. With respect to this involution it is easy to check that δ_A and $(\delta_A)^*$ commute if and only if A is normal so that the term "normal derivation" is unambiguous. It is known [**2**] that normal derivations on $\mathcal{B}(\mathcal{H})$ exhibit some of the properties of normal operators on $\mathcal{H}$, for example the orthogonality of range and kernel mentioned earlier. But Theorem (4.1) indicates that whereas a normal operator on $\mathcal{H}$ and its adjoint always have the same range, this property fails in general for normal derivations, even those induced by diagonal operators, because $z \to \bar{z}$ is not analytic. (However, the ranges of δ_A and δ_{A^*} have the same norm closure. In fact, $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)^-$ for any B in the C^*-algebra generated by the normal operator A.) This fact can be expressed in a slightly different way to provide a negative answer to a question raised in a conversation with the authors of [**3**].

COROLLARY (4.3). *There exist a diagonal operator A* (with distinct eigenvalues) *and a sequence $\{X_n\}$ in $\mathcal{B}(\mathcal{H})$ such that $\|AX_n - X_nA\| \to 0$ but $\|AX_n^* - X_n^*A\| > 1$ for each $n = 1, 2, \cdots$.*

Proof. There exists a diagonal operator A with $\mathcal{R}(\delta_A *)$ not contained in $\mathcal{R}(\delta_A)$ by the preceding remarks. For such a choice of A Corollary (3.7) implies that for each n there is an operator Z_n with $\|\delta_A * (Z_n)\| > n\,\|\delta_A(Z_n)\|$. Then $\delta_A(Z_n) \neq 0$ by the Fuglede theorem so the choice $X_n = Z_n / n\,\|\delta_A(Z_n)\|$ satisfies the required conditions.

REMARK. The sequence $\{X_n\}$ cannot be chosen to be uniformly bounded however [R. L. Moore, private communication.]

In [**3**] a counterexample is constructed to show that if A is a normal operator on $\mathcal{H}$ and P is a projection in $\{A\}''$, then in general one cannot find a positive number δ corresponding to each $\epsilon > 0$ so that the conditions $X \in \mathcal{B}(\mathcal{H})$, $\|AX - XA\| < \delta$ imply $\|PX - XP\| < \epsilon$. Or equivalently, by (3.7), the condition $P \in \{A\}''$ is not sufficient for $\mathcal{R}(\delta_P) \subset \mathcal{R}(\delta_A)$. Theorem (4.1) helps to explain this situation more fully:

COROLLARY (4.4). *Let A be a normal operator on $\mathcal{H}$ with spectral measure $E(\cdot)$ and let P be a projection on $\mathcal{H}$. Then $\mathcal{R}(\delta_P) \subset \mathcal{R}(\delta_A)$ if and only if there are disjoint closed sets Δ_0, Δ_1 with $\Delta_0 \cup \Delta_1 = \sigma(A)$ and $P = E(\Delta_1)$.*

Proof. If $\mathcal{R}(\delta_P) \subset \mathcal{R}(\delta_A)$ then by (4.1) there is a continuous function f on $\sigma(A)$ with $P = f(A)$. The spectral mapping theorem implies that $\sigma(A)$ is the union of the disjoint closed sets $\Delta_0 = f^{-1}(0)$ and $\Delta_1 = f^{-1}(1)$. Also $P = f(A) = \int f(\lambda)dE_\lambda = E(\Delta_1)$.

Conversely if $P = E(\Delta_1)$ where Δ_0, Δ_1 are disjoint closed sets whose union is the spectrum of A then by the Riesz-Dunford functional calculus $P = f(A)$ for some function f that is analytic in a neighborhood of $\sigma(A)$. Hence $\mathcal{R}(\delta_P) = \mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$ by (3.6) or by the result of Weber **[20]** mentioned in the Introduction.

The sufficiency of the condition on P may also be established directly by considering the decomposition $\mathcal{H} = E(\Delta_0)\mathcal{H} \oplus E(\Delta_1)\mathcal{H}$ and appealing to the theorem of Lumer and Rosenblum **[11]** on the solvability of the operator equations $A_1Z - ZA_0 = X$, $A_0Y - YA_1 = W$ for operators A_0, A_1 with disjoint spectra.

A theorem of Anderson **[1]** shows that if N is a normal operator then $\mathcal{R}(\delta_N) \subset \cup\, \mathcal{R}(\delta_A)$ where the union is taken over the set of all self adjoint operators A in $\mathcal{B}(\mathcal{H})$. That is, any commutator of the form $NX - XN$ can also be written $AY - YA$ for some $A = A^*$ and $Y \in \mathcal{B}(\mathcal{H})$. Theorem (4.1) implies that one cannot improve this to: $\mathcal{R}(\delta_N) \subset \mathcal{R}(\delta_A)$ for some $A = A^*$. Indeed if $\sigma(N)$ has infinite one-dimensional Hausdorff measure then we cannot have $\sigma(N) = f(\sigma(A))$, and hence cannot have $N = f(A)$, for any self-adjoint A and Lipschitz f.

Theorem (4.1) also permits a somewhat simpler proof of the theorem of Anderson and Foias **[1]** which determines when the range of a normal derivation is closed.

COROLLARY (4.5). *Let A be a normal operator on $\mathcal{H}$. Then $\mathcal{R}(\delta_A)$ is norm closed in $\mathcal{B}(\mathcal{H})$ if and only if the spectrum of A is finite.*

Proof. Suppose that $\mathcal{R}(\delta_A)$ is norm closed in $\mathcal{B}(\mathcal{H})$. If P is a norm 1 projection of $\mathcal{B}(\mathcal{H})$ onto the commutant of A then there is a constant $c > 0$ such that $\|\delta_A(X)\| \geqq c\,\|X\|$ for all $X \in \mathcal{R}(1 - P)$. For $B \in \{A\}''$ and $X \in \mathcal{B}(\mathcal{H})$ we have $P(BX - XB) = BP(X) - P(X)B = 0$ and so $\mathcal{R}(\delta_B) \subset \mathcal{R}(1 - P)$. Hence

$$c\,\|\delta_B(X)\| \leqq \|\delta_A(\delta_B(X))\| = \|\delta_B(\delta_A(X))\| \leqq \|\delta_B\|\,\|\delta_A(X)\|.$$

Therefore $\mathcal{R}(\delta_B) \subset \mathcal{R}(\delta_A)$ by (3.7) so that $B = f(A)$ for some continuous function f by (3.6). Thus $\{A\}''$, the von Neumann algebra generated by A, coincides with the C^*-algebra generated by A. It follows that $\sigma(A)$ is extremally disconnected and therefore is a finite subset of $\mathbf{C}$.

Conversely if $\sigma(A)$ is finite then A is a linear combination of orthogonal projections and this easily implies that δ_A has closed range.

REMARK. With a different argument one can show that if A is normal with infinite spectrum then $\mathcal{R}(\delta_A) + \{A\}'$ is not norm dense in $\mathcal{B}(\mathcal{H})$. (See **[2]**.) Hence $\mathcal{R}(\delta_A) \subsetneq \mathcal{R}(1-P)$ for any projection P of norm 1 from $\mathcal{B}(\mathcal{H})$ onto $\{A\}'$.

We conclude this section with an example that confirms a remark of J. Taylor [**18**; p. 29].

EXAMPLE (4.6). Let A be the operator of multiplication by the sequence $\lambda_i = i^{-1} (i \neq 0)$, $\lambda_0 = 0$ in $l^2(\mathbf{Z})$, and let f be the Lipschitz function $f(x) = x^+ = \frac{1}{2}(x + |x|)$. Then $\mathcal{R}(\delta_{f(A)}) \not\subset \mathcal{R}(\delta_A)$. That is, there does not exist a constant c such that $\|f(A)X - Xf(A)\| \leqq c\, \|AX - XA\|$ for all operators X on $l^2(\mathbf{Z})$.

This follows at once from (4.1) since f is not differentiable at $x = 0$. The result can also be proved directly by observing that

$$\gamma_{ij} = \begin{cases} (f(\lambda_i) - f(\lambda_j))(\lambda_i - \lambda_j)^{-1} & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j \end{cases}$$

cannot be an H-multiplier of $\mathcal{B}(l^2(\mathbf{Z}))$ because this would imply that the matrix $(j(i+j)^{-1})$ is an H-multiplier of $\mathcal{B}(l^2(\mathbf{Z}^+))$ which is impossible because of the relations

$$\lim_{j \to \infty} j(i+j)^{-1} = 1, \qquad \lim_{i \to \infty} j(i+j)^{-1} = 0.$$

(See the proof of (4.1).)

5. We conclude by giving a condition which is sufficient for f to satisfy the criterion of Theorem (3.6) for self-adjoint operators.

THEOREM (5.1). *Let f be a complex valued function on* $\mathbf{R}$ *with continuous third derivative. If A is a self-adjoint operator on $\mathcal{H}$ then* $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$.

Proof. By (3.5) it is sufficient to prove the theorem for the case in which A is the operator $f(x) \to x \cdot f(x)$ on $L^2(I)$ where I is a compact subinterval of $\mathbf{R}$. Define

$$\Lambda(x, y) = \begin{cases} (f(x) - f(y))(x - y)^{-1} & \text{for} \quad x \neq y \\ f'(x) & \text{for} \quad x = y. \end{cases}$$

Then Λ is a $C^{(2)}$ function on $\mathbf{R} \times \mathbf{R}$ and, without altering Λ in a neighborhood of $\sigma(A) \times \sigma(A)$ we can assume that Λ is doubly periodic with periods p, q say. The Fourier coefficients $\lambda_{kl} = p^{-1}q^{-1} \int_0^q \int_0^p \Lambda(x, y) \exp 2\pi i (kxp^{-1} + lyq^{-1}) dx\, dy$ satisfy $\{k^2\lambda_{kl}\} \in l^2(\mathbf{Z}^2)$, $\{l^2\lambda_{kl}\} \in l^2(\mathbf{Z}^2)$ because $\partial^2\Lambda/\partial x^2$ and $\partial^2\Lambda/\partial y^2$ belong to L^2. Since $\{(k^2 + l^2)^{-1}\} \in l^2(\mathbf{Z}^2)$ it follows that $\{\lambda_{kl}\} \in l^1(\mathbf{Z}^2)$.

Now if $t(x, y)$ is the kernel of a trace class operator T on $L^2(I)$ then $(\alpha_{kl} t)(x, y) = t(x, y) \exp(-2\pi i (kxp^{-1} + lyq^{-1}))$ is the kernel of the operator UTV where U and V are unitary so α_{kl} is an operator of norm 1 in $\mathcal{B}(\mathcal{T})$. Hence $\alpha = \Sigma_{k,l} \lambda_{kl} \alpha_{kl}$ is an operator in $\mathcal{B}(\mathcal{T})$ and this is $t \to \Lambda t$ because $\Lambda = \Sigma\, \lambda_{kl} \exp(-2\pi i (kxp^{-1} + lyq^{-1}))$. Thus Λt is a trace class kernel whenever t is and it follows as in the proof of Theorem (3.6) that $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$.

Note that now that we know $\mathcal{R}(\delta_{f(A)}) \subset \mathcal{R}(\delta_A)$ Theorem (3.6) shows that f also satisfies the multiplier condition of that theorem. That is, the function $\Lambda(z, w)$, taken to be 0 when $z = w$, rather than $f'(z)$, also multiplies trace class kernels.

References

1. J. H. Anderson, *Derivations, commutators, and the essential numerical range*, Ph.D. thesis, Indiana University, 1971.

2. ———, *On normal derivations*, Proc. Amer. Math. Soc., **38** (1973), 135–140.

3. J. J. Bastian, K. J. Harrison, *Subnormal weighted shifts and asymptotic properties of normal operators*, Proc. Amer. Math. Soc., **42** (1974), 475–479.

4. J. Dixmier, *Les fonctionelles lineaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert*, Ann. Math., **51** (1950), 387–408.

5. N. Dunford, J. T. Schwartz, *Linear Operators, Part II*, Interscience, New York, 1963.

6. P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

7. B. E. Johnson, *Continuity of linear operators commuting with continuous linear operators*, Trans. Amer. Math. Soc., **128** (1967), 88–102.

8. ———, *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math., **94** (1972), 685–698.

9. Y. Katznelson, *An introduction to Harmonic Analysis*, Wiley, New York, 1968.

10. V. B. Korotkov, *Integral operators with Carleman kernels*, Doklady Akad. Nauk SSSR **165** (1965), 1496–1499.

11. G. Lumer, M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc., **10** (1959), 32–41.

12. J. von Neumann, *On rings of operators III*, Ann. of Math. **41** (1940), 94–161.

13. M. A. Rieffel, *Multipliers and tensor products of L^p spaces of locally compact groups*, Studia Math. **33** (1969), 71–82.

14. R. Schatten, *A theory of cross-spaces*, Ann. Math. Studies No. 26, Princeton, 1950.

15. I. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen*, J. Reine Angew. Math., **140** (1911), 1–28.

16. J. G. Stampfli, *Derivations on $\mathcal{B}(\mathcal{H})$: The range*, Illinois J. Math., **17** (1973), 518–524.

17. J. Tomiyama, *On the projection of norm one in W^*-algbras*, Proc. Japan Acad., **33** (1957), 608–612.
18. J. L. Taylor, *Functions of several noncommuting variables*, Bull. Amer. Math. Soc., **79** (1973), 1–34.
19. R. E. Weber, *Derivation ranges*, Ph.D. thesis, Indiana University, 1972.
20. ———, *Analytic functions, ideals, and derivation ranges*, Proc. Amer. Math. Soc.
21. J. Weidmann, *Carlemanoperatoren*, Manuscripta Math. **2** (1970), 1–38.
22. J. P. Williams, *On the range of a derivation*, Pacific J. Math., **38** (1971), 273–279.
23. ———, *On the range of a derivation II*, to appear in Proc. Roy. Irish Acad.

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A DYNAMICAL CRITERION FOR CONJUGATE POINTS

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This paper presents a technique for establishing the existence of conjugate points for real fourth order differential equations defined on an interval $[\alpha, \infty)$. The point $\beta > \alpha$ is conjugate to α if there exists a nontrivial solution $y(x)$ of the equation which satisfies

$$y(\alpha) = y'(\alpha) = 0 = y(\beta) = y'(\beta).$$

An important feature of this technique is that it is not limited to equations of selfadjoint type and that the general theory applies to nonlinear equations as well.

For the special equation

$$l[y] \equiv (p_2(t)y'')'' + p_0(t)y = 0 \qquad (p_2(t) > 0) \tag{1.1}$$

criteria for the existence of conjugate points have been established by Leighton and Nehari [4] under the additional assumption $p_0(t) < 0$. Subsequent studies (see [6]) have extended parts of this theory to the general real selfadjoint equation

$$l[y] \equiv (p_2(t)y'')'' - (p_1(t)y')' + p_0(t)y = 0 \tag{1.2}$$

or the general real equation

$$l[y] \equiv (p_2(t)y'' - q_2(t)y')'' - (p_1(t)y' - q_1(t)y)' + p_0(t)y = 0. \tag{1.3}$$

replacing hypotheses on the coefficients with hypotheses specifying the nonexistence of solutions with certain orders of zeros. In this way, properties of solutions of (1.1), which were established in [4], became hypotheses which allowed the consideration of more general equations.

The present paper follows a similar pattern. In §2, we consider a second order system which can be used to represent equations of the form (1.2) or (1.3) and allows a simple dynamical interpretation in terms of a particle of unit mass in a force field. By making a number of qualitative assumptions regarding this force field which are motivated by (1.1), we demonstrate the existence of conjugate points for such systems. In §3, we establish conditions on the coefficients of the differential system which assure that these qualitative assumptions are satisfied; these conditions on the coefficients of the system are translated to conditions on the coefficients of the related fourth order equation in §4.

2. A related second order system. In this section we assume that the fourth order equation in question is represented by the second order system

$$\begin{aligned} y'' &= a(t)y + b(t)x \\ x'' &= c(t)y + d(t)x \end{aligned} \tag{2.1}$$

whose coefficients are continuous in $[\alpha, \infty)$. It has been shown by Whyburn [7] that the selfadjoint fourth order equation (1.2) can be represented in the form (2.1) with $b(t) = 1/p_2(t) > 0$ and $a(t) \equiv d(t)$. The author [1] has shown that the general real linear fourth order equation (1.3) can also be reduced to the form (2.1), with the nonselfadjointness reflected by the inequality of $a(t)$ and $d(t)$. In particular, the equation (1.1) can be represented in the form (2.1) with $b(t) = 1/p_2(t)$, $c(t) = -p_0(t)$, and $a(t) = d(t) \equiv 0$.

It will be helpful to interpret (2.1) as representing the motion of a particle of unit mass in the (x, y)-plane with t denoting time. Our objective is to impose conditions on the force field

$$\vec{F}(t) = (F_x(t), F_y(t)) = (c(t)y + d(t)x, a(t)y + b(t)x)$$

which assure the existence of a conjugate point — i.e., the existence of a trajectory C in the (x, y)-plane which is tangent to the x-axis at $t = \alpha$ and $t = \beta$.

This problem can be normalized by considering initial conditions of the form

$$y(\alpha) = y'(\alpha) = 0; \quad x(\alpha) = 1; \quad x'(\alpha) = v_0.$$

Physically this corresponds to firing a particle of unit mass from $(x, y) = (1, 0)$ tangent to the x-axis with velocity v_0 in the positive x direction. The resulting one-parameter family of trajectories will be denoted by $C(v_0)$. We also denote by I, II, III, and IV the open quadrants of the (x, y)-plane.

Motivated by the system representation of (1.1), we consider the following conditions on the force field $\vec{F}$:

(A) If for some $t_0 \geqq \alpha$ the quantities $y(t_0), y'(t_0), x(t_0)$ and $x'(t_0)$ are all nonnegative (but not all zero), then $y(t), y'(t), x(t)$ and $x'\ t)$ are all positive for $t > t_0$.

(B) No trajectory $C(v_0)$ can remain in II for arbitrarily large values of t.

(C) No trajectory in I satisfies

(i) $x(t) \downarrow x_0 \geqq 0$ and $y(t) \uparrow \infty$ as $t \to \infty$,

or

(ii) $y(t) \downarrow y_0 \geqq 0$ and $x(t) \uparrow \infty$ as $t \to \infty$,

nor can any trajectory in I tend to a finite limit point (x_0, y_0) in the closure of I as $t \to \infty$.

(D) No trajectory can go directly from II to I to II.

LEMMA 2.1. *There exist values of* v_0 *such that* $C(v_0)$ *enters the closed lower half plane*

$$\{(x, y) \mid y \leqq 0\}.$$

Proof. By **[1]** the system (2.1) represents a fourth order linear differential equation of the form $l[y] = 0$. If $\{y_i(t)\}$ $(i = 1, \cdots, 4)$ represents a fundamental set of solutions, one can find a linear combination $y(t) = \Sigma_{i=1}^4 c_i y_i(t)$ having three preassigned zeros. In particular for any $\beta > \alpha$, there exists a solution $y(t)$ satisfying $y(\alpha) = y'(\alpha) = y(\beta) = 0$.

THEOREM 2.2. *If conditions* (A)–(D) *are satisfied, then there exists a nontrivial solution* $y(t), x(t)$ *of* (2.1) *satisfying* $y(\alpha) = y'(\alpha) = 0 = y(\beta) = y'(\beta)$ *for* $\alpha < \beta < \infty$.

Proof. Solutions of (2.1) depend continuously on the initial data, and for the normalized problem under consideration the only initial parameter is v_0. It follows that in any compact interval $[\alpha, \gamma]$, $C(v_0)$ can be approximated uniformly by trajectories of the form $C(v_0 + \epsilon)$ for $|\epsilon|$ sufficiently small. Consider first

$$V_1 = \{v_0 \mid C(c_0) \text{ enters } III \cup IV\}.$$

By Lemma 2.1 there are trajectories $C(v_0)$ which are either tangent to the x-axis or enter the lower half plane, and we may therefore assume that V_1 is not empty. Furthermore, (A) implies that if $v_0 \geqq 0$, then $C(v_0)$ is "trapped" in I for all $t > \alpha$, so that V_1 is also bounded above by zero. Finally the continuous dependence of $C(v_0)$ on v_0 implies that V_1 is an open subset of **R**. Consider next

$$V_2 = \{v_0 \mid C(v_0) \text{ remains in the open upper half plane for all } t > \alpha\}.$$

By (A), V_2 contains $[0,\infty)$ and is therefore not empty. We shall show that V_2 is also open. Since by (B), $C(v_0)$ cannot remain in II for all $t > \gamma$, and by (D) no trajectory can go from II to I to II, we may restrict our attention to trajectories $C(v_0)$ for which there exists $\gamma > \alpha$ such that $C(v_0)$ remains in I for $t > \gamma$. Condition (C) rules out limit points in I as well as asymptotic trajectories for which $x'(t)$ and $y'(t)$ have opposite signs for all $t > \gamma$, so that every trajectory which remains in I eventually has positive values of $y(t), y'(t), x(t)$, and $x'(t)$. Because of the continuous dependence of $C(v_0)$ on the parameter v_0, it follows that neighboring trajectories will also eventually have positive values of $y(t), y'(t), x(t)$, and $x'(t)$ and that V_2 is an open subset of $\mathbf{R}$.

Consider now $\tilde{v} = \sup V_1$. Since $\tilde{v}$ belongs to neither V_1 nor V_2, it follows that $C(\tilde{v})$ lies in $\overline{I \cup II}$ but not in $I \cup II$ — i.e., $C(\tilde{v})$ must be tangent to the x-axis for some $x = \beta > \alpha$. This completes the proof.

3. Criteria for conjugate points.

We now consider the task of imposing conditions on the system (2.1) such that properties (A)–(D) are satisfied. A basic assumption which will be made throughout is that the coefficients of (2.1) are positive in $[\alpha,\infty)$. The reason for this assumption is the following.

THEOREM. 3.1. *If the coefficients of* (2.1) *are positive in* $[\alpha,\infty)$, *then* (A) *is satisfied.*

Proof. This follows readily from the integral representation

$$y(t) = y(t_0) + y'(t_0)(t-t_0) + \int_{t_0}^{t}\int_{t_0}^{s} [a(\tau)y(\tau) + b(\tau)x(\tau)]\,d\tau\,ds$$

$$x(t) = x(t_0) + x'(t_0)(t-t_0) + \int_{t_0}^{t}\int_{t_0}^{s} [c(\tau)y(\tau) + d(\tau)x(\tau)]\,d\tau\,ds$$

for solutions of (2.1).

In order to establish (B), we introduce a vector representation

$$Y'' = A(t)Y \tag{3.1}$$

for (2.1) where

$$Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}; \qquad A(t) = \begin{pmatrix} d(t) & c(t) \\ b(t) & a(t) \end{pmatrix}.$$

Also H will denote a constant vector

$$H = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and inner products will be denoted by $\langle \cdot, \cdot \rangle$ so that

$$\langle H, Y(t) \rangle = y(t) - x(t)$$

etc.

THEOREM 3.2. *If $c(t) \geqq a(t) > 0$ and $b(t) \geqq d(t) > 0$ in $[\alpha, \infty)$ and the equation*

$$u'' + \min\{b(t) - d(t),\ c(t) - a(t)\}u = 0 \tag{3.2}$$

is oscillatory at $t = \infty$, then (B) *is satisfied.*

Proof. If Y is a nonzero element in II then $x < 0$, $y > 0$ and

$$\langle H, Y \rangle = y - x > 0.$$

Also

$$-\langle H, AY \rangle = (b - d)(-x) + (c - a)y \geqq \min\{b - d,\ c - a\}(y - x) \geqq 0.$$

Therefore

$$-\frac{\langle H, AY \rangle}{\langle H, Y \rangle} \geqq \min\{b - d,\ c - a\} \tag{3.3}$$

for all $Y \in II$.

Define $m(t) = \min\{b(t) - d(t),\ c(t) - a(t)\}$ and let $t_1 < t_2 < \cdots$ be the zeros of an oscillatory solution of (3.2) where $t_k \uparrow \infty$. If $Y(t)$ remains in II for all $t \geqq \gamma$, then $\langle H, Y(t) \rangle > 0$ in $[\gamma, \infty)$ and a direct calculation yields

$$\frac{d}{dt}\left[uu' - u^2 \frac{\langle H, Y' \rangle}{\langle H, Y \rangle}\right] = -mu^2 - u^2 \frac{\langle H, AY \rangle}{\langle H, Y \rangle} + \left[u' - u \frac{\langle H, Y' \rangle}{\langle H, Y \rangle}\right]^2.$$

If $t_{k+1} > t_k \geqq \gamma$ we have

$$0 = \left[uu' - u^2 \frac{\langle H, Y' \rangle}{\langle H, Y \rangle}\right]_{t_k}^{t_{k+1}} \geqq -\int_{t_k}^{t_{k+1}} \left[m + \frac{\langle H, AY \rangle}{\langle H, Y \rangle}\right] u^2\, dt$$

with equality if and only if $u(t) \equiv \langle H, Y(t)\rangle$ in (t_k, t_{k+1}). Since $\langle H, Y(t)\rangle$ is assumed positive in $[\gamma, \infty)$ the above inequality must be strict. But this contradicts (3.3) and completes the proof.

Condition (C) requires that we preclude certain asymptotic paths and paths of finite length in I.

THEOREM 3.3. *If*

$$\int^{\infty} tb(t)\,dt = \infty \quad \textit{and} \quad \int^{\infty} tc(t)\,dt = \infty \tag{3.4}$$

then condition (C) *is satisfied.*

Proof. Consider first an asymptotic trajectory in I for which $y(t) \downarrow y_0 \geqq 0$ and $x(t) \uparrow \infty$ as $t \to \infty$. Since $x'' > 0$ in I there exist positive constants k and γ such that $x(t) \geqq kt$ for $t \geqq \gamma$. Since $y'' \geqq b(t)x(t)$ in I we have

$$y'(t) \geqq y'(t_0) + k \int_{t_0}^{t} tb(t)\,dt.$$

Thus the first part of (3.4) is inconsistent with such asymptotic trajectories, and the second part of (3.4) similarly precludes asymptotic trajectories for which $x(t) \downarrow x_0 \geqq 0$ and $y(t) \uparrow \infty$ as $t \to \infty$.

To deal with paths of finite length which might terminate in $\bar{I}$, we note that $x'' > 0$ and $y'' > 0$ at every point of $\bar{I}$ except (0, 0). Thus the origin is the only equilibrium point in $\bar{I}$ and the only point at which finite paths might terminate.

There are two cases to consider in completing the proof:

(i) *The trajectory never leaves* $\bar{I}$. In this case $y'' > 0$ for all $t > \alpha$ and $y(t)$ is bounded away from zero in $[\gamma, \infty)$ for every $\gamma > \alpha$.

(ii) *The trajectory leaves* $\bar{I}$ *and re-enters.* In this case x' or y' is positive at the time the trajectory crosses into I and the positivity of x'' and y'' precludes the possibility of the trajectory approaching the origin.

Finally we note that a very similar argument to that used above establishes (D). If a trajectory enters I from II at time t_0, then $x'(t_0) > 0$ when $C(v_0)$ enters I. Since $x'' > 0$ in I, $x'(t)$ is positive as long as $C(v_0)$ remains in I and therefore $C(v_0)$ cannot return directly to II from I.

Collecting all the conditions imposed above on the coefficients of (2.1) we can state our principal result.

THEOREM 3.4. *If the coefficients of* (2.1) *satisfy*

(i) $c(t) \geqq a(t) > 0$

(ii) $b(t) \geqq d(t) > 0$

in $[\alpha, \infty)$ *and*

(iii) $u'' + \min\{b(t) - d(t), c(t) - a(t)\}u = 0$ is oscillatory at $t = \infty$,

(iv) $\displaystyle\int^{\infty} tb(t)\,dt = \int^{\infty} tc(t)\,dt = \infty$,

then there exists a nontrivial solution $y(t), x(t)$ *of* (2.1) *satisfying* $y(\alpha) = y'(\alpha) = 0 = y(\beta) = y'(\beta)$ *for some* $\beta > \alpha$.

4. Application to fourth order equations. In [1] the author shows how to reduce (1.3) to the form

$$l[y] \equiv (p_2(t)y'')'' - (p_1(t)y')' + q_1(t)y' + p_0(t)y = 0 \tag{4.1}$$

and that there is a one-to-one correspondence between equations of the form (4.1) with $p_2(t) > 0$ and systems of the form (2.1) with sufficiently regular coefficients and $b(t) > 0$. This correspondence is obtained by defining

$$x(t) = \frac{1}{b(t)}\, y''(t) - \frac{a(t)}{b(t)}\, y(t)$$

and setting

$$a = \frac{p_1 - \int_\alpha^t q_1}{2p_2}$$

$$b = \frac{1}{p_2}$$

$$c = \frac{p_1^2 - \left(\int_\alpha^t q_1\right)^2}{4p_2} - \frac{p_1'' - q_1'}{2} - p_0$$

$$d = \frac{p_1 + \int_\alpha^t q_1}{2p_2}\,.$$

These equations can be solved for p_2, p_1, p_0, and q_1 to yield

$$p_2 = \frac{1}{b}$$

$$p_1 = \frac{a+d}{b}$$

$$q_1 = \left(\frac{d-a}{b}\right)'$$

$$p_0 = \frac{ad}{b} - \left(\frac{a}{b}\right)' - c.$$

This transformation makes it routine to apply Theorem 3.4 to the equation (4.1), though the transformation required to represent (1.3) in the form (4.1) makes the general application more involved.

It is of interest to examine the hypotheses of Theorem 3.4 for the special case $a(t) \equiv 0$ and $d(t) \equiv 0$ where our considerations reduce to the fourth order equation

$$(4.2) \qquad (p_2(t)y'')'' + p_0(t)y = 0 \qquad (p_0(t) < 0)$$

considered by Leighton and Nehari in Part 1 of [4]. Condition (iii) of Theorem 3.4 is then satisfied if

$$(4.3) \qquad u'' + \min\left\{\frac{1}{p_2(t)},\ -p_0(t)\right\} u = 0$$

is oscillatory at ∞. By the Sturm comparison theorem, (4.3) oscillatory implies that both $u'' + (1/p_2(t))u = 0$ and $u'' - p_0(t)u = 0$ are oscillatory at ∞, but one would not expect the converse to be true without some further hypotheses regarding the asymptotic behavior of $p_2(t)$ and $p_0(t)$. By a well known oscillation criterion of Kneser, (4.3) is oscillatory if

$$(4.4) \qquad \liminf_{t\to\infty} \left[t^2 \min\left\{\frac{1}{p_2(t)}, -p_0(t)\right\}\right] > \frac{1}{4}$$

and this gives some measure of allowable rates of decay for $1/p_2(t)$ and $p_0(t)$.

Condition (iv) of Theorem 3.4 becomes

$$\int^\infty \frac{t}{p_2(t)}\, dt = -\int^\infty t p_0(t) = \infty.$$

While these conditions also put limits on the rate of decay of $1/p_2(t)$ and $p_0(t)$, they are not sufficient to assure the oscillatory behavior of $u''+1/p_2(t)\,u=0$ and $u''-p_0(t)u=0$. For example, $u''+1/4t^2$ is nonoscillatory at ∞ but yet $\int^{\infty} 1/4t\,dt=\infty$.

A special case of Theorem 6.2 of **[4]** assures the existence of a conjugate point for any $\alpha<\infty$ if

$$\liminf_{t\to\infty} t^2\frac{1}{p_2(t)}>\frac{1}{4} \quad\text{and}\quad \liminf_{t\to\infty} t^2(-p_0(t))>\frac{1}{4} \tag{4.5}$$

while it precludes the existence of conjugate points for sufficiently large α if

$$\limsup_{t\to\infty} t^2\frac{1}{p_2(t)}<\frac{1}{4} \quad\text{and}\quad \limsup_{t\to\infty} t^2(-p_0(t))<\frac{1}{4}.$$

While (4.4) is slightly stronger than (4.5), the two conditions are roughly equivalent, and this comparison therefore suggests that the results of Theorem 3.4 are reasonably sharp even in this special case.

One is tempted to conjecture that the oscillatory behavior of both

$$u''+\frac{1}{p_2(t)}u=0 \quad\text{and}\quad u''-p_0(t)u=0$$

should insure the existence of conjugate points for (4.2) for all $\alpha<\infty$.

5. Concluding remarks. The techniques presented here are not quite as sensitive as those of **[4]** in the special case of equation (4.2). Their principal virtue is that they apply to *non*selfadjoint equations such as (4.1) and (1.3).

Several authors have used comparison theorems to establish *lower* bounds for conjugate points of nonselfadjoint equations of order $2n$, and have thereby also established criteria for this disconjugacy (see for instance **[5]**, **[6]**, and **[7]**). However, I know of no results which establish upper bounds in the nonselfadjoint case if $n>1$.

Theorem 3.4 at least gives criteria for the existence of a conjugate point in the nonselfadjoint case. The question of how to obtain specific upper bounds for such conjugate points unfortunately remains unanswered, but it is hoped that these techniques may also prove useful in this connection.

REFERENCES

1. K. Kreith, *A nonselfadjoint dynamical system*, Proc. Edinburgh Math. Soc., **19** (1974), 77–87.

2. ———, *Disconjugacy criteria for nonselfadjoint differential equations of even order*, Canad. J. Math., **23** (1971), 644–652.

3. ———, *Oscillation properties of nonselfadjoint differential equations*, Proceedings of the Conference on Difference on Differential Equations, Dundee, Scotland. Springer Verlag (1972), 294–297.

4. W. Leighton and Z. Nehari, *On the oscillation of solutions of selfadjoint linear differential equations of fourth order*, Trans. Amer. Math. Soc., **89** (1958), 325–377.

5. W. T. Reid, *Oscillation criteria for linear differential systems with complex coefi cients*, Pacific J. Math., **6** (1956), 733–751.

6. J. Ridenhour and T. Sherman, *Conjugate points for fourth order linear differential equations*, SIAM J. Appl. Math., **22** (1972), 599–603.

7. W. M. Whyburn, *On selfadjoint ordinary differential equations of the fourth order*, Amer. J. Math., **52** (1930), 171–196.

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BAIRE SPACES AND HYPERSPACES

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This paper examines the question as to when the hyperspace of a Baire space is a Baire space, and related questions. An answer is given in terms of a certain product space's being a Baire space.

A hyperspace of a space X is the space of closed subsets of X under a natural topology. In this paper we investigate what happens to Baire spaces in the formation of hyperspaces. The first section is devoted to a discussion of the basic concepts. In the second section some characterizations of Baire spaces are given which will be useful while working with hyperspaces, the study of which occurs in the third section. In particular, we shall be primarily concerned with two basic questions. If X is a Baire space, when is the hyperspace of X a Baire space? If the hyperspace of X is a Baire space, when is X a Baire space? We also look briefly at Baire spaces in the strong sense and pseudo-complete spaces.

1. Basic definitions and properties. A *Baire space* is a space in which every countable intersection of dense open subsets is dense. It can also be defined as a space such that every nonempty open subspace is of second category. The usual definition of a space of *first category* is one which can be written as a countable union of nowhere dense subsets (i.e., subsets whose closures have no interior points). A space is of *second category* then if it is not of first category. Also second category spaces can be characterized as spaces in which every countable intersection of dense open subsets is nonempty.

We now list a few of the properties which the Baire space concept enjoys. Of course the Baire Category Theorem gives a sufficient condition for a space to be a Baire space. That is, every complete metric space is a Baire space. Also every locally compact Hausdorff space is a Baire space. Clearly every nonempty open subspace of a Baire space is a Baire space. In fact a space is a Baire space if and only if every point has a neighborhood which is a Baire space. A useful property of Baire spaces is that every space which contains a dense Baire subspace is a Baire space.

The question as to which products of Baire spaces are Baire spaces has been a difficult one. There have been very few different examples discovered of Baire spaces whose product with itself is not a Baire space (see [9] and [6]). In §3, we shall be interested in having X^ω be a Baire space. This then will be the case for most known Baire spaces X. In fact it will always be true that X^ω is a Baire space if X is a Baire space having a countable base. On the otherhand if X^ω is a Baire space, then X will always be a Baire space, since an open continuous image of a Baire space is a Baire space. A thorough investigation of the properties of Baire spaces can be found in [5].

When working with hyperspaces, we shall use the notation and terminology in [7]. Briefly, if X is a topological space with topology $\mathcal{T}$, 2^X denotes the set of all nonempty closed subsets of X and $\mathscr{C}(X)$ denotes the set of all nonempty compact subsets of X. If $U_1, \cdots, U_n$ are subsets of X, then

$$\langle U_1, \cdots, U_n \rangle = \left\{ A \in 2^X \,\middle|\, A \subset \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \varnothing \text{ for each } i = 1, \cdots, n \right\}.$$

The finite topology (or Vietoris topology) on 2^X, denoted by $2^{\mathcal{T}}$, is the topology generated by the base

$$\mathscr{B}^X = \{\langle U_1, \cdots, U_n \rangle \mid U_i \in \mathcal{T},\ i = 1, \cdots, n\}.$$

The topology $\mathcal{T}^c$ on $\mathscr{C}(X)$ is the topology generated by the base $\mathscr{B}^c = \{\langle U_1, \cdots, U_n \rangle \cap \mathscr{C}(X) \mid U_i \in \mathcal{T},\ i = 1, \cdots, n\}$. It is easy to see that $\mathscr{C}(X)$ is a dense subspace of 2^X for T_1-spaces X.

If X is a T_1-space, then the natural map from X into 2^X which takes each x onto the element $\{x\}$ of 2^X is a closed embedding. Also if X is a T_1-space, so is 2^X. An investigation of the properties of hyperspaces can be found in [7]. Two such very basic properties which we shall implicityly use are the following.

> $\langle U_1, \cdots, U_n \rangle \subset \langle V_1, \cdots, V_m \rangle$ if and only if $\cup_{i=1}^n U_i \subset \cup_{i=1}^n V_i$ and for each V_i there exists a U_j such that $U_j \subset V_i$; and $\mathrm{Cl}(\langle U_1, \cdots, U_n \rangle) = \langle \bar{U}_1, \cdots, \bar{U}_n \rangle$ (where ClU and $\bar{U}$ both denote the closure of U).

We shall occasionally be concerned with *quasi-regular spaces*, that is, spaces such that every nonempty open set contains a closed subset with nonempty interior. It is not difficult to see that if X is quasi-regular, then 2^X and $\mathscr{C}(X)$ are quasi-regular.

2. Characterizations of Baire spaces. In §3, we shall need ways of looking at Baire spaces other than our definition. In this section, we establish the needed characterizations.

Let X be an arbitrary topological space. By a *pseudo-base* for X is meant a collection of nonempty open subsets of X such that each nonempty open subset of X contains a member of this collection.

Let $\mathcal{B}$ be a pseudo-base for X. Define

$$S(X,\mathcal{B})=\{f\colon \mathcal{B}\to\mathcal{B}\mid f(U)\subset U \text{ for every } U\in\mathcal{B}\}, \quad \text{and}$$

$$RS(X,\mathcal{B})=\{f\colon \mathcal{B}\to\mathcal{B}\mid \overline{f(U)}\subset U \text{ for every } U\in\mathcal{B}\}.$$

If $U\in\mathcal{B}$ and $f,g\in S(X,B)$ or $RS(X,\mathcal{B})$, define

$$[U,f,g]_1=g(U), \text{ and for } i>1,$$

$$[U,f,g]_i=\begin{cases} f([U,f,g]_{i-1}), & \text{if } i \text{ is even,}\\ g([U,f,g]_{i-1}), & \text{if } i \text{ is odd.}\end{cases}$$

The following theorem can be found in **[5]**, and similar theorems can be found in **[6]** and **[8]**.

THEOREM 2.1. *The following are equivalent.*

(i) *X is a Baire space.*

(ii) *There exists a pseudo-base $\mathcal{B}$ for X such that for every $U\in\mathcal{B}$ and $f\in S(X,\mathcal{B})$, there exists a $g\in S(X,\mathcal{B})$ such that*

$$\bigcap_{i=1}^{\infty}[U,f,g]_i\neq\varnothing.$$

(iii) *For every pseudo-base $\mathcal{B}$ for X and every $U\in\mathcal{B}$ and $f\in S(X,\mathcal{B})$, there exists a $g\in S(X,\mathcal{B})$ such that $\bigcap_{i=1}^{\infty}[U,f,g]_i\neq\varnothing$.*

There is an analogous result for spaces of second category.

THEOREM 2.2. *The following are equivalent.*

(i) *X is of second category.*

(ii) *There exists a pseudo-base $\mathcal{B}$ for X such that for every $f\in S(X,\mathcal{B})$, there exists a $U\in\mathcal{B}$ and $g\in S(X,\mathcal{B})$ such that $\bigcap_{i=1}^{\infty}[U,f,g]_i\neq\varnothing$.*

(iii) *For every pseudo-base $\mathcal{B}$ for X and every $f\in S(X,\mathcal{B})$, there exists $U\in\mathcal{B}$ and $g\in S(X,\mathcal{B})$ such that $\bigcap_{i=1}^{\infty}[U,f,g]_i\neq\varnothing$.*

The characterization which we actually use in the next section is given in the following theorem.

THEOREM 2.3. *Let X be a quasi-regular space, and let $\mathcal{B}$ be a pseudo-base for X. Then the following are equivalent.*

(i) *X is a Baire space.*

(ii) *For every $U \in B$ and $f \in S(X, \mathcal{B})$, there exists $g \in RS(X, \mathcal{B})$ such that $\bigcap_{i=1}^{\infty} [U, f, g]_i \neq \emptyset$.*

(iii) *For every $U \in \mathcal{B}$ and $f \in RS(X, \mathcal{B})$, there exists $g \in RS(X, \mathcal{B})$ such that $\bigcap_{i=1}^{\infty} [U, f, g]_i \neq \emptyset$.*

(iv) *For every $U \in \mathcal{B}$ and $f \in RS(X, \mathcal{B})$, there exists $g \in S(X, \mathcal{B})$ such that $\bigcap_{i=1}^{\infty} [U, f, g]_i \neq \emptyset$.*

Proof. For every $U \in \mathcal{B}$, $f, g \in S(X, \mathcal{B})$ with $f(U) \neq U$ and $\overline{g(U)} \subset U$, define $\hat{f}, \hat{g} \in RS(X, \mathcal{B})$ as follows. Since X is quasi-regular, for every $V \in \mathcal{B}$, there exists $\hat{f}(V) \in B$ such that $\overline{\hat{f}(V)} \subset f(V)$. If $V = f([U, \hat{f}, g]_{2n-1})$ for some $n = 1, 2, \cdots$, let $\hat{g}(V) = [U, \hat{f}, g]_{2n+1}$; if $V = U$, let $\hat{g}(V) = g(V)$; and otherwise let $\hat{g}(V)$ be an element of $\mathcal{B}$ such that $\overline{\hat{g}(V)} \subset V$. Now observe that $[U, f, \hat{g}]_1 = \hat{g}(U) = g(U) = [U, \hat{f}, g]_1$. Suppose that $[U, f, \hat{g}]_{2k-1} = [U, \hat{f}, g]_{2k-1}$ for every $k = 1, \cdots, n$. Then $[U, f, \hat{g}]_{2n+1} = \hat{g}(f([U, f, \hat{g}]_{2n-1})) = \hat{g}(f([U, \hat{f}, g]_{2n-1})) = [U, \hat{f}, g]_{2n+1}$. Therefore by induction $[U, f, \hat{g}]_{2n-1} = [U, \hat{f}, g]_{2n-1}$ for every n.

To see that (i) imples (ii), suppose that (ii) does not hold. Then there exists $U \in \mathcal{B}$ and $f \in S(X, \mathcal{B})$ such that for every $g \in RS(X, \mathcal{B})$, $\bigcap_{i=1}^{\infty} [U, f, g]_i = \emptyset$. Let g be an arbitrary element of $S(X, \mathcal{B})$. We may assume without loss of generality that $f(U) \neq U$ and $\overline{g(U)} \subset U$. Now $\bigcap_{i=1}^{\infty} [U, \hat{f}, g]_i = \bigcap_{i=1}^{\infty} [U, f, \hat{g}]_i = \emptyset$, so that X cannot be a Baire space by Theorem 2.1.

Clearly (ii) implies (iii) and (iii) implies (iv), so that it remains to show that (iv) implies (i). Suppose X is not a Baire space. Then there exists $U \in \mathcal{B}$ and $f \in S(X, \mathcal{B})$ such that for every $g \in S(X, \mathcal{B})$, $\bigcap_{i=1}^{\infty} [U, f, g]_i = \emptyset$. Let g be an arbitrary element of $S(X, \mathcal{B})$. Again we may assume without loss of generality that $f(U) \neq U$ and $\overline{g(U)} \subset U$. Then $\bigcap_{i=1}^{\infty} [U, \hat{f}, g]_i = \bigcap_{i=1}^{\infty} [U, f, \hat{g}]_i = \emptyset$, which establishes the negation of statement (iv) by Theorem 2.1 again.

Just as Theorem 2.1 has Theorem 2.2 as an analog for spaces of second category, Theorem 2.3 has its analog for spaces of second category, which we shall not state since it can easily be induced.

3. Applications to hyperspaces. First we establish that for a T_1-space X, if the hyperspace of X is a Baire space, then X must be a Baire space. Since the following two theorems have essentially the same proofs, we only give the proof for the second theorem.

THEOREM 3.1. *If $\mathscr{C}(X)$ is a Baire space (space of second category, respectively), then X is a Baire space (space of second category, respectively).*

THEOREM 3.2. *If X is a T_1-space and 2^X is a Baire space (space of second category, respectively), then X is a Baire space (space of second category, respectively).*

Proof. Suppose that X is not a Baire space. Then there exists an open U in X and a sequence $\{U_i\}$ of dense open subsets of X such that $U \cap (\bigcap_{i=1}^{\infty} U_i) = \varnothing$. Let $G = \langle U \rangle$, and for each i, let $G_i = \langle U_i \rangle$. To see that each G_i is dense in 2^X, let $\langle V_1, \cdots, V_n \rangle$ be an element of $\mathscr{B}^X$. Now each $V_j \cap U_i \neq \varnothing$ for $j = 1, \cdots, n$. So let $x_j \in V_j \cap U_i$ for each such j, and let $A = \{x_1, \cdots, x_n\}$. Since X is a T_1-space, $A \in 2^X$. Also A is easily seen to be in $G_i \cap \langle V_1, \cdots, V_n \rangle$, so that each G_i is dense in 2^X. Finally, it is clear that $G \cap (\bigcap_{i=1}^{\infty} G_i) = \varnothing$, so that 2^X could not be a Baire space.

COROLLARY 3.3. *If X is a first category space, then so is $\mathscr{C}(X)$. If in addition X is a T_1-space, then 2^X is a first category T_1-space.*

The T_1-space hypothesis in Theorem 3.2 cannot be omitted since the space of natural numbers, N, with the topology consisting of right rays is a space of first category, while 2^N has the indiscrete topology, so is a Baire space. Note that this is also an example of a space X such that 2^X is quasi-regular, but X is not quasi-regular.

Before examining the converses of Theorem 3.1 and 3.2, we wish to introduce a new space. Let X^ω denote the Cartesian product of a countably infinite number of copies of X. Instead of putting the usual product topology on X^ω, we shall need to consider a topology $\mathscr{T}^*$ on X^ω which lies strictly between the product topology and the box topology. If $U_1, \cdots, U_n$ are subsets of X, let $\Pi(U_1, \cdots, U_n)$ denote the subset $(\Pi_{i=1}^{n} U_i) \times (\Pi_{i=n+1}^{\infty} (\bigcup_{j=1}^{n} U_j))$ of X^ω. Let

$$\mathscr{B}^* = \{\Pi(U_1, \cdots, U_n) \mid U_i \text{ is open in } X, i = 1, \cdots, n\}.$$

Finally, let $\mathscr{T}^*$ be the topology on X^ω generated by $\mathscr{B}^*$.

LEMMA 3.4. *The family $\mathscr{B}^*$ is a base for $(X^\omega, \mathscr{T}^*)$.*

Proof. Let $U = \Pi(U_1, \cdots, U_n)$ and $V = \Pi(V_1, \cdots, V_m)$ be arbitrary members of $\mathscr{B}^*$. We may suppose that $n \geqq m$. Then

$$U \cap V = \Pi\left(U_1 \cap V_1, \cdots, U_m \cap V_m, U_{m+1} \cap \left(\bigcup_{i=1}^{m} V_i\right), \cdots, U_n\right.$$

$$\left. \cap \left(\bigcup_{i=1}^{m} V_i\right), \left(\bigcup_{i=1}^{n} U_i\right) \cap \left(\bigcup_{i=1}^{m} V_i\right)\right),$$

which is a member of $\mathscr{B}^*$.

Lemma 3.4 now gives us the following fact.

LEMMA 3.5. *Each projection map on $(X^\omega, \mathscr{T}^*)$ is continuous and open.*

It is easy to see that open continuous functions preserve Baire spaces and spaces of second category (see [4]). The next lemma then follows from Lemma 3.5.

LEMMA 3.6. *If $(X^\omega, \mathscr{T}^*)$ is a Baire space (space of second category, respectively), then X is a Baire space (space of second category, respectively).*

A modification of the proofs of 2.5 and 2.6 in [9] establishes the following fact.

LEMMA 3.7. *If X is a Baire space (space of second category, respectively) having a countable pseudo-base, then $(X^\omega, \mathscr{J}^*)$ is a Baire space (pace of second category, respectively).*

We now introduce some notation which will be used in the proof of the next two theorems. We shall be working with the three spaces $(X^\omega, \mathscr{T}^*)$, $(2^X, 2^{\mathscr{T}})$, and $(\mathscr{C}(X), \mathscr{T}^c)$ having bases $\mathscr{B}^*, \mathscr{B}^X$, and $\mathscr{B}^c$, respectivley. If $U = \Pi(U_1, \cdots, U_n) \in \mathscr{B}$, then define $U^X = \langle U_1, \cdots, U_n\rangle$, which is an element of $\mathscr{B}^X$, and define $U^c = \langle U_1, \cdots, U_n\rangle \cap \mathscr{C}(X)$, which is an element of $\mathscr{B}^c$. On the other hand if $U = \langle U_1, \cdots, U_n\rangle \in \mathscr{B}^X$ or $U = \langle U_1, \cdots, U_n\rangle \cap \mathscr{C}(X) \in \mathscr{B}^c$, define $U^* = \Pi(U_1, \cdots, U_n)$, which is an element of $\mathscr{B}^*$. Also for each $f \in RS(X^\omega, \mathscr{B}^*)$, define $f^X \in RS(2^X, \mathscr{B}^X)$ and $f^c \in RS(\mathscr{C}(X), \mathscr{B}^c)$ as follows. If $U = \langle U_1, \cdots, U_n\rangle$ is an arbitrary element of $\mathscr{B}^X$, define $f^X(U) = (f(U^*))^X$, and if $U = \langle U_1, \cdots, U_n\rangle \cap \mathscr{C}(X)$ is an arbitrary element of $\mathscr{B}^c$, define $f^c(U) = (f(U^*))^c$. Finally, for each $f \in RS(2^X, \mathscr{B}^X)$ or $f \in RS(\mathscr{C}(X), \mathscr{B}^c)$, define $f^* \in RS(X^\omega, \mathscr{B}^*)$ as follows. Let $f \in RS(2^X, \mathscr{B}^X)$. If $U = \langle U_1, \cdots, U_n\rangle \in \mathscr{B}^X$, then we may assume that $f(U)$ is written as $\langle V_1, \cdots, V_m\rangle$, where $m \geqq n$ and for each $i = 1, \cdots, n$, $\bar{V}_i \subset U_i$. Now if $U = \Pi(U_1, \cdots, U_n)$ is an arbitrary element of $\mathscr{B}^*$, then define $f^*(U) = (f(U^X))^*$. A similar definition is to be given for f^* if $f \in RS(\mathscr{C}(X), \mathscr{B}^c)$.

THEOREM 3.8. *If X is quasi-regular and $(X^\omega, \mathcal{T}^*)$ is a Baire space (space of second category, respectively), then 2^X is a Baire space (space of second category, respectively).*

Proof. Let $U \in \mathcal{B}^X$ and let $f \in RS(2^X, \mathcal{B}^X)$. Since $(X^\omega, \mathcal{T}^*)$ is a Baire space, by Theorem 2.3, there exists $g \in RS(X^\omega, \mathcal{B}^*)$ such that $\bigcap_{i=1}^\infty [U^*, f^*, g]_i$ contains some element (x_i) of X^ω. Now $[U, f, g^X]_2 = f(g^X(U)) = f((g(U^*))^X)$, and $[U^*, f^*, g]_2 = f^*(g(U^*)) = (f((g(U^*))^X))^*$. Also for each $n > 1$,

$$\begin{aligned}
[U, f, g^X]_{2n} &= f(g^X([U, f, g^X]_{2n-2})) \\
&= f((g([U, f, g^X]^*_{2n-2}))^X), \quad \text{and} \\
[U^*, f^*, g]_{2n} &= f^*(g([U^*, f^*, g]_{2n-2})) \\
&= (f((g([U^*, f^*, g]_{2n-2}))^X))^*.
\end{aligned}$$

So that by induction, $[U^*, f^*, g]_{2n} = ([U, f, g^X]_{2n})^*$ for every n. Therefore since $(x_i) \in [U^*, f^*, g]_{2n}$ for every n, then $\overline{\{x_i\}} \in [U, f, g^X]_{2n}$ for every n. Thus $\bigcap_{i=1}^\infty [U, f, g^X]_i \neq \emptyset$, so that 2^X must be a Baire space by Theorem 2.3 again.

COROLLARY 3.9. *If X is a quasi-regular Baire space (space of second category, respectively) having a countable pseudo-base, then 2^X is a Baire space (space of second category, respectively).*

If we consider the space $\mathcal{C}(X)$ instead of 2^X, then we obtain the converse of Theorem 3.8.

THEOREM 3.10. *If X is quasi-regular and $\mathcal{C}(X)$ is a Baire space (space of second category, respectively), then $(X^\omega, \mathcal{T}^*)$ is a Baire space (space of second category, respectively).*

Proof. Let $U \in \mathcal{B}^*$ and let $f \in RS(X^\omega, \mathcal{B}^*)$. Since $\mathcal{C}(X)$ is a Baire space, by Theorem 2.3, there exists a $g \in RS(\mathcal{C}(X), \mathcal{B}^c)$ such that $\bigcap_{i=1}^\infty [U^c, f^c, g]_i$ contains some element A of $\mathcal{C}(X)$. Now $[U, f, g^*]_2 = f(g^*(U)) = f((g(U^c))^*)$, and $[U^c, f^c, g]_2 = f^c(g(U^c)) = f((g(U^c))^*))^c$. Also for each $n > 1$,

$$\begin{aligned}
[U, f, g^*]_{2n} &= f(g^*([U, f, g^*]_{2n-2})) \\
&= f((g([U, f, g^*]^c_{2n-2}))^*), \quad \text{and} \\
[U^c, f^c, g]_{2n} &= f^c(g([U^c, f^c, g]_{2n-2})) \\
&= (f((g([U^c, f^c, g]_{2n-2}))^*))^c.
\end{aligned}$$

So that by induction, $[U^c, f^c, g]_{2n} = ([U, f, g^*]_{2n})^c$ for every n. Suppose that for each $n, [U^c, f^c, g]_{2n} \doteq \langle U_1^n, \cdots, U_{m(n)}^n \rangle \cap \mathscr{C}(X)$. Then each $[U, f, g^*]_{2n} = (U_1^n, \cdots, U_{m(n)}^n)$, so that $m(1) \leqq m(2) \leqq \cdots$, and for each $i = 1, \cdots, m(n)$, $U_i^{n+1} \subset U_i^n$.

Let i be a fixed positive integer. If there exists an n such that $i \leqq m(n)$, let $n(i)$ be the smallest such n. In this case, for each $j = n(i), n(i)+1, \cdots$, let $a_i^j \in A \cap U_i^j$. Otherwise, for each $j = 1, 2, \cdots$, let $a_i^j \in A \cap (\bigcup_{k=1}^{m(j)} U_k^j)$. In either case, since A is compact, $\{a_i^j \mid j = 1, 2, \cdots\}$ has a cluster point, say a_i. Now letting i vary, this defines the point $(a_i) \in X^\omega$. It can be seen that $(a_i) \in \bigcap_{i=1}^\infty [U, f, g^*]_i$, so that $(X^\omega, \mathscr{T}^*)$ must be a Baire space by Theorem 2.3 again.

We might observe that when X is a quasi-regular space, Theorem 3.1 now follows from Theorem 3.10 and Lemma 3.6.

Using Theorem 3.10, we can now obtain an example of a metric Baire space K such that $\mathscr{C}(K)$ is not a Baire space. In **[9]**, Oxtoby gave an example of a Baire space whose square is not a Baire space. This space unfortunately needs the continuum hypothesis in its construction. Later in **[6]**, Krom constructed a new space K from Oxtoby's example which is a metric Baire space whose square is not a Baire space. Now if X^2 is not a Baire space, it is easy to see by a modification of Lemma 3.6 that $(X^\omega, \mathscr{T}^*)$ is not a Baire space. Thus by Theorem 3.10, $\mathscr{C}(K)$ is not a Baire space.

It would be of interest to know whether 2^K is a Baire space, where K is Krom's example discussed above. One is tempted to try to answer this by investigating the relationship between 2^X and $\mathscr{C}(X)$ in terms of being a Baire space. First, since any extension of a Baire space is a Baire space, we immediately get the following.

THEOREM 3.11. *If $\mathscr{C}(X)$ is a Baire space, then so is 2^X.*

On the other hand, Aarts and Lutzer in **[1]** gave a test to determine when a dense subspace of a Baire space is a Baire space. A modification of this theorem (see **[5]**) is that if X is a dense subspace of the Baire space Y, then X is a Baire space if and only if every somewhere dense G_δ-subset of Y intersects X (a set being somewhere dense if its closure has nonempty interior). Therefore, as partial converse of Theorem 3.11, we have the following.

THEOREM 3.12. *If 2^X is a Baire space and every somewhere dense G_δ-subset of 2^X intersects $\mathscr{C}(X)$, then $\mathscr{C}(X)$ is a Baire space.*

However, the full converse of Theorem 3.9 is false, as we shall now extablish. Let P be a dense Baire subspace of the irrationals with the usual metric topology having the property that every compact subset has an isolated point (see for example [1], example 2.4; or [5], Theorem 2.6). Let $\mathcal{I}$ be the set of all intervals in P having rational end points and diameter less than one. For each pairwise disjoint collection $\{I_1, \cdots, I_k\}$ of elements of $\mathcal{I}$ and for each natural number n, let

$$G_n(I_1, \cdots, I_k) = \langle I_1^1, \cdots, I_1^n, I_2^1, \cdots, I_2^n, \cdots, I_k^1, \cdots, I_k^n \rangle,$$

where for each $i = 1, \cdots, k$, the I_i^j's are pairwise disjoint intervals such that $I_i = \bigcup_{j=1}^n I_i^j$ and diam $I_i^j = 1/n$ diam I_i for each $j = 1, \cdots, n$. Now for each n, let $G_n = \bigcup\{G_n(I_1, \cdots, I_k) \mid I_1, \cdots, I_k$ are pairwise disjoint elements of $\mathcal{I}\}$, which is a dense open subset of 2^P. To see that $(\bigcap_{n=1}^\infty G_n) \cap \mathscr{C}(P) = \varnothing$, let $A \in \mathscr{C}(P)$. Now A must contain some isolated point x. Let m be a natural number such that no other point of A is within $3/m$ of x. If for some $I_1, \cdots, I_k \in \mathcal{I}$, $A \in G_m(I_1, \cdots, I_k)$, then $x \in I_i^j$ for some $i = 1, \cdots, k$ and $j = 1, \cdots, m$. But then for some $j' = 1, \cdots, m$, $A \cap I_i^{j'} = \varnothing$ — which contradicts A being in $G_m(I_1, \cdots, I_k)$. Therefore $(\bigcap_{n=1}^\infty G_n) \cap \mathscr{C}(P) = \varnothing$, so that $\mathscr{C}(P)$ must be of first category. Note also that $\bigcap_{n=1}^\infty G_n$ is a dense G_δ-subset of 2^P. Finally since P is second countable, 2^P will be a Baire space by Corollary 3.9.

For the final two theorems, we shall be concerned with two properties closely related to Baire spaces — Baire spaces in the strong sense and pseudo-complete spaces.

A space X is called a *Baire space in the strong sense* (or *totally non-meagre space*) provided that every nonempty closed subspace of X is of second category. This turns out to be equivalent to the property that every nonempty closed subspace is a Baire space. Discussions of Baire spaces in the strong sense can be found in [3] and [2]. We have the following immediate theorem.

THEOREM 3.13. *Let X be a T_1-space. If either 2^X or $\mathscr{C}(X)$ is a Baire space in the strong sense, then so is X.*

A space X is called *pseudo-complete* if it is quasi-regular and there exists a sequence $\{\mathscr{B}_i\}$ of pseudo-bases for X such that if for each $i = 2, \cdots$, $U_i \in \mathscr{B}_i$ and $\overline{U_{i+1}} \subset U_i$, then $\bigcap_{i=1}^\infty U_i \neq \varnothing$. It is not difficult to see that every pseudo-complete space is a Baire space. Properties of pseudo-complete spaces can be found in [9] and [1]. One such interesting property of pseudo-complete spaces (found in the latter reference) is that for a metric space X, X is pseudo-complete if and only if it has a dense completely metrizable subspace.

THEOREM 3.14. *If X is pseudo-complete, then so is 2^X.*

Proof. Let $\{\mathcal{B}_i\}$ be a sequence of pseudo-bases for X such that if for each $i = 1, 2, \cdots, U_i \in \mathcal{B}_i$ and $\overline{U_{i+1}} \subset U_i$, then $\bigcap_{i=1}^{\infty} U_i \neq \emptyset$. For each i, let $\mathcal{B}_i^X = \{\langle U_1, \cdots, U_n \rangle \mid U_1, \cdots, U_n \in \mathcal{B}_i\}$, which is a pseudo-base for 2^X. Now for each i, let $G_i = \langle U_1^i, \cdots, U_{n(i)}^i \rangle \in \mathcal{B}_i^X$ such that $\overline{G_{i+1}} \subset G_i$. We may assume that $n(1) \leqq n(2) \leqq \cdots$, and that for each i, $U_j^{i+1} \subset U_j^i$ for $j = 1, \cdots, n(i)$. Also we know for each i that $\bigcup_{j=1}^{n(i+1)} \overline{U_j^{i+1}} \subset \bigcup_{j=1}^{n(i)} U_j^i$. Let $n(0) = 1$. For each nonnegative integer k such that $n(k) < n(k+1)$ and each integer j such that $n(k) < j \leqq n(k+1)$, choose $x_j \in \bigcap_{i=k+1}^{\infty} U_j^i$. Let $L = \infty$ if $\lim_{k \to \infty} n(k)$ is infinite, and let $L = 1 + \lim_{k \to \infty} n(k)$ if $\lim_{k \to \infty} n(k)$ is finite. Then define $A = \{x_j \mid 1 \leqq j < L\}$. Now $\bar{A} \in G_i$ for every i, so that $\bar{A} \in \overline{G_{i+1}} \subset G_i$ for every i. Therefore $\bar{A} \in \bigcap_{i=1}^{\infty} G_i$, so that 2^X is pseudo-complete.

The converse of Theorem 3.14 is easily seen to be false since the space N of natural numbers with the right ray topology is not quasi-regular, while 2^X has the indiscrete topology and is hence pseudo-complete.

REFERENCES

1. J. M. Aarts and D. J. Lutzer, *Pseudo-completeness and the product of Baire spaces*, Pacific J. Math., **48** (1973), 1–10.

2. ———, *The product of totally nonmeagre spaces*, Proc. Amer. Math. Soc., **38** (1973), 198–200.

3. N. Bourbaki, *General Topology*, part 2, Addison-Wesley Publishing Company, Reading, Massachusetts, 1966.

4. Z. Frolik, *Remarks concerning the invariance of Baire spaces under mappings*, Czech. Math. J., **11** (1961), 381–385.

5. R. C. Haworth and R. A. McCoy, *Baire spaces*, (to appear).

6. M. R. Krom, *Cartesian products of metric Baire spaces*, Proc. Amer. Math. Soc., **42** (1973), 588–594.

7. E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc., **71** (1951), 152–182.

8. J. C. Oxtoby, *The Banach-Mazur game and Banach Category Theorem, Contribution to the theory of games* Vol. III, (Annals of Mathematics Studies number 39), Princeton University Press, Princeton, 1957.

9. ———, *Cartesian products of Baire spaces*, Fund. Math., **49** (1961), 157–166.

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ISOMETRIES OF THE DISK ALGEBRA

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In this paper we are concerned with the problem, posed by R. R. Phelps, of describing the into isometries of the disk algebra. We show that, in a certain sense, every isometry can be approximated by convex combinations of isometries of the form $f \to k(f \circ \phi)$. We also give some sufficient conditions for an isometry to be of the form $f \to k(f \circ \phi)$.

Let D and Γ denote, respectively, the open unit disk and the unit circle. The disk algebra, i.e., the algebra of all complex valued functions which are continuous on $D \cup \Gamma$ and analytic on D, will be denoted by A. It will be assumed that A is equipped with the sup-norm.

Operators of the form

$$Tf = k(f \circ \phi) \tag{1}$$

are isometries of A: if $k \in A$, if $\|k\| = 1$, and if $\phi: D \cup \Gamma \to D \cup \Gamma$ is analytic on D, continuous on $D \cup \Gamma \sim k^{-1}(0)$, and satisfies $\phi(k^{-1}(\Gamma)) \supset \Gamma$. In fact, if T is a surjective linear isometry of A, then it must be of the form (1) with k being a constant, and ϕ being a Mobius transformation. (See **[3**, pp. 142–148].) Rochberg **[8]** has shown that if T is an isometry such that $T1 = 1$, and $T(A)$ is a sub-algebra of A, then T is of the form (1) with $k \equiv 1$.

Note that any bounded linear operator $T: A \to A$ which satisfies (1) also satisfies.

$$T1\,T(fg) = TfTg \tag{2}$$

for all f and g in A. Moreover, we have the following.

PROPOSITION 1.1. *A bounded linear operator $T: A \to A$ satisfies* (2) *for all $f, g \in A$ iff it is of the form* (1).

Proof. It is only necessary to show that, if T satisfies (2) for all $f, g \in A$, then it satisfies (1).

Suppose that w is a point of D where $T1$ is not 0. Consider the linear functional defined on A by

$$L_w(f) = (T1(w))^{-1}Tf(w).$$

By (2), L_w is a multiplicative. Hence, there is a v in $D \cup \Gamma$ such that

$L_w(f) = f(v)$. Since $v = (T1(w))^{-1}TZ(w)$, where Z is the identity function on $D \cup \Gamma$, it follows that the function $\phi = (T1)^{-1}TZ$ is bounded on D. Thus, the singularities of ϕ in D are removable. Let S be the operator defined on A by

$$Sf = T1(f \circ \phi).$$

It follows easily from (2) that $SZ^n = TZ^n$ for $n = 0, 1, \cdots$. Since the polynomials in Z are dense in A, the operators T and S are the same. If $T1 \equiv 0$, then, by (2), $(Tf)^2 = T1Tf^2 \equiv 0$. It follows that T is of the form (1) with $k \equiv 0$.

For an example of an isometry which fixes 1 but is not multiplicative, see **[8]**.

For the remainder of this section, T will denote an arbitrary isometry of A. Consider the closed set $\Gamma(T) = \{z \in \Gamma \,|\, |Tl(z)| = 1$ and there is a point $\hat{T}(z)$ in Γ such that $Tf(z) = T1(z)f(\hat{T}(z))$ for all $f \in A\}$. Since A separates the points of Γ, it follows that the mapping $z \to \hat{T}(z)$, denoted by $\hat{T}$, is well defined and continuous on $\Gamma(T)$. In **[5]**, we showed that $\hat{T}$ maps $\Gamma(T)$ onto Γ. The following proposition gives a simple description of $\Gamma(T)$.

Proposition 1.2.

$$\Gamma(T) = \{w \in \Gamma \,|\, |T1(w)| = 1 \quad \textit{and} \quad |TZ(w)| = 1\}.$$

Proof. It is enough to show that if $|T1(z_1)| = |TZ(z_1)| = 1$, then $z_1 \in \Gamma(T)$. By the Hahn-Banach theorem, there is a measure μ on Γ having total variation $\leqq 1$ such that $Tf(z_1) = \int fd\mu$ for all $f \in A$. Let $a = \int 1d\mu$ and $b = \int Zd\mu$, where Z is the identity on $D \cup \Gamma$. Since $\bar{a}\mu$ has total variation $\leqq 1$ and $\int \bar{a}d\mu = 1$, it follows that $\bar{a}d\mu$ is nonnegative. Note that $\int \mathrm{Re}(1 - a\bar{b}Z)\bar{a}d\mu = 0$. Thus, $\mathrm{Re}(1 - a\bar{b}Z)$ is 0 on the support of μ. Hence the support of μ consists of a single point, i.e., $\hat{T}(z_1)$.

Theorem 1.1. *Suppose $m(\Gamma(T)) > 0$, where m denotes Lesbegue measure on Γ. Then T is of the form* (1).

Proof. For $f, g \in A$, we have

$$T1(z)\, T(fg)(z) = Tf(z)\, Tg(z)$$

for every $z \in \Gamma(T)$. Any two functions in A which agree on a subset of Γ having positive Lesbeque measure are equal. (See [3, p. 52].) Thus $T1\, T(fg) = Tf\, Tg$. It follows by Proposition 1.1 that T is of the form (1).

THEOREM 1.2. *Assume that $T1$ is an inner function. Suppose that $T(A)$ contains a function G having the following properties*: $\|G\| = 1$, $m(G^{-1})(\Gamma)) > 0$, *the set of connected components of $G^{-1}(\Gamma)$ is countab e, and G is not a constant multiple of $T1$. Then T is of the form* (1).

Proof. Let $H = \overline{T1}G$. Note that $H^{-1}(\Gamma) = G^{-1}(\Gamma)$. Let $\{J_1, J_2, \cdots\}$ denote the collection of connected components of $H^{-1}(\Gamma)$. Suppose it can be shown that, for some q, $m(H(J_q \cap \Gamma(T))) > 0$. Then J_q is necessarily a nontrivial sub-arc of Γ. By a form of the Schwartz reflection principle (See, e.g. [2, p. 187].), G can be continued analytically across the interior of J_q. It follows that the restriction of H to the interior of J_q is continuously differentiable. If H were constant on J_q, then we would have $G = cT1$ where c is a constant. Thus, H is not constant and, hence, $m(J_q \cap \Gamma(T)) > 0$. It now follows by Theorem 1.1 that T is of the form (1).

It remains to be shown that $m(H(J_q \cap \Gamma(T))) > 0$ for some q. It is claimed that

$$H(H^{-1}(\Gamma)) = H(H^{-1}(\Gamma) \cap \Gamma(T)).$$

For each $z \in \Gamma$, there exists a measure μ_z, having total variation $\leqq 1$, such that $\int f d\mu_z = Tf(z)$ for each $f \in A$. In particular, we have $1 = \int \overline{T1(z)} d\mu_z$. It follows that the measure $\overline{T1(z)}\mu_z$ is nonnegative. Suppose that z is choosen so that $|G(z)| = |H(z)| = 1$. Let F be the unique function in A such that $G = TF$. Then

$$\int \operatorname{Re}(1 - \overline{H(z)}F)\overline{T1(z)} d\mu = 0.$$

It follows that $H(z) = F(w)$ for each w in the support of μ_z. Since the mapping $\hat{T}$ is onto, there exists a $z_1 \in \Gamma(T)$ such that

$$\begin{aligned} H(z) &= F(\hat{T}(z_1)) \\ &= \overline{T1(z_1)}T1(z_1)F(\hat{T}(z_1)) \\ &= H(z_1). \end{aligned}$$

Next it is claimed that $m(H(H^{-1}(\Gamma)))>0$. If $m(H(H^{-1}(\Gamma)))=0$, then H is constant on all of the J_n's. Since at least one of the J_n's is a nontrivial sub-arc of Γ, it follows that $G = cT1$ for some constant c – a contradiction to the hypothesis that G not be a scalar multiple of $T1$. Finally, we have

$$0<m(H(H^{-1}(\Gamma)))\leqq \Sigma m(H(J_n\cap\Gamma(T))).$$

It follows that $m(H(J_q\cap\Gamma(T)))>0$ for some q.

COROLLARY. *Suppose that $T1$ is an inner function. If TA contains an inner function which is not a scalar multiple of Tl then T is of the form* (1).

REMARK. Let $\mathscr{A}$ denote the sub-algebra of A consisting of functions which are analytic in a neighborhood of $D\cup\Gamma$. By arguments similar to those used to prove Theorem 1.2, one can show that every isometry of $\mathscr{A}$ must be of the form (1).

2. Approximation of arbitrary isometries. As in the previous section, T will denote an arbitrary isometry of A. Let B denote the space of bounded linear operators: $A\to A$ and let B_1 denote the set of members of B having norm $\leqq 1$. As in **[5]**, we define $E(T)=\{U\in B_1 \mid Uf(z)=Tf(z)$ for every $z\in\Gamma(T)$ and every $f\in A\}$. In **[5]** we showed that $E(T)$ is a face of B_1, that $E(T)$ is closed in the weak operator topology, and that each member of $E(T)$ is an isometry. Thus, the set of isometries of A is the union of weak operator-closed faces of B_1. It follows from Proposition 1.2, that

$$E(T)=\{U\in B_1 \mid UZ\mid\Gamma(T)=TZ\mid\Gamma(T) \quad\text{and}\quad U1\mid\Gamma(T)=T1\mid\Gamma(T)\},$$

where Z denotes the identity function on $D\cup\Gamma$. If $m(\Gamma(T))>0$, it follows that $E(T)=\{T\}$. Suppose that $m(\Gamma(T))=0$. Let A_1 denote the unit ball in A, let $S_1=\{f\in A_1 \mid f\mid\Gamma(T)=\hat{T}\}$, and let $S_2=\{g\in A_1 \mid g\mid\Gamma(T)=T1\mid\Gamma(T)\}$. By a result due to Rudin **[9]**, both S_1 and S_2 have infinitely many members. Let $h\in S_1$ and $k\in S_2$. The operator U defined by $Uf=k(f\circ h)$ is in $E(T)$. Thus, $E(T)$ contains infinitely many elements iff $m(\Gamma(T))=0$. For the remainder of the paper, we will consider only isometries T for which $m(\Gamma(T))=0$.

Let $F(T)=\{U\in E(T) \mid U$ is of the form (1)$\}$. In view of **[5**, Th. 3**]**, it is natural to ask whether $E(T)$ is the closed convex hull of $F(T)$, where the closure is taken in the weak operator topology? Although we are unable to answer this question, we will show that there is a family $\mathfrak{S}$

of locally convex Hausdorff topologies on B with the following properties: for each $\mathcal{T} \in \mathfrak{S}$, $E(T)$ is the $\mathcal{T}$-closed convex hull of $F(T)$, and the weakest topology containing all the members of $\mathfrak{S}$ is the weak operator topology.

The weak operator topology on B is the weakest topology in which all linear functionals of the form $H \to \int Hf\,d\mu$, where f is in A and μ is a Baire measure on Γ, are continuous. It follows that the space B^* of weak operator continuous linear functionals on B is the direct sum of sub-spaces $\mathcal{A}$ and $\mathcal{S}$, where $\mathcal{A}$ is the sub-space of B^* spanned by linear functionals of the form $H \to \int (Hf)g\,dm$ with $g \in L_1(m)$, and where $\mathcal{S}$ is the sub-space of B^* spanned by functionals of the form $H \to \int Hf\,d\nu$ with ν being singular with respect to m. (See [**1**, p. 421]). Let $L_1^+(m)$ denote $\{g \in L_1(m) | g \geqq 0 \text{ a.e.}\}$. For each $g \in L_1^+(m)$ we define the *$\mathcal{S}g$-topology* on B to be the weakest topology in which the linear functionals of the form $H \to \int (Hf)g\,dm$ with f in A, and the linear functionals in $\mathcal{S}$, are continuous. Set $\mathfrak{S} = \{\mathcal{S}g \,|\, g \in L_1^+(m)\}$. Let $\mathcal{W}$ denote the weak operator topology on B. Note that $\mathcal{S}g \subseteq \mathcal{W}$ for each $g \in L_1^+(m)$. By [**1**, p. 421], the $\mathcal{S}g$-continuous linear functionals on B are those of the form $l(H) = \int Hfg\,dm + \Sigma_{i=1}^n \int Hf_i\,d\mu_i$, where the measures μ_i, $i = 1, 2, \cdots, n$, are singular with respect to m and $f, f_1, f_2 \cdots f_n \in A$. Let $\mathcal{U}$ denote the smallest locally convex topology on B which contains all members of $\mathfrak{S}$. Any functional of the form $L(H) = \int Hf\,d\nu$, where $f \in A$, and ν is a regular Baire measure, can be written in the form

$$L(H) = \int Hf\,d\mu + \sum_{n=1}^{4} \int H(i^n f)g_n\,dm,$$

where μ is singular with respect to m, and $g_1, g_2, g_3, g_4 \in L_1^+(m)$. It follows that L is $\mathcal{U}$-continuous. Hence, by the definition of $\mathcal{W}$, we have $\mathcal{U} \subseteq \mathcal{W}$.

THEOREM 2.1. *For each $g \in L_1^+(m)$, $E(T)$ is the $\mathcal{S}g$-closed convex hull of $F(T)$.*

REMARK. It is not possible to prove Theorem 2.1 by using arguments based on the Krein-Milman theorem. For in order for the Krein-Milman theorem to apply to $E(T)$ it would be necessary for

$E(T)$ to be compact in the $\mathcal{S}g$-topology, but the following argument shows that $E(T)$ is not $\mathcal{S}g$-compact for any $g \in L_1^+(m)$: Let K be a "Cantor" subset of Γ which is disjoint from $\Gamma(T)$. Let $C_1(K)$ denote the set of continuous complex valued functions on K having absolute value $\leqq 1$. Define $j: E(T) \to C_1(K)$ by $j(U) = UZ|K$. If $C_1(K)$ is equipped with the topology of pointwise convergence, then j is $\mathcal{S}g$-continuous for each $g \in L_1^+(m)$. By **[9]**, the map j is onto. Since $C_1(K)$ is not compact in the topology of pointwise convergence, it follows that $E(T)$ is not compact in the $\mathcal{S}g$-topology.

Our proof of Theorem 3 will depend on the following two lemmas:

LEMMA 2.1. *For z in D, and $t \in [-\pi, \pi]$ let*

$$P_z(e^{it}) = \operatorname{Re}((z + e^{it})\ (z - e^{it})^{-1}),$$

i.e., $P_z(e^{it})$ is the Poisson kernel. Consider the set $V = \{\Sigma_{i=1}^n c_i P_{z_i} \mid c_i \geqq 0$ and $z_i \in D$ for $i = 1, 2, \cdots, n\}$. Then the L_1-closure of V is $L_1^+(m)$.

Proof. Suppose that $g_1 \in L_1^+(m)$, but g_1 is not in the closure of V. Then there exists an h in $L_\infty(m)$, such that $\int g_1 h\,dm > 0$ and $\int v h\,dm \leqq 0$ for every $v \in V$. In particular

$$\int P_z(e^{it}) h(e^{it})\,dm(e^{it}) \leqq 0.$$

for all z in D. Fatou's Theorem **[3**, p. 34**]** implies that $h \leqq 0$ almost everywhere with respect to m. Hence, $\int g\,h\,dm \leqq 0$. Thus, we have reached a contradiction.

LEMMA 2.2. *Let E be a closed subset of Γ such that $m(E) = 0$. Let $\phi_0: E \to D \cup \Gamma$ be continuous. Consider $z_1, z_2, \cdots, z_n$, $w \in D$. There is a function ϕ in the unit ball A_1 of A which extends ϕ_0 and satisfies $\phi(z_i) = w$ for $i = 1, 2, \cdots, n$.*

Proof. Suppose that $w = 0$. Let

$$B(Z) = \Pi_{i=1}^n [(z - \bar{z}_i)\ (1 - \bar{z}_i z)^{-1}].$$

For each u in Γ, we have $|B(u)| = 1$. Define β_0 on E by $\beta_0(u) = \overline{B(u)}\phi_0(u)$. The function β_0 has an extension β in A_1. It follows that

$B\beta$ satisfies the assertion of the lemma in the case where $w = 0$. If $w \neq 0$, we choose a Mobius transformation τ of D such that $\tau(0) = w$, and apply the preceeding argument to obtain a function ϕ_1 in A_1 which extends $\tau^{-1} \circ \phi_0$ and maps $z_1, \cdots, z_n$ into 0. Thus, the function $\phi = \tau \circ \phi_1$, extends ϕ_0, lies in A_1, and statisfies $\phi(z_i) = w$ for $i = 1, 2, \cdots n$.

Proof of Theorem 2.1. (The argument used here is an adaptation of one due to Morris and Phelps [6, Th. 2.1].)

Suppose $U \in E(T)$ but it is not in the $\mathcal{S}g$-closed convex hull of $F(T)$. By [1, Th. 9, p. 421], there are functions $f, f_1, f_2, \cdots, f_n \in A$, measures $\mu_1, \mu_2, \cdots, \mu_n$ on Γ which are singular with respect to m, and a real number $r > 0$ such that

$$\mathrm{Re}\left(\int (Uf)\, g dm + \sum_{i=1}^{n} \int Uf_i d\mu_i\right) \geqq \mathrm{Re}\left(\int (Ff)\, g dm + \sum_{i=1}^{n} \int Ff_i d\mu_i\right) + r,$$

for every F in $E(T)$.

By Lemma 2.1, there are points $z_1, z_2, \cdots, z_p \in D$ and nonnegative real numbers $c_1, c_2, \cdots, c_p$ such that

$$\begin{aligned} &\mathrm{Re}\left(\sum_{j=1}^{p} c_j Uf(z_j) + \sum_{i=1}^{n} \int Uf_i d\mu_i\right) \\ (3) \qquad &\geq \mathrm{Re}\left(\sum_{j=1}^{p} c_j Ff(z_j) + \sum_{i=1}^{n} \int Ff_i d\hat{\mu i}\right) + \frac{r}{2}, \end{aligned}$$

for every F in $F(T)$. We can assume without loss of generality that $\mu_i \geqq 0$ for $i = 1, 2, , \cdots, n$. Since $Uf_i = Ff_i$ on $\Gamma(T)$ for $i = 1, 2, \cdots, n$ and $F \in F(T)$, we can also assume that $\mu_i(\Gamma(T)) = 0$ for $i = 1, 2, \cdots, n$. Let $\nu = \sum_{i=1}^{n} \mu_i$. Given $\epsilon > 0$, there is a closed subset Y of $\Gamma \sim \Gamma(T)$ such that $m(Y) = 0$ and $\nu(\Gamma \sim Y) < \epsilon$. Let h_i denote the Radon-Nikodym derivative of μ_i with respect to ν for $i = 1, 2, \cdots, n$. Choose continuous functions h'_i on Γ such that $0 \leqq h'_i \leqq 1$ and $\int |h_i - h'_i| d\nu < \epsilon$ for $i = 1, 2, \cdots, n$.

Let $g = \sum_{i=1}^{n} h'_i Uf_i$. For each $y \in \Gamma$, define $k_y = \sum_{i=1}^{n} h'_i(y) f_i$. Then $g(y) = Uk_y(y)$. $g(y)$ is also equal to $(U^* e_y)\,(k_y)$, where U^* is the adjoint of U and e_y represents the "evaluation at y" functional on A. Let S denote the unit ball in the dual space of A. Since U^* maps S into S, it follows that $U^* e_y \in S$. The function $W(\rho) = \mathrm{Re}\, \rho(k_y)$ is weak* continuous on S and $\sup W(S) \geqq \mathrm{Re}\, U^* e_y(k_y) = \mathrm{Re}\, g(y)$. The extreme points of S are exactly the functionals ce_y, where $c, y \in \Gamma$. It follows from the Krein-Milman theorem that, for each $y \in \Gamma$, there exist $\phi(y)$, $c(y) \in \Gamma$, such that

$$\operatorname{Re}\left(\sum_{i=1}^{n} c(y)h_i'(y)f_i(\phi(y))\right) > \operatorname{Re} g(y) - \epsilon.$$

For each $y \in Y$ choose an open neighborhood V_y of y such that $V_y \cap \Gamma(T) = \phi$ and

$$\operatorname{Re}\left(\sum_{i=1}^{n} c(y)h_i'(w)f_i(\phi(y))\right) > \operatorname{Re} g(w) - 2\epsilon$$

for every $w \in V_y$. Let $\{V_{y_1}, \cdots, V_{y_p}\}$ be a finite collection of V_y's which covers y. We can easily find another open cover $\{U_1, \cdots, U_p\}$ of Y such that $U_i \subset V_{y_i}$ and $\nu(\{y \mid y \text{ is in more than one } U_j\}) < \epsilon$. Consider the sets

$$H_j = (Y \cap U_j) \sim \cup \{U_i \mid i \neq j\}, \quad j = 1, 2, \cdots, p.$$

Then H_j's are closed and disjoint and $\nu(Y \sim \cup_{j=1}^{p} H_j) < \epsilon$.

Define mappings θ_0, k_0: $\Gamma(T) \cup [\cup_{i=1}^{p} H_i] \to \Gamma$ by

$$\theta_0(y) = \begin{cases} \phi(y_j) & \text{if} \quad y \in H_j \\ \hat{T}(y) & \text{if} \quad y \in \Gamma(T). \end{cases}$$

$$k_0(y) = \begin{cases} c(u_j) & \text{if} \quad y \in H_j \\ T1(y) & \text{if} \quad y \in \Gamma(T). \end{cases}$$

Note that $m(\Gamma(T) \cup [\cup_{i=1}^{p} H_i]) = 0$. Since U is an isometry, there are points w_0 and w_1 in D such that $\operatorname{Re} w_0 f(w_1) \geqq \operatorname{Re} Uf(z_i) - \epsilon$ for $i = 1, 2, \cdots, p$. By Lemma 2.2, there are extensions θ and k, of θ_0 and k_0 respectively, which lie in A_1 and satisfy $\theta(z_j) = w_1$ and $k(z_j) = w_0$ for $j = 1, 2, \cdots, n$. Define the linear operator F_1: $A \to A$ by $F_1 h = k(h \circ \theta)$. Clearly, we have $F_1 \in F(T)$. By a straightforward argument, we can find a constant $M > 0$ independent of ϵ such that

$$\operatorname{Re}\left(\sum_{j=1}^{p} c_j F_1 f(z_j) + \sum_{i=1}^{n} \int F_1 f_i d\mu_i\right)$$
$$> \operatorname{Re}\left(\sum_{j=1}^{p} c_j Uf(z_j) + \sum_{i=1}^{n} \int Uf_i d\mu_i\right) - M\epsilon.$$

We can obtain a contradiction to (3) by taking ϵ to be sufficient small.

COROLLARY 2.1. *Suppose* $T1 \equiv 1$. *Let* $E_1(T) = \{U \mid U \in E(T)$ *and* $U1 = 1\}$, *and let* $F_1(T) = E_1(T) \cap F(T)$. *Then for each* $g \in L_1^+(m)$, *the set* $E_1(T)$ *is the closed convex hull of* $F_1(T)$ *where the closure is taken in the* $\mathcal{S}g$*-topology.*

Proof. Let $S_1 = \{L \in S \mid L(1) = 1\}$. The adjoint T^* of T maps S_1 to S_1. The extreme points of S_1 are the functionals of the form e_y with $y \in \Gamma$. Thus, in the proof of Theorem 2.1 we may take $c(y) = 1$. Also, it is clear that, in this case, we may take $w_0 = 1$. Consequently, it can be asumed that the function k is identically 1.

3. The case $T1 = 1$. In this section it will be assumed that $T1 = 1$. We will investigate the closure in the weak operator topology of the set $\operatorname{cov} F_1(T)$.

Let H_∞ denote the space of bounded analytic functions on D and let $B(H_\infty)$ denote the space of bounded linear operators on H_∞. Denote by $\mathcal{P}$ the weakest topology on $B(H_\infty)$ such that all linear functionals of the form $M \to Mg(z)$, where $g \in H_\infty$ and $z \in D$, are continuous. The following property of $B(H_\infty)$ will be very useful in this section: *The unit ball of* $B(H_\infty)$ *is* $\mathcal{P}$*-compact.* To verify this property it sufficies to use a result due to Kadison **[4]** together with the fact that the unit ball H_∞^1 of H_∞ is compact in the topology of pointwise convergence.

Let $A_T = \{\phi \in A_1 \mid \phi \mid \Gamma(T) = \hat{T}\}$. Let H_T denote the closure of A_T in the topology of pointwise convergence on D. Since $H_T \subseteq H_\infty^1$, it follows that H_T is compact in the topology of pointwise convergence on D. Each $F \in F_1(T)$ is of the form $Ff = f \circ \phi$ for all $f \in A$, where $\phi \in A_T$. Thus, F has an extension to H_∞ denoted F^* which is defined by $F^*g = g \circ \phi$ for every $g \in H_\infty$. Similarly, each $V \in \operatorname{cov} F_1(T)$ has an extension V^* lying in $\operatorname{cov} F_1^*(T)$, where $F_1^*(T) = \{F^* \mid F \in F_1(T)\}$. Since $F_1^*(T)$ is contained in the unit ball of $B(H_\infty)$, it follows that the $\mathcal{P}$-closed convex hull of $F_1^*(T)$, denoted by R, is compact in the $\mathcal{P}$-topology. Let Q denote the $\mathcal{P}$-closure of $F_1^*(T)$. Suppose that $W \in R$. By the integral form of the Krein-Milman Theorem **[7**, p. 6**]**, there is a probability measure μ_W supported by Q such that

$$Wg(z) = \int_Q W'g(z)\,d\mu_W(W')$$

for every $g \in H_\infty$ and every $z \in D$. Note that $Q = \{W \mid Wg = g \circ \phi$, where $\phi \in H_T\}$. Thus, Q may be identified with H_T. Consequently, we can write

$$Wg(z) = \int_{H_T} g \circ \phi(z)\,d\mu_W(\phi)$$

for all $g \in H_\infty$ and all $z \in D$. Suppose now that U is in the weak operator closed convex hull of $F_1(T)$. Then there exists a net $\{V_\alpha\}$ in cov $F_1(T)$ which converges in the weak operator topology to U. In particular, $Uf(z) = \lim V_\alpha f(z)$ for each $f \in A$ and each $z \in D$. The net V_α^* has a subnet V_β^* which converges to some $U^* \in R$. By the definition of the $\mathcal{P}$-topology, we have $U^*f(z) = Uf(z)$ for $f \in A$ and $z \in D$. Thus, we have proved the following:

THEOREM 3.1. *Let U be in the closure of* cov $F_1(T)$ *in the weak operator topology. Then there exists a probability measure μ on H_T such that*

$$Uf(z) = \int_{H_T} f \circ \phi(z) d\mu(\phi)$$

for each $f \in A$ and each $z \in D$.

We will now use Theorem 3.1 to derive another sufficient condition for an isometry to be of the form (1).

THEOREM 3.2. *Suppose U is in the weak operator closure of* cov $F_1(T)$. *If there is a nonconstant inner function G such that UG is an extreme point of A_1, then U is of the form* (1).

Our proof of Theorem 3.2 depends upon the following technical lemma.

LEMMA 3.1. *Let G be a nonconstant inner function in A.* (a) *Suppose that $k \in A$ is of the form $k = G \circ h$ on D, where $h \in H_\infty^1$. Then h has an extension to $D \cup \Gamma$ which is continuous.* (b) *Let h_1, $h_2 \in A_1$. Consider the set*

$$S = \{z \in D \cup \Gamma \mid h_1(z) = h_2(z)\}.$$

Suppose that $h_1(S)$ is infinite. Suppose also that $G \circ h_1 = G \circ h_2$. Then $h_1 = h_2$.

Proof. Since G is an inner function and is a member of A, it follows that G is of the form

$$G(z) = e^{ix} \prod_{n-1}^{N} (z - z_n)/(1 - \bar{z}_n z),$$

where the z_n's are (not necessarily distinct) points of D. It follows that, given any point $u_0 \in \Gamma \cup D$, there exists a disk D_0 about u_0 and analytic functions $g_1, g_2, \cdots, g_n$ defined on D_0, such that if $G(w) = u \in D_0$ then $w = g_j(u)$ for some j. Suppose that $u_0 = k(z_0)$, where $z_0 \in \Gamma$. Choose a set W containing z_0 which is open relative to $\Gamma \cup D$ and satisfies $k(W) \subseteq D_0$. On $W \cap D$, we have $k = G \circ h$. It follows that for some j, $h(z) = g_j \circ k(z)$ for all $z \in W$. Thus, h can be extended continuously to $W \cap \Gamma$. A simple compactness argument now shows that h can be extended continuously to all of Γ.

Consider the set $Y = \{z \in D \cup \Gamma \mid G'(h_1(z)) \neq 0$ and $h_1(z) = h_2(z)\}$. We will show that Y is open relative to $D \cup \Gamma$. Since Y is nonempty, it will follow that $h_1 = h_2$. Let $z_0 \in Y$. Since $G'(h_1(z_0)) \neq 0$, there exists an open disk D_0 about $h_1(z_0)$ such that G is one-to-one on D_0. Choose a set N, which is open relative to $D \cup \Gamma$, such that $h_1(N) \subseteq D_0$ and $h_2(N) \subseteq D_0$. Then, for $z \in N$, $G(h_1(z)) = G(h_2(z))$. It follows that $h_1 = h_2$ on N.

Proof of Theorem 3.2: By Theorem 3.1, we may write

$$Uf(z) = \int_{H_T} f \circ \phi(z) d\mu(\phi)$$

for all $f \in A$ and all $z \in D$. For each $z \in D$, let

$$J_z = \{\phi \in H_T \mid \operatorname{Re} UG(z) < \operatorname{Re} G \circ \phi(z)\}.$$

Suppose that for some $u \in D$, we have $c = \mu(J_u) > 0$. Define measures μ_1 and μ_2 on H_T by

$$\mu_1(K) = c^{-1}\mu(K \cap J_u)$$
$$\mu_2(K) = (1-c)^{-1}\mu(K \cap (H_T \sim J_u)).$$

By [7, Prop. 1.1], there are operators U_1 and U_2 in R such that

$$U_i f(z) = \int_{H_T} f \circ \phi(z) d\mu_i \qquad i = 1,2,$$

for each $f \in A$ and each $z \in D$. (Note that for $f \in A$, $U_i f$ is not necessarily in A.) It follows that

$$Uf(z) = cU_1 f(z) + (1-c)U_2 f(z)$$

for $f \in A$ and $z \in D$. Since UG is an extreme point of A_1, it is also extreme point of H^1_∞. (See [3, p. 139].) Thus, we have $UG = U_1 G = U_2 G$ on D, but

$$\operatorname{Re} UG(u) = \int_{H_T} \operatorname{Re} UG(u)\, d\mu_1(\phi)$$

$$< \int_{H_T} \operatorname{Re} G \circ \phi(u)\, d\mu_1(\phi) = \operatorname{Re} U_1 G(u),$$

a contradiction. It follows that for each $z \in D$ we have $\mu\{\phi \mid \operatorname{Re} UG(z) < \operatorname{Re} G \circ \phi(z)\}) = 0$. Similarly, we can show that

$$\mu(\{\phi \mid \operatorname{Re} UG(z) > \operatorname{Re} G \circ \phi(z)\}) = \mu(\{\phi \mid \operatorname{Im} UG(z) \neq \operatorname{Im} G \circ \phi(z)\}) = 0.$$

Thus, $UG(z) = G \circ \phi(z)$ for all ϕ in the support of μ. It follows that the support of μ consists of finitely many functions $\phi_1, \cdots, \phi_m \in H_T$, where each ϕ_i satisfies $G \circ \phi_i = UG$ on D. By Lemma 3.1, each ϕ_i is continuous on $D \cup \Gamma$. Thus, there exist positive numbers $c_1, \cdots, c_m$ such that $\Sigma c_i = 1$ and

$$Uf(z) = \sum c_i f \circ \phi_i(z)$$

for each $f \in A$ and each $z \in D \cup \Gamma$. For $z \in \Gamma(T)$, we have $\phi_i(z) = \hat{T}(z)$ for $i = 1, 2, \cdots, m$. It follows by Lemma 3.1, that $\phi_1 = \phi_2 = \cdots = \phi_m$. Hence $U \in F_1(T)$.

Remark. Theorem 3.2 provides a possible approach to the problem of finding an isometry T such that $T1 = 1$ and $E_1(T)$ is not the weak operator closure of $\operatorname{cov} F_1(T)$. If an isometry T can be found such that: $T1 = 1$, T is not of the form (1), and TG is an extreme point of A_1 for some nonconstant inner function $G \in A$, then it will follow from Theorem 3.2 that $T \notin$ weak operator closure of $\operatorname{cov} F_1(T)$.

References

1. N. Dunford and J. T. Schwartz, *Linear Operators*, Vol. I, Interscience, New York, 1958.
2. E. Hille, *Analytic Function Theory*, Vol. I, Blaisdell, New York, 1959.
3. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, New Jersey, 1962.
4. R. V. Kadison, *The trace in finite operator algebras*, Proc. Amer. Math. Soc., **12** (1961).
5. J. N. McDonald, *Isometries of function algebras*, Illinois J. Math., **17**, No. 3, pp. 579–583.
6. P. D. Morris and R. R. Phelps, *Theorems of Krein-Milman type for certain convex sets of operators*, Trans. Amer. Math. Soc., **150** (1970), 183–200.
7. R. R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand, 1966.
8. R. Rochberg, *Which linear maps of the disk algebra are multiplicative*, Pacific J. Math., **38** (1971).
9. W. Rudin, *Boundary values of continuous analytic functions*, Proc. Amer. Math. Soc., **7** (1956).

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DOUBLY STOCHASTIC MATRICES WITH MINIMAL PERMANENTS

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A simple elementary proof is given for a result of D. London on permanental minors of doubly stochastic matrices with minimal permanents.

A matrix with nonnegative entries is called *doubly stochastic* if all its row sums and column sums are equal to 1. A well-known conjecture of van der Waerden [3] asserts that the permanent function attains its minimum in Ω_n, the set of $n \times n$ doubly stochastic matrices, uniquely for the matrix all of whose entries are $1/n$. The conjecture is still unresolved.

A matrix A in Ω_n is said to be *minimizing* if

$$\operatorname{per}(A) = \min_{S \in \Omega_n} \operatorname{per}(S).$$

The properties of minimizing matrices have been studied extensively in the hope of finding a lead to a proof of the van der Waerden conjecture.

Let $A(i\,|\,j)$ denote the submatrix obtained from A by deleting its ith row and its jth colum. Marcus and Newman [3] have obtained inter alia the following two results.

THEOREM 1. *A minimizing matrix A is fully indecomposable,* i.e.,

$$\operatorname{per}(A(i\,|\,j)) > 0$$

for all i and j.

In other words, if A is a minimizing $n \times n$ matrix then for any (i,j) there exists a permutation σ such that $j = \sigma(i)$ and $a_{s,\sigma(s)} > 0$ for $s = 1,\cdots,i-1,i+1,\cdots,n$.

THEOREM 2. *If $A = (a_{ij})$ is a minimizing matrix then*

$$\operatorname{per}(A(i\,|\,j)) = \operatorname{per}(A) \tag{1}$$

for any (i,j) for which $a_{ij} > 0$.

The result in Theorem 2 is of considerable interest. For, if it could be shown that (1) holds for all permanental minors of A, the van der Waerden conjecture would follow. London [2] obtained the following result.

THEOREM 3. *If A is a minimizing matrix, then*

$$\operatorname{per}(A(i\,|\,j)) \geqq \operatorname{per}(A) \tag{2}$$

for all i and j.

London's proof of Theorem 3 depends on the theory of linear inequalities. Another proof of London's result is due to Hedrick [1]. In this paper I give an elementary proof of the result that is considerably simpler than either of the above noted proofs.

Proof of Theorem 3. Let $A = (a_{ij})$ be an $n \times n$ minimizing matrix. Let σ be a permutation on $\{1, \cdots, n\}$ and $P = (p_{ij})$ be the corresponding permutation matrix. For $0 \leqq \theta \leqq 1$, define

$$f_P(\theta) = \operatorname{per}((1-\theta)A + \theta P).$$

Since A is a minimizing matrix, we have

$$f_P'(0) \geqq 0$$

for any permutation matrix P. Now

$$\begin{aligned} f_P'(0) &= \sum_{s,t=1}^{n} (-a_{st} + p_{st}) \operatorname{per}(A(s\,|\,t)) \\ &= \sum_{s,t=1}^{n} p_{st} \operatorname{per}(A(s\,|\,t)) - n \operatorname{per}(A) \\ &= \sum_{s=1}^{n} \operatorname{per}(A(s\,|\,\sigma(s))) - n \operatorname{per}(A). \end{aligned}$$

Hence,

$$\sum_{s=1}^{n} \operatorname{per}(A(s\,|\,\sigma(s))) \geqq n \operatorname{per}(A) \tag{3}$$

$$\sum_{s=1}^{n} \operatorname{per}(A(s\,|\,\sigma(s))) \geqq n \operatorname{per}(A)$$

for any permutation σ. Since A is a minimizing matrix and thus, by Theorem 1, fully indecomposable, we can find for any given (i,j) a permutation σ such that $j=\sigma(i)$ and $a_{s,\sigma(s)}>0$ for $s = 1,\cdots,i-1,i+1,\cdots,n$. But then by Theorem 2,

$$\operatorname{per}(A(s\,|\,\sigma(s))) = \operatorname{per}(A)$$

for $s = 1,\cdots,i-1,i+1,\cdots,n$, and it follows from (3) that

$$\operatorname{per}(A(i\,|\,j)) \geqq \operatorname{per}(A).$$

References

1. M. B. Hedrick, *P-minors of the doubly stochastic matrix at which the permanent achieves a minimum*, (to appear).
2. David London, *Some notes on the van der Waerden conjecture*, Linear Algebra and Appl., **4** (1971), 155–160.
3. Marvin Marcus and Morris Newman, *On the minimum of the permanent of a doubly stochastic matrix*, Duke Math. J., **26** (1959), 61–72.

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COVERING THE VERTICES OF A GRAPH BY VERTEX-DISJOINT PATHS

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Define the path-covering number $\mu(G)$ of a finite graph G to be the minimum number of vertex-disjoint paths required to cover the vertices of G. Let $g(n,k)$ be the minimum integer so that every graph, G, with n vertices and at least $g(n,k)$ edges has $\mu(G) \leqq k$. A relationship between $\mu(G)$ and the degree sequence for a graph G is found; this is used to show that

$$\tfrac{1}{2}(n-k)(n-k-1)+1 \leqq g(n,k) \leqq \tfrac{1}{2}(n-1)(n-k-1)+1$$

A further extremal problem is solved.

1. Introduction. A *graph* G is a finite collection $\mathscr{V}(G)$ of vertices (or points) some pairs of which are joined by a single edge; the collection of edges is denoted by $\mathscr{E}(G)$. H is a *subgraph* of G if $\mathscr{V}(H) \subseteq \mathscr{V}(G)$ and $\mathscr{E}(H) \subseteq \mathscr{E}(G)$. If H and K are two vertex-disjoint graphs, $H \cup K$ is the graph with $\mathscr{V}(H \cup K) = \mathscr{V}(H) \cup \mathscr{V}(K)$ and $\mathscr{E}(H \cup K) = \mathscr{E}(H) \cup \mathscr{E}(K)$; $H + K$ is $H \cup K$ together with all $|\mathscr{V}(H)|\ |\mathscr{V}(K)|$ possible choices of edges joining a vertex of H to a vertex of K. $\bar{G}$ denotes the complement of G; Γ_n denotes the complete graph with n vertices and $\Gamma_{m,n}$ denotes the complete bipartite graph, $\bar{\Gamma}_m + \bar{\Gamma}_n$.

Let G be a graph. A path of length n in G is an ordered sequence $P = \langle a_1, a_2, \cdots, a_n \rangle$ of distinct points, where if $n \geqq 2$, a_i is adjacent to a_{i+1} for $1 \leqq i \leqq n-1$. $\langle a_1, a_2, \cdots, a_n \rangle$ is the same path as $\langle a_n, a_{n-1}, \cdots, a_1 \rangle$. If P and Q are paths, by $P * Q$ we shall mean that one end-point, a of P, is adjacent to one end-point, b of Q, and that $P * Q$ is formed by joining a to b. More specifically we may write $Pa * bQ$ or $P * bQ$ or $Pa * Q$ to specify, in varying degrees, which end-point of P is joined to which end-point of Q. Also, $\langle a_1, a_2, \cdots, a_n \rangle * \langle b_1, b_2, \cdots, b_m \rangle = \langle a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m \rangle$ where a_n must be adjacent to b_1. A *Hamilton-path* is a path of length $|\mathscr{V}(G)|$. A *path-cover* of G is a collection, $\mathscr{S}$, of vertex-disjoint paths such that every vertex of G lies on some path in $\mathscr{S}$. The *path-covering number*, denoted by $\mu(G)$, of G is defined by:

$$\mu(G) = \mathrm{Min}\{|\mathscr{S}| : \mathscr{S} \text{ is a path-cover of } G\}.$$

A *minimal path-cover* (M.P.C.) of G is a path-cover, $\mathcal{S}$ of G, with $|\mathcal{S}| = \mu(G)$.

We note that $\mu(G)$ is an invariant of G and remark that a graph, G, has a Hamilton-path if and only if $\mu(G) = 1$. It has been shown by Nash-Williams [1] and others that the problem of classifying all Hamiltonian graphs is equivalent to that of classifying all graphs which have a Hamilton-path. Thus a classification of all graphs with $\mu(G) = k$ ($k = 1,2,3,\cdots$) would also solve the Hamiltonian problem as a special case.

Historically, O, Ore [3] first introduced the graphical invariant μ. In [2] some elementary properties of μ are derived. In §2 we generalize a result of O. Ore (Theorem 2.1 in [3]) and in §3 we consider two extremal problems involving μ.

2. Valency considerations. In this section we derive a connection between the path-covering number and the degree sequence of a graph. We begin with some definitions:

DEFINITION 2.1. Let A be a finite set and f a real-valued function defined on the collection of subsets of A. For $B \subseteq A$ and for any integer i with $1 \leqq i \leqq |B|$, define the function S_i by:

$$S_i(f, B) = \sum_{\substack{C \subseteq B \\ |C| = i}} f(C).$$

DEFINITION 2.2. If G is a graph, $B \subseteq \mathcal{V}(G)$, and either $H \subseteq \mathcal{V}(G)$ or H is a subgraph of G, then define the generalized valence function, ρ, by

$$\rho_H(B) = \text{the number of vertices of } H \text{ which are adjacent to every member of } B.$$

If x is a vertex of G, then we write $\rho(x)$ for $\rho_G(\{x\})$.

DEFINITION 2.3. Let G be a graph and $X \subseteq \mathcal{V}(G)$ with $|X| = k \geqq 2$. Define:

$$D(G, X) = \frac{1}{k} S_1(\rho_G, X) - \sum_{i=1}^{k} (-1)^i \left(\frac{k-i}{k}\right) S_i(\rho_G, X).$$

The following lemma is easily verified:

LEMMA 2.4. *If* $X = \{x_1, x_2, \cdots, x_k\}$, *and* $1 \leqq m \leqq k-1$, *then*

$$\sum_{i=1}^{k} S_m(f, X - \{x_i\}) = (k-m) S_m(f, X).$$

We now state the main result of this section:

THEOREM 2.5. *Let G be a graph with $\mu = \mu(G) \geqq 2$, $|\mathscr{V}(G)| = n$ and k an integer with $2 \leqq k \leqq \mu$, then there exists a set X consisting of k mutually non-adjacent vertices of G, satisfying*:

$$\mu \leqq n - D(G, X). \tag{2.6}$$

Note that the case $k = 2$ reduces to the result of Ore (Theorem 2.1 in [3]):

$$\mu \leqq n - \rho(x_1) - \rho(x_2).$$

Proof. Let $\mathscr{S} = \{P_1, P_2, \cdots, P_\mu\}$ be a M.P.C. for G. For each $1 \leqq i \leqq k$, let x_i be an end-vertex of P_i. Since $\mathscr{S}$ is a M.P.C., x_i is not adjacent to x_j for $i \neq j$.

Let $X = \{x_1, x_2, \cdots, x_k\}$. We first show that for $1 \leqq i \leqq k$ and $1 \leqq j \leqq \mu$, the inequality:

$$\rho_{P_j}(\{x_i\}) \leqq |P_j| - \left(1 - \sum_{l=1}^{k-1} (-1)^l S_l(\rho_{P_j}, X - \{x_i\})\right) \tag{2.7}$$

holds. Let P_j be the path $\langle a_1, a_2, \cdots, a_t \rangle$, let $1 \leqq m \leqq k$, $m \neq i$, and consider the following cases:

(i) $i = j$. In this case assume that $x_i = a_1$.

(ii) $m = j$. In this case assume that $x_m = a_t$.

(iii) $m \neq j$ and $i \neq j$.

Let

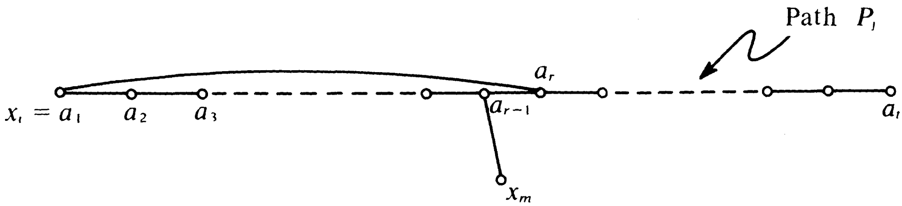
$$A = \{r\colon a_r \text{ is adjacent to } x_i\},$$

$$B_m = \{r\colon a_{r-1} \text{ is adjacent to } x_m\}$$

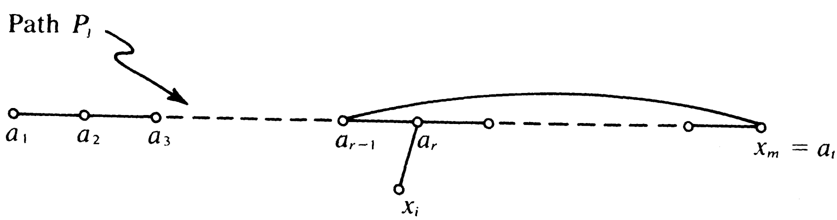
and

$$B = \bigcup_{\substack{1 \leqq m \leqq k \\ m \neq i}} B_m.$$

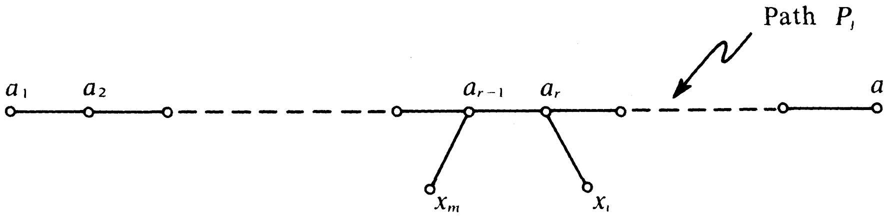
We claim that $A \cap B_m = \phi$, for if $r \in A \cap B_m$, then in each case we can construct a path-cover, $\mathcal{T}$ for G, as follows (see Figure 2.8):



Case (i)



Case (ii)



Case (iii)

FIGURE 2.8

In case (i), let:

$$\mathcal{T} = \mathcal{S} \cup \{\langle a_t, a_{t-1}, \cdots, a_r, x_i, a_2, a_3, \cdots, a_{r-1}\rangle * x_m P_m\} - \{P_i, P_m\}.$$

In case (ii), let:

$$\mathcal{T} = \mathcal{S} \cup \{\langle a_1, a_2, \cdots, a_{r-1}, x_m, a_{t-1}, a_{t-2}, \cdots, a_r\rangle * x_i P_i\} - \{P_t, P_m\}.$$

In case (iii), let:

$$\mathcal{T} = \mathcal{S} \cup \{\langle a_1, \cdots, a_{r-1}\rangle * x_m P_m, \langle a_t, a_{t-1}, \cdots, a_r\rangle * x_i P_i\} - \{P_t, P_j, P_m\}.$$

In either case, $|\mathcal{T}| = |\mathcal{S}| - 1 < |\mathcal{S}|$, contradicting the minimality of $\mathcal{S}$. Hence $A \cap B_m = \phi$. Also, in each case $a_1 \notin A$; so $A \subseteq P_j - B \cup \{a_1\}$. This gives $|A| \leqq |P_j| - |B \cup \{a_1\}|$, since $B \cup \{a_1\} \subseteq P_j$. But then, since $a_1 \notin B$, we get:

$$|A| \leqq |P_j| - (1 + |B|). \tag{2.9}$$

For $1 \leqq m \leqq k$, let:

$$C_m = \{r\colon a_r \text{ is adjacent to } x_m\}.$$

Then since x_m is not adjacent to a_1, $|C_m| = |B_m|$ and:

$$|B| = \Bigl| \bigcup_{\substack{1 \leqq m \leqq k \\ m \neq i}} B_m \Bigr| = \Bigl| \bigcup_{\substack{1 \leqq m \leqq k \\ m \neq i}} C_m \Bigr|$$

$$= \sum_{l=1}^{k-1} (-1)^{l+1} \sum_{\substack{1 \leqq m_1 < m_2 < \cdots < m_l \leqq k \\ m_1, m_2, \cdots, m_l \neq i}} |C_{m_1} \cap C_{m_2} \cap \cdots \cap C_{m_l}|$$

$$= -\sum_{l=1}^{k-1} (-1)^l S_l(\rho_{P_j}, X - \{x_i\}). \tag{2.10}$$

So since $|A| = \rho_{P_j}(\{x_i\})$, (2.7) follows from (2.9) and (2.10). Summing (2.7) for $1 \leqq i \leqq k$ and applying Lemma 2.4, we get:

$$S_1(\rho_{P_j}, X) \leqq k|P_j| - \Bigl(k - \sum_{l=1}^{k-1} (-1)^l (k-l) S_l(\rho_{P_j}, X)\Bigr). \tag{2.11}$$

Summing (2.11) for $1 \leqq j \leqq \mu$, we get:

$$S_1(\rho_G, X) \leqq kn - \left(k\mu - \sum_{l=1}^{k-1} (-1)^l (k-l) S_l(\rho_{G_l}, X)\right).$$

from which (2.6) follows.

3. Extremal problems.

DEFINITION 3.1. Let k and n be integers with $1 \leqq k \leqq n$. Define:

$$g(n,k) = \operatorname{Min}\{m\colon \text{ every graph, } G, \text{ with } |\mathscr{V}(G)| = n \text{ and } |\mathscr{E}(G)| \geqq m \text{ has } \mu(G) \leqq k\}.$$

In this section we determine bounds for $g(n,k)$. See **[4]** for techniques in proving the following:

LEMMA 3.2.

$$\sum_{i=1}^{k=1} (-1)^i \left(\frac{k-i}{k}\right) \binom{k}{i} = -1 \quad \textit{if} \quad k \geqq 2, \tag{3.3}$$

$$\sum_{i=2}^{k} (-1)^i (k-i+1) \binom{k}{i-1} = k \quad \textit{if} \quad k \geqq 2, \tag{3.4}$$

$$\sum_{i=2}^{j} (-1)^i (k-i+1) \binom{j-1}{i-1} = k \quad \textit{if} \quad 3 \leqq \mathrm{j} \leqq \mathrm{k}. \tag{3.5}$$

LEMMA 3.6. *Let K be a graph with $|\mathscr{V}(K)| = s \geqq 1$, and let k be an integer with $k \geqq 2$, and suppose $H = \bar{\Gamma}_k + K$, then:*

$$D(H, \mathscr{V}(\bar{\Gamma}_k)) = 2s.$$

Proof. For $1 \leqq i \leqq k-1$ and $B \subseteq \mathscr{V}(\bar{\Gamma}_k)$ with $|B| = i$, each member of B is adjacent to every member of $\mathscr{V}(K)$. There are $\binom{k}{i}$ choices for B and $|\mathscr{V}(K)| = s$; thus:

$$S_i(\rho_H, \mathscr{V}(\bar{\Gamma}_k)) = s\binom{k}{i}.$$

This gives:

$$D(H, \mathscr{V}(\bar{\Gamma}_k) = \frac{s}{k}\binom{k}{1} - \sum_{i=1}^{k-1}(-1)^i s\left(\frac{k-i}{k}\right)\binom{k}{i}$$

$$= s\left[1 - \sum_{i=1}^{k-1}(-1)^i\left(\frac{k-i}{k}\right)\binom{k}{i}\right]$$

$$= 2s, \quad \text{using} \quad (3.3).$$

THEOREM 3.7. *For* $1 \leqq k \leqq n$,

$$g(n,k) \leqq \tfrac{1}{2}(n-1)(n-k-1)+1. \tag{3.8}$$

Proof. Let G be a graph with $|\mathscr{V}(G)| = n$, and $|\mathscr{E}(G)| \geqq \frac{1}{2}(n-1)(n-k-1)+1$. Suppose $\mu(G) > k$ and $X = \{x_1, x_2, \cdots, x_k, x_{k+1}\}$ is a set of mutually nonadjacent vertices of G.

G may be considered to have been obtained from the complete graph Γ_n through the elimination of at most:

$$\tfrac{1}{2}n(n-1) - \tfrac{1}{2}(n-1)(n-k-1) - 1 = \tfrac{1}{2}(n-1)(k+1) - 1$$

edges. $\frac{1}{2}k(k+1)$ are removed in obtaining, from Γ_n, the graph H in which only members of X are nonadjacent. Thus, to obtain G from H, at most:

$$\tfrac{1}{2}(n-1)(k+1) - 1 - \tfrac{1}{2}k(k+1) = \tfrac{1}{2}(n-k-1)(k+1) - 1 \tag{3.9}$$

edges are removed from H.

We wish to compute $D(G, X)$. By Lemma 3.6,

$$D(H, X) = 2(n-k-1). \tag{3.10}$$

Now suppose that at some stage in the transformation from H to G, we have obtained a graph K with $\mathscr{E}(H) \supseteq \mathscr{E}(K) \supseteq \mathscr{E}(G)$ and $\mathscr{V}(K) = \mathscr{V}(H) = \mathscr{V}(G)$. Let $L = K - e$ where $e \in \mathscr{E}(K) - \mathscr{E}(G)$. We wish to know the effect, $f(e) = D(L, X) - D(K, X)$, on D, of removing the edge e. Since e is an edge of H, it cannot join two points of X. If neither end-point of e is in X, then $f(e) = 0$ since $S_i(\rho_K, X) = S_i(\rho_L, X)$ for $1 \leqq i \leqq k$. Now suppose that one end-point, y_1, of e is in X and that the other end-point, v, is not in X. Let $y_1, y_2, \cdots, y_j$ be the points of X which are adjacent to v in the graph K. Note that $1 \leqq j \leqq k+1$.

If $1 \leqq i \leqq j$ and $B \subseteq \{y_2, y_3, \cdots, y_j\}$ with $|B| = i - 1$, and $C = B \cup \{y_1\}$, then $|C| = i$ and v is adjacent to every member of C in the graph K but not in the graph L. There are $\binom{j-1}{i-1}$ choices for such a set C. Furthermore, for any other combination of a vertex, t, and a set $A \subseteq X$ with $|A| = i$, t is adjacent to every member of A in the graph L. Thus:

$$S_i(\rho_L, X) - S_i(\rho_K, X) = \begin{cases} -\binom{j-1}{i-1} & \text{for} \quad i \leqq i \leqq j \\ 0 & \text{for} \quad i > j. \end{cases}$$

This gives:

$$f_j = f(e) = D(L, X) - D(K, X)$$

$$= \begin{cases} -\left[\dfrac{1}{k+1} - \sum_{i=1}^{k} (-1)^i \left(\dfrac{k-i+1}{k+1}\right) \binom{k}{i-1}\right] & \text{if} \quad j = k+1 \\[2ex] -\left[\dfrac{1}{k+1} - \sum_{i=1}^{j} (-1)^i \left(\dfrac{k-i+1}{k+1}\right) \binom{j-1}{i=1}\right] & \text{if} \quad 1 \leqq j \leqq k \end{cases}$$

$$= \begin{cases} -\dfrac{1}{k+1}\left[k+1 - \sum_{i=2}^{k} (-1)^i (k-i+1) \binom{k}{i-1}\right] & \text{if} \quad j = k+1 \\[2ex] -\dfrac{1}{k+1}\left[k+1 - \sum_{i=2}^{k} (-1)^i (k-i+1) \binom{j-1}{i-1}\right] & \text{if} \quad 2 \leqq j \leqq k \\ -1 & \text{if} \quad j = 1 \end{cases}$$

$$= \begin{cases} -\dfrac{1}{k+1} & \text{if} \quad 3 \leqq j \leqq k+1 \\[2ex] -\dfrac{2}{k+1} & \text{if} \quad j = 2 \\ -1 & \text{if} \quad j = 1 \end{cases}$$

using (3.4) and (3.5).

Notice that $f_1 \leqq f_2 \leqq \cdots \leqq f_k \leqq f_{k+1} < 0$ and that in order to realize the effect f_j, edges with effects $f_{k+1}, f_k, \cdots, f_{j+1}$ must first be removed. Hence when $(k+1)$ edges are removed, the combined effect is at least:

$$\sum_{i=1}^{k+1} f_i = -2.$$

So if r edges are removed in obtaining G from H,

$$D(G,X)-D(H,X)\geqq -\frac{2r}{k+1}. \tag{3.11}$$

Using (3.9) and (3.10) in (3.11) now gives:

$$D(G,X)\geqq [2(n-k-1)-(n-k-1)+2/(k+1)]>n-k-1. \tag{3.12}$$

But Theorem 2.5 guarantees the existence of a set X as constructed above, and satisfying:

$$D(G,X)\leqq n-\mu(G)\leqq n-k-1.$$

This contradicts (3.12) and completes the proof of the theorem.

COROLLARY 3.13. *For* $n\geqq 4$, $g(n,n-3)=n$.

Proof. The bipartite graph $\Gamma_{1,n-1}$ is a graph with n vertices, $(n-1)$ edges and path-covering number $(n-2)$. Thus $g(n,n-3)\geqq n$. The reverse inequality is given by Theorem 3.7.

To obtain a lower bound for $g(n,k)$, consider the graph $G=\Gamma_{n-k}\cup\bar{\Gamma}_k$; then $\mu(G)=k+1$, while $|\mathscr{V}(G)|=n$ and $|\mathscr{E}(G)|=\frac{1}{2}(n-1)(n-k-1)$. This gives:

PROPOSITION 3.14. *For* $n>k\geqq 1$

$$g(n,k)\geqq\tfrac{1}{2}(n-k)(n-k-1)+1. \tag{3.15}$$

The following proposition gives some results that are easily verified:

PROPOSITION 3.15.

(i) $g(n,n)=0$, $g(n+1,n)=1$, $g(n+2,n)=2$ *for* $n\geqq 1$
(ii) $g(6,2)=7$
(iii) $g(n+1,k+1)\geqq g(n,k)$ *for* $n\geqq k\geqq 1$.

Part (iii) can be seen by letting $G=H\cup\{x\}$ where H is a graph with n vertices, $g(n,k)-1$ edges, and $\mu(H)=k+1$, and x is an isolated vertex with $x\notin|$ ith $x\notin\mathscr{V}(H)$. Then G has $(n+1)$ vertices, $g(n,k)-1$ edges, and $(G)=k+2$.

In the case $k = 1$, the upper bound in (3.8) is seen to be the same as the lower bound in (3.15) and hence equality holds for $g(n, k)$ in both inequalities. However, Corollary 3.13 shows that the upper bound in (3.8) and not the lower bound in (3.15) is achieved in the case $k = n - 3$. Part (ii) of Proposition 3.15 shows a case where the lower bound and not the upper bound is achieved. It is conjectured that for small values of k, $g(n, k)$ is close to the lower bound in (3.15), while for large values of $k, g(n, k)$ is closer to the upper bound in (3.8).

We now turn to another extremal problem. Let v and n be integers with $0 \leqq v \leqq n$. Define:

$$h(n, v) = \mathrm{Min}\{k\colon \text{every graph, } G, \text{ with } |\mathcal{V}(G)| = n \text{ and } \rho(x) \geqq v \text{ for every } x \in \mathcal{V}(G), \text{ has } \mu(G) \leqq k\}.$$

THEOREM 3.16.

$$(n, v) = \begin{cases} 1 & \text{if } \quad v \geqq \dfrac{n}{2} \\ n - 2v & \text{if } \quad v < \dfrac{n}{2}. \end{cases}$$

Proof. The case $v \geqq \frac{n}{2}$ and the upper bound $h(n, v) \leqq n - 2v$ if $v < \frac{n}{2}$ follows from 0. Ore's result (the note to Theorem 2.5). If $v < \frac{n}{2}$, let $K = \Gamma_{v, n-v}$. Then clearly $|\mathcal{V}(K)| = n$ and $\rho(x) \geqq v$ for every $x \in \mathcal{V}(G)$; and in [2] (Theorem 2.2.10) we show that $\mu(K) = n - 2v$. Hence

$$h(n, v) \geqq n - 2v$$

completing the proof of the theorem.

REFERENCES

1. C. St. J. A. Nash-Williams, *Hamilton arcs and circuits, Recent Trends in Graph Theory*, Berlin-Heidelberg-New York, Springer-Verlag, (1971), 197–210.
2. S. Noorvash, *Covering the vertices of a graph by paths, Hamilton cycles and 1–factors, with related tree enumeration problems*, (Ph.D. Thesis, U.C.L.A.) 1972, 8–38.
3. O. Ore, *Arc coverings of graphs*, Ann. Mat. Ser. IV **55** (1961), 315–321.
4. J. Riordan, *An Introduction to Combinatorial Analysis*, New York, Wiley, 1958.

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JORDAN *-HOMOMORPHISMS BETWEEN REDUCED BANACH *-ALGEBRAS

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A number of known results on Jordan *-homomorphism between B^*-algebras are generalized to Jordan *-homomorphisms between reduced Banach *-algebras. However the main results presented here are new even for maps between B^*-algebras. We state these results briefly. For any *-algebra $\mathfrak{A}$, let $\mathfrak{A}_{qU}$ be the set of quasi-unitary elements. Let $\mathfrak{A}$ and $\mathfrak{B}$ be reduced Banach *-algebras ($= A^*$-algebras). Let $\varphi: \mathfrak{A} \to \mathfrak{B}$ be a linear map. Then φ is a Jordan *-homomorphism if and only if $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$. If φ is bijective these conditions are equivalent to φ being a weakly positive isometry with respect to the Gelfand-Naimark norms of $\mathfrak{A}$ and $\mathfrak{B}$.

The main results of this note are contained in Theorems 3 and 4. Theorem 1 is merely a restatement of results in **[11]**, and Theorem 2 contains a generalization to the context of reduced Banach *-algebras of results previously known for B^*-algebras. Several of these results have been recently used by the author to characterize *-homomorphisms **[13]**. Further comments on the results, and their history, will be given when they are stated. First we introduce our terminology and notation. Any terms not explained here are used in the sense defined in C. E. Rickart's book **[17]**.

We use $\mathbf{C}, \mathbf{R}$, and $\mathbf{N}$ to denote the sets of complex numbers, real numbers, and natural numbers respectively. We use λ^* to denote the complex conjugate of $\lambda \in \mathbf{C}$. All algebras have complex scalars.

Any associative algebra $\mathfrak{A}$ can be made into a Jordan algebra by defining a product

$$a \circ b = 2^{-1}(ab + ba) \qquad \forall\ a, b \in \mathfrak{A}.$$

A linear map $\varphi: \mathfrak{A} \to \mathfrak{B}$ is called a Jordan homomorphism if it preserves the Jordan structure of the algebra. Thus a linear map $\varphi: \mathfrak{A} \to \mathfrak{B}$ is a Jordan homomorphism iff

$$\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a) \qquad \forall\ a, b \in \mathfrak{A}.$$

It is easy to check that this condition can be replaced by

$$\varphi(a^2) = \varphi(a)^2 \qquad \forall\ a \in \mathfrak{A}.$$

The terms Jordan algebra and Jordan homomorphism derive from a generalization of the formalism of quantum mechanics due to P. Jordan [6] which was further discussed by P. Jordan, J. von Neumann, and E. Wigner [7]. The term Jordan homomorphism seems to have been used first in two fundamental papers by N. Jacobson and C. E. Rickart [4, 5]. Under other names, Jordan homomorphims had been considered earlier in purely algebraic contexts.

If $\mathfrak{A}$ and $\mathfrak{B}$ are *-algebras, a linear map $\varphi: \mathfrak{A} \to \mathfrak{B}$ is called a *-map if it preserves (i.e., commutes with) the involutions. A Jordan *-homomorphism between *-algebras is simply a Jordan homomorphism which is also a *-map. Jordan *-homomorphisms between B^*-algebras preserve the quantum mechanical structure of the algebras. They have been called C^*-homomorphisms by R. V. Kadison [8] and others.

For any *-algebra $\mathfrak{A}$ the set of hermitian elements is denoted by $\mathfrak{A}_H$. It is trivial to check that a linear *-map $\varphi: \mathfrak{A} \to \mathfrak{B}$ is a Jordan *-homomorphism if and only if it satisfies

$$\varphi(h^2) = \varphi(h)^2 \qquad \forall\ h \in \mathfrak{A}_H.$$

This is the condition we will use.

In any algebra we denote an identity element by 1. A linear map φ between algebras with identity elements is called unital if $\varphi(1) = 1$.

A map $\varphi: \mathfrak{A} \to \mathfrak{B}$ between *-algebras is called weakly positive if it satisfies

$$\varphi(h^2) \in \mathfrak{B}_+ \qquad \forall\ h \in \mathfrak{A}_H.$$

Here $\mathfrak{B}_+$ is the set $\{\Sigma_{j=1}^n b_j^* b_j : b_j \in \mathfrak{B}\}$. One of the important differences between reduced Banach *-algebras and B^*-algebras is the failure of the equality $\{h^2: h \in \mathfrak{B}_H\} = \mathfrak{B}_+$ in the former case. This complicates calculations with Jordan *-homomorphisms. In particular Jordan *-homomorphisms between Banach *-algebras are weakly positive but not usually positive.

One of the fundamental properties of Banach *-algebras (we do not require the involution to be continuous) is that they have a universal *-representation which includes (in a certain weak sense) all other *-representations. The norm carried back from this *-representation is the largest submultiplicative pseudo-norm on the Banach *-algebra which satisfies the B^*-condition ($\|a^*a\| = \|a\|^2$). It is called the Gelfand-Naimark pseudo-norm, and is denoted by γ. The Gelfand-Naimark pseudo-norm on a *-algebra $\mathfrak{A}$ can also be described by

$$\gamma(a) = \sup\{\|T_a\|: T \text{ is is *-representation of } \mathfrak{A}\}.$$

Hence it is clear that the *-ideal of elements in $\mathfrak{A}$ whch are represented by zero in all *-representations of $\mathfrak{A}$ (which is called the reducing ideal, and is denoted by $\mathfrak{A}_R$) is given by

$$\mathfrak{A}_R = \{a\colon \gamma(a) = 0\}.$$

If $\mathfrak{A}_R = \{0\}$ the *-algebra $\mathfrak{A}$ is said to be reduced. Clearly γ is a norm rather than just a pseudo-norm if and only if the *-algebra is reduced. We use the terms "γ-isometry", "γ-contraction" and "γ unit ball" to abbreviate "isometry relative to the Gelfand-Naimark pseudo-norms", etc. A Banach *-algebra is a B^*-algebra if and only if its complete norm equals γ.

For any *-algebra $\mathfrak{A}$ a state is a positive linear functional ω such that there is a *-representation T of $\mathfrak{A}$ and a topologically cyclic unit vector x in the Hilbert space on which T acts satisfying

$$\omega(a) = (T_a x, x) \qquad \forall\ a \in \mathfrak{A}.$$

The Gelfand-Naimark pseudo-norm can be described in terms of states:

$$\gamma(a) = \sup\{\omega(a^*a)^{\frac{1}{2}}\colon \omega \text{ is a state of } \mathfrak{A}\}.$$

Conversely in a Banach *-algebra $\mathfrak{A}$ with an identity element states can be described in terms of the Gelfand-Naimark pseudo-norm:

$$\{\text{States on } \mathfrak{A}\} = \{\text{linear functionals } \omega \text{ on } \mathfrak{A} \text{ such that}$$
$$\omega(1) = 1 = \|\omega\|_\gamma\}$$

where $\|\omega\|_\gamma = \sup\{|\omega(a)|\colon$ a belongs to the γ-unit ball$\}$. For a reduced *-algebra $\mathfrak{A}$, there are enough states to separate points, and in particular an element $h \in \mathfrak{A}$ is hermitian if and only if $\omega(h)$ is real for each state on $\mathfrak{A}$.

An element u of the γ-unit ball of $\mathfrak{A}$ is called a vertex if the set of linear functionals ω on $\mathfrak{A}$ such that $\omega(u) = 1 = \|\omega\|_\gamma$ separates points of $\mathfrak{A}$. In the course of proving Theorem 2 we will extend a result of H. F. Bahnenblust and S. Karlin [1] to show that an element in a reduced Banach *-algebra $\mathfrak{A}$ is a vertex of the γ-unit ball if and only if it is unitary. We denote the set of unitary elements in a *-algebra $\mathfrak{A}$ by $\mathfrak{A}_U = \{u \in \mathfrak{A}\colon u^*u = uu^* = 1\}$. The set $\{v \in \mathfrak{A}\colon v^*v = vv^* = v + v^*\}$ of quasi-unitary elements is denoted by $\mathfrak{A}_{qU}$. For a *-algebra with an identity element the involutive map $v \to 1 - v$ carries the set of quasi-unitary elements onto the set of unitary elements and *visa-versa*. The set of quasi-unitary elements is a group under quasi-multiplication and the involution is the (quasi-) inverse map in this group. The next theorem, which is one of our major tools, explains the importance of quasi-unitary elements.

THEOREM 1. *Let $\mathfrak{A}$ be a Banach *-algebra. For each $a \in \mathfrak{A}$,*

$$\gamma(a) = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : a = \sum_{j=1}^{n} \lambda_j v_j,\ 0 = \sum_{j=1}^{n} \lambda_j \text{ where } n \in \mathbf{N},\right.$$
$$\left.\lambda_j \in \mathbf{C},\ v_j \in \mathfrak{A}_{qU}\right\}.$$

If $\mathfrak{A}$ has an identity element then for each $a \in \mathfrak{A}$,

$$\gamma(a) = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : a = \sum_{j=1}^{n} \lambda_j u_j \text{ where } n \in \mathbf{N},\ \lambda_j \in \mathbf{C},\right.$$
$$\left. u_j \in \mathfrak{A}_U\right\}.$$

*Hence if $\mathfrak{A}$ and $\mathfrak{B}$ are Banach *-algebras and $\varphi: \mathfrak{A} \to \mathfrak{B}$ is a linear map satisying either $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$ or* (when $\mathfrak{A}$ has an identity element) *$\varphi(\mathfrak{A}_U) \subseteq \mathfrak{B}_U$ then φ is a γ-contraction.*

Proof. See **[11]**, especially the remark at the bottom of page 63.

We remark that if $\mathfrak{A}$ and $\mathfrak{B}$ are Banach *-algebras, $\mathfrak{B}$ is reduced, and $\varphi: \mathfrak{A} \to \mathfrak{B}$ is a γ-contraction then φ is continuous with respect to the complete norms of $\mathfrak{A}$ and $\mathfrak{B}$. This follows from a standard application of the closed graph theorem since γ is always continuous with respect to the complete norm.

Next we extend some results known previously for B^*-algebras. In applying condition (b) of this theorem the following remark is sometimes useful. If $\varphi: \mathfrak{A} \to \mathfrak{B}$ is a Jordan homomorphism, $\mathfrak{A}$ has an identity element, and $\mathfrak{B}$ is a topological algebra, which is the closure of the algebra generated by $\varphi(\mathfrak{A})$, then φ is unital **[14,** 0.10.3]. It is easy to prove, starting from (b), that $\operatorname{Ker}(\psi)$ is a closed *-ideal [**14,** 0.10.8]. This is also an immediate consequence of Theorem 3(c) below.

THEOREM 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be reduced Banach *-algebras with identity elements. Let $\psi: \mathfrak{A} \to \mathfrak{B}$ be a linear map. Then the following are equivalent.*

(a) $\psi(\mathfrak{A}_U) \subseteq \mathfrak{B}_U$.

(b) *There is a unitary element* $u \in \mathfrak{B}$ *and a unital Jordan *-homomorphism* $\varphi: \mathfrak{A} \to \mathfrak{B}$ *satisfying*

$$\psi(a) = u\varphi(a) \qquad \forall\ a \in \mathfrak{A}.$$

If ψ *is a bijection these conditions are also equivalent to*:

(c) ψ *is a* γ*-isometry.*

REMARK. We could prove this theorem by extending ψ to a map between B^*-algebras and then quoting known theorems. Instead we will indicate how to modify and piece together various known proofs to cover the present situation. In the process we give a proof for the B^*-algebra case which we believe is easier than any proof which has previously been written down in one place. We begin by modifying a proof due to A. L. T. Paterson **[15]** to prove (a) implies (b). In the B^*-algebra case this result is due to B. Russo and H. A. Dye **[18]**. The implication (b) $\Rightarrow$ (a) is easy algebra which is essentially an observation of N. Jacobson and C. E. Rickart **[4]**. When ψ is a bijection the implication ((a) and (b)) $\Rightarrow$ (c) follows from Theorem 1. In the B^*-algebra case the implication is due to B. Russo and H. A. Dye **[18]** and now has an easy proof due to L. A. Harris **[3]**. We use a result of P. Miles **[10]** and modify an argument due to H. F. Bohnenblust and S. Karlin **[1]** to show (c) $\Rightarrow$ (a). In the B^*-algebra case the implication (c) $\Rightarrow$ (a) is due to R. V. Kadison **[8]**.

Proof. Suppose ψ satisfies (a). Denote $\psi(1)$ by u and define $\varphi: \mathfrak{A} \to \mathfrak{B}$ by $\varphi(a) = u^*\psi(a)$ for each $a \in \mathfrak{A}$. Then it is enough to show that φ is a Jordan *-homomorphism.

First we show that φ is a linear *-map. It is obviously linear and it is a γ-contraction by Theorem 1. Let ω be an arbitrary state of $\mathfrak{B}$. Then $\omega(1) = 1$ and $|\omega(b)| \leqq \gamma_{\mathfrak{B}}(b)$ holds for all $b \in \mathfrak{B}$. Thus $\varphi^*(\omega)(1) = \omega\varphi(1) = \omega(1) = 1$ and $|\varphi^*(\omega)(a)| = |\omega(\varphi(a))| \leqq \gamma_{\mathfrak{B}}(\varphi(a)) \leqq \gamma_{\mathfrak{A}}(a)$ hold for all $a \in \mathfrak{A}$. Hence $\varphi^*(\omega)$ is a state of $\mathfrak{A}$. Therefore $\omega(\varphi(h)) = \varphi^*(\omega)(h)$ is real for all $h \in \mathfrak{A}_H$. Since $\mathfrak{B}$ is reduced and ω was an arbitrary state, this implies $\varphi(h)$ is hermitian. Thus φ is a *-map.

Next we show that $\varphi(h^2) = \varphi(h)^2$ for all $h \in \mathfrak{A}_H$. The involution in $\mathfrak{A}$ is norm continous since $\mathfrak{A}$ is reduced. Hence e^{ith} is a unitary element of $\mathfrak{A}$ for each $t \in \mathbf{R}$, and $h \in \mathfrak{A}_H$. Hence $\varphi(e^{ith})$ is unitary so $\varphi(e^{ith})\varphi(e^{-ith}) = \varphi(e^{ith})\varphi((e^{ith})^*) = \varphi(e^{ith})\varphi(e^{ith})^* = 1$. Expanding the first few terms of this identity shows

$$\|1-[1+it\varphi(h)-2^{-1}t^2\varphi(h^2)][1-it\varphi(h)-2^{-1}t^2\varphi(h^2)]\| = O(t^3)$$

as t approaches zero. We conclude

$$t^2\|\varphi(h)^2-\varphi(h^2)\| = O(t^3)$$

which implies $\varphi(h^2)=\varphi(h)^2$. This implies φ is a Jordan *-homomorphism. Hence (a) implies (b).

Now suppose (b) holds. In order to prove (a) it is obviously sufficient to show $\varphi(\mathfrak{A}_U)\subseteq\mathfrak{B}_U$ holds. For any unitary element $w\in\mathfrak{A}_U$ let $h,k\in\mathfrak{A}_H$ satisfy $w=h+ik$. Then h and k commute and $h^2+k^2=1$. Hence $\varphi(h)^2+\varphi(k)^2=\varphi(h^2+k^2)=\varphi(1)=1$. Thus $\varphi(w)=\varphi(h)+i\varphi(k)$ is unitary if $\varphi(h)$ and $\varphi(k)$ commute. However a calculation shows $0=\varphi((hk-kh)^2)=(\varphi(h)\varphi(k)-\varphi(k)\varphi(h))^2$ (cf. **[4]**). Since a skew hermitian element in a reduced *-algebra (such as $\mathfrak{B}$) is zero if its square is zero, $\varphi(h)$ and $\varphi(k)$ commute. Hence (b) implies (a).

Now suppose ψ is a bijection. If (b) holds, the map φ is a bijection and hence a Jordan *-isomorphism. Thus both ψ and ψ^{-1} satisfy (a) so $\psi(\mathfrak{A}_U)=\mathfrak{B}_U$. Hence by Theorem 1 ψ is a γ-isometry. Therefore (b) implies (c).

Assume that ψ is a γ-isometry. We will show that an element u in a reduced Banach *-algebra is a vertex of the γ-unit ball if and only if it is unitary. Since an isometry obviously preserves vertices it will follow that $\psi(\mathfrak{A}_U)=\mathfrak{B}_U$.

P. Miles **[10]**, generalizing a result of R. V. Kadison **[8]**, shows that for any *-algebra $\mathfrak{A}$ and any (not necessarily complete) B^*-norm γ on $\mathfrak{A}$, an element $v\in\mathfrak{A}$ is an extreme point of the γ-unit ball if and only if it satisfies

$$(1-v^*v)\mathfrak{A}(1-vv^*)=\{0\}.$$

If v satisfies this condition it is a partial isometry since $(v-vv^*v)^*(v-vv^*v)=(1-v^*v)v^*(1-vv^*)v=0$ holds. Thus any γ-vertex is at least a partial isometry.

Choose a faithful, γ-isometric *-representation T of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$. H. F. Bohnenblust and S. Karlin [**1**, Theorem 11] show that for every partial isometry $v\in\mathfrak{A}$ the set of linear functionals ω on $\mathfrak{A}$ satisfying $\omega(v)=1$ and $|\omega(a)|\leqq\gamma(a)$ for all $a\in\mathfrak{A}$ is the weak* closed convex hull of the set of linear maps of the form

$$a\to(T_ax,\ T_vx)$$

where x belongs to $\mathfrak{H}$ and $\|x\| = \|T_v x\| = 1$ holds. However all these linear functionals vanish on $1 - vv^*$. Thus if v is a γ-vertex then $vv^* = 1$. However if v is a γ-vertex then v^* is also a γ-vertex so $v^*v = 1$ also holds. Thus v is unitary. Hence (c) implies (a).

In the next theorem, condition (a) has not previously been considered. However it is natural in a number of contexts **[14]**. The equivalence of (b) and (c) is essentially due to R. V. Kadison **[9]** when the *-algebras are B^*-algebras. When the *-representation T of condition (c) is faithful, the condition says that φ is essentially the sum of a *-homomorphism and a *-anti-homomorphism.

THEOREM 3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be reduced Banach *-algebras. Then the following are equivalent for a linear map $\varphi: \mathfrak{A} \to \mathfrak{B}$.*

(a) $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$.

(b) *φ is a Jordan *-homomorphism.*

(c) *The von Neumann algebra $\mathfrak{B}'$ generated by any *-representation of the closed *-subalgebra of $\mathfrak{B}$ generated by $\varphi(\mathfrak{A})$ contains a central projection e satisfying*

$$\left.\begin{aligned} T_{\varphi(ab)}\, e &= T_{\varphi(a)\varphi(b)}\, e \\ T_{\varphi(ab)}(1-e) &= T_{\varphi(b)\varphi(a)}(1-e) \end{aligned}\right\} \quad \forall\, a, b \in \mathfrak{A}.$$

When these conditions hold φ is a weakly positive γ-contraction.

Proof. Assume (a) holds. Whether or not $\mathfrak{A}$ already has an identity element we adjoin a new one. That is, we construct the Banach *-algebra $\mathfrak{A}^1$ with $\mathbf{C} \oplus \mathfrak{A}$ as linear space, $(\lambda \oplus a)^* = \lambda^* \oplus a^*$ as involution, $(\lambda \oplus a)(\mu \oplus b) = \lambda\mu \oplus \lambda b + \mu a + ab$ as product and $\|\lambda \oplus a\| = |\lambda| + \|a\|$ as norm. Then $\mathfrak{A}^1$ is still reduced since

$$T^1_{\lambda \oplus a} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + T_a \end{pmatrix}$$

is a faithful *-representation of $\mathfrak{A}^1$ on $\mathfrak{H} \oplus \mathfrak{H}$ when T is a faithful *-representation of $\mathfrak{A}$ on $\mathfrak{H}$. Construct $\mathfrak{B}^1$ similarly. Define $\varphi^1: \mathfrak{A}^1 \to \mathfrak{B}^1$ by $\varphi^1(\lambda \oplus a) = \lambda \oplus \varphi(a)$. It is easy to check that an element in $\mathfrak{A}^1$ is unitary if and only if it has the form $\zeta(1 \oplus (-v))$ for some quasi-unitary element v in $\mathfrak{A}$ and some complex number ζ of norm one. Since a similar statement holds for $\mathfrak{B}$, $\varphi^1(\mathfrak{A}^1_U) \subseteq \mathfrak{B}^1_U$ holds. Hence Theorem 2 shows φ^1 (which is obviously unital) is a

Jordan *-homomorphism. Its restriction φ is also a Jordan *-homomorphism. Thus (a) implies (b).

A trivial modification of the proof that (b) implies (a) in Theorem 2 shows that (b) implies (a) here also. (Notice that this proof does not use the hypothesis that $\mathfrak{A}$ is reduced.)

When T is chosen faithful, condition (c) implies (b) in a trivial way. Now assume (a) and hence (b) hold. Replace $\mathfrak{B}$ by the closed subalgebra generated by $\varphi(\mathfrak{A})$. (It is a *-subalgebra.) Extend φ to $\varphi^1: \mathfrak{A}^1 \to \mathfrak{B}^1$ as in the proof that (a) implies (b). Then $\varphi^1(\mathfrak{A}^1_U) \subseteq \mathfrak{B}^1_U$ so φ^1 and hence φ are γ-contractions by Theorem 1 and Jordan *-homomorphisms by Theorem 2. Let $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ be the B^*-enveloping algebras of $\mathfrak{A}^1$ and $\mathfrak{B}^1$ respectively. These are simply the completions of the incomplete normed algebras $(\mathfrak{A}^1, \gamma_{\mathfrak{A}^1})$ and $(\mathfrak{B}^1, \gamma_{\mathfrak{B}^1})$. The γ-contraction φ can be extended by continuity to $\bar{\varphi}: \bar{\mathfrak{A}} \to \bar{\mathfrak{B}}$. Obviously $\bar{\varphi}$ is still a Jordan *-homomorphism and a contraction. Thus we extend $\bar{\varphi}$ once more to its double adjoint map $\bar{\varphi}^{**}: \bar{\mathfrak{A}}^{**} \to \bar{\mathfrak{B}}^{**}$ which is just the extension of $\bar{\varphi}$ by continuity in the $\bar{\mathfrak{A}}^*$-topology. The double dual spaces $\bar{\mathfrak{A}}^{**}$ and $\bar{\mathfrak{B}}^{**}$ are the W^*-enveloping algebras of $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ under *-algebra operations inherited from their interpretation as the weak closures of the universal representations of $\mathfrak{A}$ and $\mathfrak{B}$ [**2**, §12; or **14**, §4.5]. A Lemma of R. V. Kadison [**9**, Lemma 2.4] shows that $\bar{\varphi}^{**}$ is again a Jordan *-homomorphism. Arguments of R. V. Kadison [**8**, Theorem 10; 9, Theorem 2.6] based on a fundamental result of N. Jacobson and C. E. Rickart [**4**] show that $\bar{\varphi}^{**}$, and hence φ, have the asserted form with $\mathfrak{B}' = \bar{\mathfrak{B}}^{**}$. (In [**8**] and [**9**] Kadison actually assumes φ is surjective, but this is not necessary for the proof.) Now if T is any *-representation of $\bar{\mathfrak{A}}$ then there is an extension $\bar{T}$ of T to be a *-representation of $\bar{\mathfrak{A}}^{**}$ on the same Hilbert space so that the image of $\bar{\mathfrak{A}}^{**}$ under $\bar{T}$ is the von Neumann algebra generated by $T_{\bar{\mathfrak{A}}}$, and also generated by $T_{\mathfrak{A}}$. Since every *-representation of $\mathfrak{A}$ is the restriction of a *-representation T of $\bar{\mathfrak{A}}$, this proves (c).

We have already shown that φ is a γ-contraction. A Jordan *-homomorphism such as φ satisfies $\varphi(h^2) = \varphi(h)^2 \in \mathfrak{B}_+$ so it is obviously weakly positive.

For B^*-algebras the equivalence of conditions (b) and (c) in the next theorem is implicit in [**8**].

THEOREM 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach *-algebras with $\mathfrak{A}$ or $\mathfrak{B}$ reduced. Let $\varphi: \mathfrak{A} \to \mathfrak{B}$ be a linear bijection. Then the following are equivalent.*

(a) $\varphi(\mathfrak{A}_{qU}) = \mathfrak{B}_{qU}$.
(b) *φ is a weakly positive γ-isometry.*
(c) *φ is a Jordan *-isomorphism.*

If $\mathfrak{A}$ and $\mathfrak{B}$ have identity elements then these conditions are also equivalent to:
(d) *$\varphi(\mathfrak{A}_U) = \mathfrak{B}_U$ and φ is weakly positive.*
(e) *φ is a unital Jordan *-isomorphism.*

Proof. Suppose φ satisfies (a). Theorem 1 shows that both φ and φ^{-1} are γ-contractions. Thus φ is a γ-isometry. Hence both $\mathfrak{A}$ and $\mathfrak{B}$ are reduced. Now Theorem 3 applies and shows that φ (or φ^{-1}) is a bijective Jordan *-homomorphism and hence a Jordan *-isomorphism. Therefore φ is weakly positive. Thus (a) implies (b) and (c).

Suppose φ satisfies (c). If $\mathfrak{B}$ is reduced, the proof that (b) implies (a) in Theorem 3 shows that $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$ holds. In this case φ is a γ-contraction as well as a bijection so $a \in \mathfrak{A}_R$ implies $\gamma_{\mathfrak{B}}(\varphi(a)) \leqq \gamma_{\mathfrak{A}}(a) = 0$ which in turn implies $\varphi(a)$, and hence a, are zero. Thus $\mathfrak{A}$ is also reduced. By symmetry it follows also that $\mathfrak{B}$ is reduced if $\mathfrak{A}$ is reduced. Hence Theorem 3 shows that $\varphi(\mathfrak{A}_{qU}) \subseteq \mathfrak{B}_{qU}$ and $\varphi^{-1}(\mathfrak{B}_{qU}) \subseteq \mathfrak{A}_{qU}$ both hold. This verifies condition (a).

Suppose (b) holds. Extend φ by continuity to an isometry $\bar{\varphi}: \bar{\mathfrak{A}} \to \bar{\mathfrak{B}}$ where $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ are the B^*-enveloping algebras of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. The set of positive elements in a B^*-algebra such as $\bar{\mathfrak{B}}$ is closed. Hence by continuity $\bar{\varphi}(h^2) \in \bar{\mathfrak{B}}_+$ for any $h \in \bar{\mathfrak{A}}_H$. Since $\bar{\mathfrak{A}}$ is a B^*-algebra $\bar{\varphi}$ is positive.

We now extend $\bar{\varphi}$ again by taking its double dual map $\bar{\varphi}^{**}: \bar{\mathfrak{A}}^{**} \to \bar{\mathfrak{B}}^{**}$. The double dual space $\bar{\mathfrak{A}}^{**}(\bar{\mathfrak{B}}^{**})$ is natural identified with the closure in the weak operator topology of the universal *-representation of $\bar{\mathfrak{A}}(\bar{\mathfrak{B}})$. From this interpretation it is clear that $\bar{\varphi}^{**}$ is again a positive map. However Theorem 2 shows that $\bar{\varphi}^{**}(1)$ is unitary. It is also positive since $\bar{\varphi}^{**}$ is positive. Hence $\bar{\varphi}^{**}(1)$ is 1. Then Theorem 2 shows that $\bar{\varphi}^{**}$, and hence its restriction φ, are Jordan *-homomorphisms. Thus (b) implies (c).

Theorem 2 shows the equivalence of (b) and (d). If (d) holds then Theorem 2 shows $\varphi(1)$ is a positive unitary element and hence $\varphi(1) = 1$. Thus (d) and (e) are equivalent.

The reader of this paper will be interested in two recent papers by A. L. T. Paterson and A. M. Sinclair **[16]** and by K. Ylinen **[19]** which deal with Jordan *-homomorphisms between B^*-algebras without identity elements. All of their results can be reformulated as theorems about reduced Banach *-algebras. Except for those already given, the reformulations which the author has been able to prove are unpleasantly technical. It appears to be unknown whether the statement of Theorem 1 in [16] remains valid when "C^*-algebra" is simply replaced by "reduced Banach *-algebra".

REFERENCES

1. H. F. Bohnenblust and S. Karlin, *Geometrical properties of the unit sphere of Banach algebras*, Annals of Math., **62** (1955), 217–229.
2. J. Dixmier, *Les C^*-algèbres et leurs représentations*, Gauthiers-Villars, Paris, 1969.
3. L. A. Harris, *Banach algebras with involution and Mobius transformations*, J. Functional Analysis, **11** (1972), 1–16.
4. N. Jacobson and C. E. Rickart, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc., **69** (1950), 479–502.
5. ———, *Homomorphisms of Jordan rings of self-adjoint elements*, Trans. Amer. Math. Soc., **72** (1952), 310–322.
6. P. Jordan, *Über eine Klasse nichtassociativer hypercomplexer Algebren*, Nachr. Ges. Wiss. Götingen (1932), 569–575.
7. P. Jordan, J. von Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. of Math., (2) **35** (1934), 29–64.
8. R. V. Kadison, *Isometries of operator algebras*, Ann. of Math., **54** (1951), 325–338.
9. ———, *Transformation of states in operator theory and dynamics*, Topology 3 Suppl., **2** (1965), 177–198.
10. P. Miles, *B^*-algebra unit ball extremal points*, Pacific J. Math., **14** (1964), 627–637.
11. T. W. Palmer, *The Gelfand-Naimark pseudo-norm on Banach *-algebras*, J. London Math. Soc., (2) **2** (1970), 89–96.
12. ———, *-Representations of U*-algebras, Indiana Univ. Math. J., **20** (1971), 929–933.
13. ———, *Characterization of *-homomorphisms and expectations*, Proc. Amer. Math. Soc., **46** (1974), 265–272.
14. ———, *The General Theory of *-Algebras*, monograph in preparation.
15. A. L. T. Paterson, *Isometries between B^*-algebras*, Proc. Amer. Math. Soc., **22** (1969), 570–572.
16. A. L. T. Paterson and A. M. Sinclair, *Characterizations of isometries between C^*-algebras*, J. London Math. Soc., (2) **5** (1972), 755–761.
17. C. E. Rickart, *General Theory of Banach Algebras*, van Nostrand, Princeton, 1960.
18. B. Russo and H. A. Dye, *A note on unitary operators in C^*-algebras*, Duke Math. J., **33** (1966), 413–416.
19. K. Ylinen, *Vector space isomorphisms of C^*-algebras*, Studia Math., **96** (1973), 31–34.

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ON THE SEMISIMPLICITY OF GROUP RINGS OF SOME LOCALLY FINITE GROUPS

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We consider the semisimplicity problem for group rings of some locally finite groups. In particular we study locally solvable groups and linear groups in the mixed characteristic case. While the results here are by no means definitive, we hope the techniques constitute a first step in the complete solution.

Our notation follows that of [2] and [4] and all groups considered are assumed to be locally finite unless otherwise stated. If K is a field of characteristic 0 then in this case $K[G]$ is trivially seen to be semisimple. Thus we assume throughout that $p > 0$ is a fixed prime and that K is a fixed field of characteristic p.

1. Group ring lemmas.

The following few results are basic for handling nil ideals in group rings.

LEMMA 1.1. *Let*

$$\alpha = 1 + \sum a_i x_i \in JK[G]$$

with $x_i \in G$, $x_i \neq 1$ and let $x \in G$. Then there exists n, i such that x^{p^n} is conjugate to $(x_i x)^{p^n}$ in G. In particular if σ is a set of primes and if x is a σ-element then $x_i x$ is a $\sigma \cup \{p\}$-element.

Proof. We have $\alpha x \in JK[G]$ so αx is nilpotent and hence $(\alpha x)^{p^n} = 0$ for some n. Thus by Lemma 3.4 of [2]

$$0 = (\alpha x)^{p^n} = x^{p^n} + \sum a_i^{p^n} (x_i x)^{p^n} + \beta$$

with $\beta \in [K[G], K[G]]$, the commutator subspace. Since the sum of the coefficients in β over any conjugacy class is zero it then follows that the x^{p^n} term must be partially cancelled by some conjugate of $(x_i x)^{p^n}$ for some i. Hence x^{p^n} is conjugate to $(x_i x)^{p^n}$ and the result follows.

LEMMA 1.2. *Let P be a normal p-subgroup of G, let $\pi_P\colon K[G] \to K[P]$ denote the natural projection and suppose that*

$$\alpha = 1 + \sum a_i x_i \in JK[G]$$

with $x_i \in G$, $x_i \neq 1$ satisfies $\pi_P(\alpha) \notin JK[P]$. If $x \in G$ then there exists n, i such that $x_i \notin P$ and x^{p^n} is conjugate modulo P to $(x_ix)^{p^n}$. In particular if σ is a set of primes and if x is a σ-element then x_ix is a $\sigma \cup \{p\}$-element.

Proof. Let $-: K[G] \to K[G/P]$ be the natural homomorphism and observe that the kernel of this map is precisely $JK[P] \cdot K[G]$ since P is a p-group. Then $\pi_P(\alpha)$ is by assumption a nonzero scalar, say b, and

$$b^{-1}\bar{\alpha} = \bar{1} + \sum{}' (b^{-1}a_i)\bar{x}_i \in JK[\bar{G}]$$

where the sum Σ' is over all $x_i \notin P$. Thus Lemma 1.1 applied to the group $\bar{G}$ implies that for some n, i we have $\bar{x}^{p^n}$ conjugate in $\bar{G}$ to $\overline{x_ix}^{p^n}$. Since P is a p-group this clearly yields the result.

LEMMA 1.3. *Let $G = NH$ be finite with $N \lhd G$ and $H \cap N = \langle 1 \rangle$. If $JK[G] \cap K[H] \neq 0$ then every p'-conjugacy class of N is normalized by an element of H of order p.*

Proof. By assumption we may choose

$$\alpha = 1 + \sum a_ix_i \in JK[G]$$

with $x_i \in H$, $x_i \neq 1$. If $x \in N$ is a p'-element then by Lemma 1.1 there exists n, i with x^{p^n} conjugate to $(x_ix)^{p^n}$. If $g \in G$ with $g^{-1}(x^{p^n})g = (x_ix)^{p^n}$ then we see that x^{p^n} is centralized by $g(x_ix)g^{-1}$. Hence since x is a p'-element, $\langle x \rangle = \langle x^{p^n} \rangle$ so x is centralized by $g(x_ix)g^{-1}$.

Write $g(x_ix)g^{-1} = yh$ with $y \in N$, $h \in H$. Then since $N \lhd G$ we have modulo N

$$h^{p^n} \equiv (yh)^{p^n} = g(x_ix)^{p^n}g^{-1} = x^{p^n} \equiv 1$$

so $h^{p^n} \in H \cap N = \langle 1 \rangle$ and h is a p-element of H. Furthermore $h \neq 1$ since $yh = g(x_ix)g^{-1} \notin N$. Finally $x^{yh} = x$ shows that h normalizes the N-conjugacy class of x and the lemma is proved.

LEMMA 1.4. *Let G have two finite subgroups N and H. Suppose $N_0 \lhd N$ with N/N_0 an abelian p'-group and suppose that H normalizes both N and N_0. If $H \cap N = \langle 1 \rangle$ and $JK[G] \cap K[H] \neq 0$ then*

$$N/N_0 = \bigcup_h \mathbf{C}_{N/N_0}(h)$$

where h runs through all elements of H of order p.

Proof. By Lemma 16.9 of [2] we may assume that $G = NH$. If $\bar{x} \in N/N_0$ then since N/N_0 is a p'-group there exists $x \in N$, a p'-element, with $\bar{x} = xN_0/N_0$. Now by the preceding lemma there exists $h \in H$ of order p which normalizes the N-conjugacy class of x and hence the N/N_0-conjugacy class of $\bar{x}$. Finally since N/N_0 is abelian, h centralizes $\bar{x}$.

The following is a partial converse.

LEMMA 1.5. *Let $G = NH$ be finite with $N \lhd G$. Suppose that $N = \bigcup_h \mathbf{C}_N(h)$ where h runs through all elements of H of order p. Then $JK[G] \cap K[H] \neq 0$.*

Proof. Set $\alpha = \hat{H} = \Sigma_{h \in H} h$. We show that $\alpha \in JK[G]$ and in fact we show that $K[G]\alpha$ is a left ideal of square zero. Since $h\alpha = \alpha$ for $h \in H$, this ideal has as a spanning set elements of the form $x\alpha$ with $x \in N$ and it suffices to show that for all such $x, \alpha x \alpha = 0$.

Given $x \in N$ by assumption there exists $y \in H$ of order p which centralizes it. If $Y = \langle y \rangle$ then $\alpha = \hat{H} = \hat{Y}\beta$ where β is a sum of right coset representatives for Y in H. Since x and y commute and $|Y| = p$ we then have

$$\begin{aligned} \alpha x \alpha &= \alpha x \hat{Y} \beta = \alpha \hat{Y} \cdot x\beta \\ &= |Y| \alpha \cdot x\beta = 0 \end{aligned}$$

and the result follows.

In locally finite groups the concept of locally finite index is trivial but the following does seem to be of interest. Let N be a subgroup of G. We say that N is *almost normal* in G if for every finite subgroup H of G we have $[\langle N, H\rangle : N] < \infty$. Clearly every normal subgroup of G is almost normal and indeed we have

LEMMA 1.6. *Let N be a subgroup of G. Then N is almost normal in G if and only if every finite subgroup H of G normalizes some normal subgroup of N of finite index.*

Proof. Let H be a finite subgroup of G. If N is almost normal in G then $[\langle N, H\rangle : N] < \infty$ and both H and N normalize the core of N in $\langle N, H\rangle$.

Conversely suppose H normalizes N_0 with $N_0 \lhd N$ and of finite index. Then $N_0 \lhd \langle N, H\rangle$ and $\langle N, H\rangle / N_0$ is a locally finite group generated by the finite groups N/N_0 and N_0H/N_0. Thus $[\langle N, H\rangle : N] \leqq [\langle N, H\rangle : N_0] < \infty$.

Recall that if H is a subgroup of G then

$$\mathbf{D}_G(H) = \{x \in G \mid [H : \mathbf{C}_H(x)] < \infty\}$$

is the almost centralizer of H in G. Thus in particular $\mathbf{D}_G(G) = \Delta(G)$ in the f.c. subgroup of G.

LEMMA 1.7. *Let N be an almost normal subgroup of G. Then $D = \mathbf{D}_G(N)$ is normal in G. Furthermore if $JK[N]$ is nilpotent then D carries the radical of G, that is*

$$JK[G] = JK[D] \cdot K[G].$$

Proof. Let H be an finite subgroup of G. Then by assumption N has finite index in $M = \langle N, H\rangle$. Thus clearly

$$D \cap M = \mathbf{D}_M(N) = \Delta(M) \lhd M$$

and it follows easily that $D \lhd G$.

Now suppose further that $JK[N]$ is nilpotent. Since $D \lhd G$ and G is locally finite we have

$$\pi_D(JK[G]) \cdot K[G] \supseteq JK[G] \supseteq JK[D] \cdot K[G]$$

where $\pi_D : K[G] \to K[D]$ is the natural projection. Thus it suffices to show that the ideal $\pi_D(JK[G])$ of $K[D]$ is nil. Let $\alpha \in JK[G]$ and take $H = \langle \operatorname{Supp} \alpha \rangle$ in the above. Then $\alpha \in JK[G] \cap K[M] \subseteq JK[M]$ by Lemma 16.9 of **[2]**. Also $[M : N] < \infty$ and $JK[N]$ is nilpotent so $JK[M]$ is nilpotent by Lemma 16.8 of **[2]**. Hence Theorem 20.2 of **[2]** yields $JK[M] = JK[\Delta(M)] \cdot K[M]$ so $\pi_{\Delta(M)}(\alpha)$ is nilpotent. Finally $\Delta(M) = D \cap M$ so $\pi_D(\alpha) = \pi_{\Delta(M)}(\alpha)$ is nilpotent and the result follows.

We remark that not every subgroup of a locally finite group is almost normal. For example let N be a infinite locally finite group and let $H \neq \langle 1\rangle$ be finite. Then $G = H \wr N$ is locally finite but $[\langle H, N\rangle : N] = [G : N] = \infty$.

2. Locally solvable groups. The next result is a key lemma in the study of Sylow intersections in solvable groups (see [1], for example).

LEMMA 2.1. *Let P be a finite p-group which acts faithfully on a finite abelian p'-group Q. If either P is abelian or both $|P|$ and $|Q|$ are odd, then there exists $x \in Q$ with $\mathbf{C}_P(x) = \langle 1 \rangle$.*

Proof. We proceed by induction on $|Q|$. Suppose $Q = Q_1 \times Q_2$ and each factor is nontrivial and P-invariant. Then there exist $x_i \in Q_i$ with $\mathbf{C}_P(x_i) = \mathbf{C}_P(Q_i)$ so if $x = x_1x_2$ then $\mathbf{C}_P(x) = \mathbf{C}_P(Q_1) \cap \mathbf{C}_P(Q_2) = \langle 1 \rangle$. Thus we may assume that Q is indecomposable as a P-module and hence Q is a q-group for some $q \neq p$. Also P acts faithfully on $\Omega_1(Q)$ so we may take Q to be elementary abelian and then P acts irreducibly on Q. If P is abelian then by Schur's lemma P acts semiregularly on Q. Hence for all $x \in Q - \{1\}$, $\mathbf{C}_P(x) = \langle 1 \rangle$.

We now assume that both $|P|$ and $|Q|$ are odd and prove that Q contains at least two orbits under the action of P of elements x with $\mathbf{C}_P(x) = \langle 1 \rangle$. First if P is cyclic then P acts semiregularly on $Q^{\#} = Q - \{1\}$. The number of such orbits is then $(|Q| - 1)/|P|$, a nonzero even number since both $|P|$ and $|Q| \neq 1$ are odd.

Now suppose P is not cyclic so, since $p > 2$, P has a normal abelian (p, p)-subgroup U. If $H = \mathbf{C}_P(U)$ then $H \lhd P$, $[P: H] = p$ and $P = \langle H, y \rangle$ for some element $y \in P$. If L is a noncentral (in P) subgroup of U of order p and if $V = \mathbf{C}_Q(L)$ then

$$Q = V \times V^{y} \times V^{y^2} \times \cdots \times V^{y^{p-1}}$$

is a direct product of H-submodules of Q. This all follows from Schur's lemma since U cannot act semiregularly. If N is the kernel of the action of H on V then by induction there exist two H-orbits $A, B \subseteq V^{\#}$ with the property that $x \in A, B$ implies that $\mathbf{C}_H(x) = N$.

Consider the two subsets of Q given by

$$S = A \times B^{y} \times B^{y^2} \times \cdots \times B^{y^{p-1}}$$

$$T = A \times A^{y} \times B^{y^2} \times \cdots \times B^{y^{p-1}}.$$

If $x \in S, T$ then clearly $\mathbf{C}_H(x) = \bigcap N^{y^i} = \langle 1 \rangle$. Then also $\mathbf{C}_P(x) = \langle 1 \rangle$ since $hy \in \mathbf{C}_P(x)$ for some $h \in H$ would imply using $p > 3$ that A and B are the same H-orbit. Finally it is clear from $P = \langle H, y \rangle$ that no element of S can be P-conjugate to an element of T. Thus Q does indeed have at least two such orbits of elements x with $\mathbf{C}_P(x) = \langle 1 \rangle$ and the result follows.

We remark that the above lemma is false in many instances if the prime 2 is present. Indeed the following three examples are typical of what occurs.

First let $p = 2$ and suppose $q = 2^n - 1$ is a Mersenne prime. Then the dihedral group P of order 2^{n+1} acts faithfully on Q, an abelian group of type (q, q). If x, y are distinct noncentral involutions of P then clearly $|\mathbf{C}_Q(x)| = q$ and $\mathbf{C}_Q(x) \cap \mathbf{C}_Q(y) = \langle 1 \rangle$ since the cyclic subgroup of P of index 2 acts semiregularly. Thus since P has 2^n noncentral involutions x we have

$$\left| \bigcup_x \mathbf{C}_Q(x)^\# \right| = 2^n(q-1) = (q+1)(q-1) = |Q^\#|$$

and every element of $Q^\#$ is fixed by some involution of P.

Now let $p = 2$ and suppose $q = 2^n + 1$ is a Fermat prime. If P_0 is cyclic of order $2^n = q - 1$ then P_0 acts faithfully and transitively on $V^\#$ where $V \cong Z_q$ is cyclic of order q. Thus $P = P_0 \wr Z_2$ acts faithfully on $Q = V_1 \times V_2$, a direct product of two copies of V. Write $P = \langle P_1, P_2, x \rangle$ where P_i is cyclic of order $q - 1$ and acts transitively on $V_i^\#$ and where x interchanges V_1 and V_2. If $v = (v_1, v_2) \in Q$ and say $v_i = 1$ then $\mathbf{C}_P(v) \supseteq P_j$ for $j \neq i$. On the other hand if $v_i \neq 1$ for $i = 1, 2$ then by transitivity there exists $y_i \in P_i$ with $v_i^{y_i} = v_j$ $(j \neq i)$, viewed as elements of V, so that $y_1 y_2 x$ centralizes v.

Finally let $q = 2$ and let $p = 2^n - 1$ be a Mersenne prime. Then Z_p acts faithfully and transitively on $V^\#$ where V is elementary abelian of order 2^n and hence $P = Z_P \wr Z_P$ acts faithfully on $Q = V_1 \times V_2 \times \cdots \times V_p$ a direct product of p copies of V. As in the preceding example the transitivity of Z_p on $V^\#$ implies easily that every element of Q has a nontrivial centralizer in P.

As an indication of the basically different behavior with respect to semisimplicity of odd and even order finite solvable groups we prove the following.

PROPOSITION 2.2. *Let G be a finite solvable group and let P be a p-subgroup of G. Suppose that either P is abelian or $|G|$ is odd. Then $JK[G] \cap K[P] \neq 0$ if and only if $P \cap \mathbf{O}_p(G) \neq \langle 1 \rangle$.*

Proof. Suppose first that $L = P \cap \mathbf{O}_p(G) \neq \langle 1 \rangle$. Then for the augmentation ideal $\omega(K[L]) \subseteq K[P]$ we have

$$0 \neq \omega(K[L]) \subseteq \omega(K[\mathbf{O}_p(G)]) \subseteq JK[G]$$

so $JK[G] \cap K[P] \neq 0$.

Conversely suppose that $P \cap \mathbf{O}_p(G) = \langle 1 \rangle$ and define $N \lhd G$ by $N \supseteq \mathbf{O}_p(G)$ and $N/\mathbf{O}_p(G) = \mathrm{Fit}(G/\mathbf{O}_p(G))$. By Fitting's theorem P acts faithfully on $N/\mathbf{O}_p(G)$ and hence on N/N_0, the Frattini quotient of the nilpotent p'-group $N/\mathbf{O}_p(G)$. Now according to Lemma 1.4 we must have

$$N/N_0 = \bigcup_h \mathbf{C}_{N/N_0}(h)$$

for all $h \in P^{\#}$ but since either P is abelian or $|G|$ is odd this violates Lemma 2.1. The result follows.

On the other hand if $G = QP$ for any of the three examples given above then G is solvable, $P \cap \mathbf{O}_p(G) = \langle 1 \rangle$ since P acts faithfully on Q and $JK[G] \cap K[P] \neq 0$ by Lemma 1.5.

LEMMA 2.3. *Let G be a finite group with subgroups H, H_1 and H_2.*

(i) *Suppose that for all $g \in G$, $H_1^g \cap H_2$ contains an element of order p. Then there exists an element $x \in H_1$ of order p with $[G: \mathbf{C}(x)] \leqq |H_1| \cdot |H_2|$.*

(ii) *Suppose G acts transitively as permutations on Ω and that for each $\alpha \in \Omega$, H contains an element of order p fixing α. Then there exists an element $x \in H$ of order p with $[G: \mathbf{C}(x)] \leqq |H| \cdot |G_\alpha|$.*

Proof. We consider (i). Let X be the set of elements of H_1 of order p and let Y be those of H_2. Then by assumption for each $g \in G$ there exist $x \in X$, $y \in Y$ with $x^g = y$. Thus g belongs to a certain right coset of $\mathbf{C}(x)$ depending on x and y. We therefore have

$$G = \bigcup_{x,y} \mathbf{C}(x) g_{x,y}$$

and hence for some $x \in X$, $[G: \mathbf{C}(x)] \leqq |X| \cdot |Y|$. Since $X \subseteq H_1$, $Y \subseteq H_2$ this part follows.

Finally for (ii) we merely apply (i) with $H_1 = H$, $H_2 = G_\alpha$. For each $g \in G$ we have by assumption an element of order p in

$$H \cap G_{\alpha g^{-1}} = H \cap (G_\alpha)^{g^{-1}}$$

so there is an element of order p in $H^g \cap G_\alpha = H_1^g \cap H_2$.

LEMMA 2.4. *Let G be a locally finite group with $\mathbf{O}_q(G) = \langle 1 \rangle$ for all primes q. If H is a finite subgroup of G then there exists a subgroup G^* of G with $G^* \supseteq H$ and such that G^* is the ascending union of the finite groups $H \subseteq G_1 \subseteq G_2 \subseteq \cdots$. Furthermore for $i > j$, $G_j \cap \mathrm{Fit}(G_i) = \langle 1 \rangle$.*

Proof. We first find such a sequence of groups G_i with $G_i \cap \mathrm{Fit}(G_{i+1}) = \langle 1 \rangle$. Set $G_1 = H$ and suppose we have found $G_1, G_2, \cdots, G_n$. Now $\mathbf{O}_q(G) = \langle 1 \rangle$ for all primes so for each $x \in G_n$, $x \neq 1$ the normal closure $\langle x \rangle^G$ is not locally nilpotent. Thus there exists a finite group L with $\langle x \rangle^L$ not nilpotent. We merely let G_{n+1} be the group generated by G_n and those finitely many L's, one for each $x \in G_n$ $x \neq 1$. Clearly $G_n \cap \mathrm{Fit}(G_{n+1}) = \langle 1 \rangle$.

Finally let $i > j$ so $i \geqq j + 1$. Then

$$\begin{aligned} G_j \cap \mathrm{Fit}(G_i) &= G_j \cap (G_{j+1} \cap \mathrm{Fit}(G_i)) \\ &\subseteq G_j \cap \mathrm{Fit}(G_{j+1}) = \langle 1 \rangle \end{aligned}$$

and the lemma is proved with $G^* = \bigcup G_i$.

We now come to our main result on locally solvable groups. The oddness hypothesis is obviously too restrictive here and the conclusion is not strong enough. Never-the-less we do show that $JK[G] \neq 0$ implies the existence of some nontrivial global structure on G, certainly a first step towards the complete solution.

THEOREM 2.5. *Let K be a field of characteristic $p > 0$ and let G be a locally finite, locally solvable group. Suppose that either all p-subgroups of G are abelian or that G is a $2'$-group. Then $JK[G] \neq 0$ implies $\mathbf{O}_q(G) \neq \langle 1 \rangle$ for some prime q.*

Proof. We assume that $\mathbf{O}_q(G) = \langle 1 \rangle$ for all primes q and show that $JK[G] = 0$. Suppose by way of contradiction that $JK[G] \neq 0$ and let $\alpha \in JK[G]$ with $1 \in \mathrm{Supp}\,\alpha$. Set $H = \langle \mathrm{Supp}\,\alpha \rangle$ and apply Lemma 2.4. By Lemma 16.9 of [2] $\alpha \in JK[G^*]$ so we may assume that $G = G^* = \bigcup G_i$ since clearly $\mathbf{O}_q(G^*) = \langle 1 \rangle$ for all q. Set $F_i = \mathrm{Fit}(G_i)$ and write $F_i = P_i \times Q_i$ where $P_i = \mathbf{O}_p(F_i)$ and $Q_i = \mathbf{O}_{p'}(F_i)$.

Let $Q = \langle Q_1, Q_2, \cdots \rangle$. Since Q_i normalizes Q_j for $j \geqq i, Q$ is clearly a p'-group. This group can best be visualized as the acending union of the n-fold semidirect products $Q_n Q_{n-1} \cdots Q_1$. Now G_i normalizes $\langle Q_i, Q_{i+1}, \cdots \rangle$ a normal subgroup of Q of finite index so since $G = \bigcup G_i$ we conclude from Lemma 1.6 that Q is almost normal in G. Furthermore Q is a p'-group so $JK[Q] = 0$ and hence by Lemma 1.7 $D = \mathbf{D}_G(Q)$ carries the radical, that is $JK[G] =$

$JK[D]\cdot K[G]$. Thus $\pi_D(JK[G])\subseteq JK[G]$ so replacing α by $\pi_D(\alpha)\neq 0$ if necessary we may assume that $H\subseteq D$. Now $H\subseteq G_1$ so H normalizes all Q_i and since $G_i\cap Q_j=\langle 1\rangle$ for $j>i$ it follows easily that for any $h\in H$

$$\mathbf{C}_Q(h)=\langle \mathbf{C}_{Q_1}(h),\mathbf{C}_{Q_2}(h),\cdots\rangle$$

and

$$[Q:\mathbf{C}_Q(h)]=\Pi_i[Q_i:\mathbf{C}_{Q_i}(h)].$$

Thus since $H\subseteq D$ we have $[Q:\mathbf{C}_Q(h)]<\infty$ and it follows that h centralizes all Q_i after awhile and hence since H is finite, H centralizes all Q_i for i sufficiently large.

Now set $R_n=\langle P_1,P_2,\cdots,P_n\rangle$ so that R_n is a p-subgroup of G_n. We also define $S_{n+1}/P_{n+1}=\mathrm{Fit}(G_{n+1}/P_{n+1})$. Then S_{n+1}/P_{n+1} is a nilpotent p'-group and we let $\bar{S}_{n+1}=(S_{n+1}/P_{n+1})/\Phi(S_{n+1}/P_{n+1})$ be its Frattini quotient. Observe that $H\subseteq G_1$ implies that H normalizes R_n and that R_nH acts on $\bar{S}_{n+1}$. We will use this action to show that for some element $h\in H^{\#}$ we have $[R_n:\mathbf{C}_{R_n}(h)]\leqq|H|^2$.

Now R_n is a p-subgroup of G_n so $R_n\cap P_{n+1}=\langle 1\rangle$ and hence by Fitting's theorem, since G_{n+1}/P_{n+1} is solvable, we see that R_n acts faithfully on S_{n+1}/P_{n+1}. Hence R_n also acts faithfully on $\bar{S}_{n+1}$. If H does not act faithfully on $\bar{S}_{n+1}$ and if $h\in H^{\#}$ acts trivially then $(R_n,h)\subseteq R_n$ act trivially so h centralizes R_n and $[R_n:\mathbf{C}_{R_n}(h)]=1\leqq|H|^2$. Thus we may assume that H acts faithfully on $\bar{S}_{n+1}$ and therefore that $H\cap S_{n+1}=\langle 1\rangle$ since S_{n+1} acts trivially on $\bar{S}_{n+1}$.

By assumption either R_n is abelian or both R_n and S_{n+1} have odd order. Hence we conclude from Lemma 2.1 that there exists $x\in\bar{S}_{n+1}$ with $\mathbf{C}_{R_n}(x)=\langle 1\rangle$. We consider the action of $L=R_nH$ on the L-orbit Ω of x. By the above $\mathbf{C}_L(x)\cap R_n=\langle 1\rangle$ so $|\mathbf{C}_L(x)|\leqq|H|$ for this particular $x\in\bar{S}_{n+1}$. Furthermore since $H\cap S_{n+1}=\langle 1\rangle$ Lemma 1.4 implies that every element of $\bar{S}_{n+1}$ is centralized by some element of H of order p. Thus by Lemma 2.3 (ii) there exists $h\in H^{\#}$ with

$$[L:\mathbf{C}_L(h)]\leqq|H|\cdot|\mathbf{C}_L(x)|\leqq|H|^2.$$

Since $[R_n:\mathbf{C}_{R_n}(h)]\leqq[L:\mathbf{C}_L(h)]$ this fact follows.

Let $P=\langle P_1,P_2,\cdots\rangle$. Then since P is the ascending union of the groups R_n and since H is finite, it follows from the above that there exists $h\in H^{\#}$ with $[P:\mathbf{C}_P(h)]\leqq|H|^2$. Again we have

$$\mathbf{C}_P(h)=\langle \mathbf{C}_{P_1}(h),\mathbf{C}_{P_2}(h),\cdots\rangle$$

and

$$|H|^2 \geqq [P: \mathbf{C}_P(h)] = \Pi_i [P_i: \mathbf{C}_{P_i}(h)]$$

so h centralizes all P_j with j sufficiently large. Since h also centralizes all Q_j with j sufficiently large, it follows that h centralizes F_j for some $j > 1$. But G_j is solvable and $h \in G_j$, $h \notin F_j$ so we have a contradiction by Fitting's theorem. This completes the proof.

We remark finally on locally solvable groups which are not necessarily locally finite. If G is such a group and if H is a finitely generated subgroup of G, then H is of course a finitely generated solvable group. Thus by a theorem of Zalesskii [7] (or see [4] Theorem 4.2) $JK[H] = NK[H]$ and hence by Lemma 4.1 of [4] we have $JK[G] = N^*K[G]$. Now by Theorem 1.6 of [4]

$$N^*K[G] = JK[\Lambda^+(G)] \cdot K[G]$$

where $\Lambda^+(G)$ is a certain locally finite characteristic subgroup of G. Clearly $\Lambda^+(G)$ is locally finite and locally solvable so Theorem 2.5 applied to $\Lambda^+(G)$ yields results on $JK[G]$.

3. Linear group reductions. We now begin our work on locally finite linear groups over fields of finite characteristic $q \neq p$. The cases $q = 0$ and $q = p$ have already been considered in [3] and [4]. In the following, unless otherwise indicated, q will be a fixed prime different from p and all groups will be locally finite linear groups in characteristic q. The first lemma is well known. We let $GL_n(q^\infty)$ denote the general linear group over $GF(q^\infty)$, the algebraic closure of $GF(q)$.

LEMMA 3.1. *Let G be an irreducible subgroup of $GL_n(F)$ with F algebraically closed. Then G is conjugate in $GL_n(F)$ to a subgroup of $GL_n(q^\infty)$.*

Proof. Since F is algebraically closed we have $F \supseteq GF(q^\infty)$ and since G acts irreducibly the linear span FG is the whole matrix ring F_n.

Since $FG = F_n$ choose $x_1, x_2, \cdots, x_m \in G$ which form a basis for the matrix ring F_n. Then $H = \langle x_1, x_2, \cdots, x_m \rangle$ is a finite subgroup of G and the embedding of H in F_n is clearly an absolutely irreducible representation for H in characteristic q. Now H is finite so all such representations are realizeable over $GF(q^\infty)$ and hence there exists a nonsingular matrix $s \in GL_n(F)$ with $s^{-1}Hs \subseteq GL_n(q^\infty)$. Replacing G by $s^{-1}Gs$ we may clearly assume that $H \subseteq GL_n(q^\infty)$.

We now proceed as in the proof of Burnside's lemma. Let tr denote the usual matrix trace so that tr defines a nondegenerate bilinear form on F_n. Hence the matrix $[\operatorname{tr} x_i x_j]$ is nonsingular. Now let $x \in G$. Since the x_i's span F_n we have

$$x = \sum_i a_i x_i$$

for suitable $a_i \in F$. Hence multiplying by x_j and taking traces yields

$$\operatorname{tr} xx_j = \sum_i a_i \operatorname{tr} x_i x_j \qquad j = 1, 2, \cdots, m.$$

Observe that xx_j and $x_i x_j$ are elements of G. Thus they are periodic matrices and have traces contained in $GF(q^\infty)$. Therefore the above is a set of m equations over $GF(q^\infty)$ in the m unknowns $a_1, a_2, \cdots, a_m$ with nonzero determinant. The solution is therefore in $GF(q^\infty)$ so $a_i \in GF(q^\infty)$ for all i and hence $x \in GL_n(q^\infty)$.

In view of earlier work on linear groups it is reasonable to expect that $\mathbf{O}_p(G) = \langle 1 \rangle$ implies $JK[G]$ nilpotent. Thus the following few lemmas are relevant.

LEMMA 3.2. *Let $G \subseteq GL_n(F)$ with $\mathbf{O}_p(G) = \langle 1 \rangle$. Suppose that $G_0 = G \cap SL_n(F)$ and $JK[G_0]$ is nilpotent. Then $JK[G]$ is nilpotent*

Proof. Now $G_0 \lhd G$ and $JK[G_0]$ is nilpotent so by Lemma 1.7, $JK[G] = JK[D] \cdot K[G]$ where $D = \mathbf{D}_G(G_0)$. It therefore suffices to show that $JK[D]$ is nilpotent.

Now $D_0 = D \cap G_0 = \Delta(G_0)$ and since D is a linear group, Lemma 1.2 (i) of [3] implies that D_0 has a subgroup Z of finite index which is central in D. Since $D/D_0 \subseteq GL_n(F)/SL_n(F)$ we have D/D_0 abelian. This implies that $H = \mathbf{C}_D(D_0/Z)$ is a nilpotent normal subgroup of D of finite index. Now $D \lhd G$ so $\mathbf{O}_p(D) = \langle 1 \rangle$ and hence $\mathbf{O}_p(H) = \langle 1 \rangle$. But H is nilpotent so H is a p'-group and $JK[H] = 0$. Finally $[D: H] < \infty$ so we conclude from Lemma 16.8 of [2] that $JK[D]$ is nilpotent.

LEMMA 3.3. *Let $G \subseteq GL_n(F)$ and suppose we know that for all $H \lhd G$ if $\dim FH < \dim FG$ then $JK[H]$ is nilpotent. Let $M \lhd G$ with $\dim FM < \dim FG$. Then either $JK[G]$ is nilpotent or $[M: M \cap \mathbf{Z}(G)] < \infty$.*

Proof. Since $\dim FM < \dim FG$ we know that $JK[M]$ is nilpotent. Hence by Lemma 1.7, $D = \mathbf{D}_G(M)$ carries the radical. Now $D \lhd G$ and if $\dim FD < \dim FG$ then $JK[D]$ would be nilpotent and hence so would $JK[G]$. On the other hand if $\dim FD = \dim FG$ then $G \subseteq FD$ so by Lemma 1.2 (i) of **[3]** M has a subgroup of finite index central in D and hence in G. Thus $[M: M \cap \mathbf{Z}(G)] < \infty$.

Let r be a prime. By a Sylow r-subgroup of G we mean a maximal r-subgroup. Thus by definition, every r-subgroup is certainly contained in a Sylow r-subgroup of G. Now suppose G is a locally finite linear group. Then by a theorem of Platonov (see **[6]** Theorem 9.10), for each prime r, the Sylow r-subgroups of G are conjugate in G. We will use this result implicitly in the remainder of this paper. Furthermore we have

LEMMA 3.4. *Let $G \subseteq GL_n(F)$ and let P be a Sylow p-subgroup of G. Then P contains a normal abelian divisible subgroup A of finite index, Moreover if F is algebraically closed then A can be diagonalized.*

Since the existence of subgroups of finite index is frequently annoying the following is useful. We use the subgroup $\mathcal{S}(G)$ as defined in **[5]** §5.

LEMMA 3.5. *Let $G \subseteq GL_n(F)$. Then G has a characteristic subgroup G_0 such that G/G_0 is a p' by finite group and such that G_0 has no proper subgroups of finite index. Moreover if $JK[G_0]$ is nilpotent then so is $JK[G]$ and if $\mathcal{S}(G_0)$ carries $JK[G_0]$ then $\mathcal{S}(G)$ carries $JK[G]$.*

Proof. For any group G let $R(G)$ be the intersection of all its normal subgroups of finite index and let $S(G)$ be given by $S(G)/R(G) = \mathbf{O}_{p'}(G/R(G))$. Then clearly $R(G)$ and $S(G)$ are characteristic subgroups of G. We show first that $G \subseteq GL_n(F)$ implies $[G: S(G)] < \infty$.

Let P and A be given as in Lemma 3.4. If H is a normal subgroup of G of finite index then $H \supseteq A$ since A has no subgroup of finite index. Since G has Sylow theorems it follows that P maps onto a Sylow p-subgroup of G/H so $|G/H|_p \leqq [P: A]$. Now choose $H \lhd G$ of finite index so that $|G/H|_p$ is as large as possible. Then $G \supseteq H \supseteq R(G)$ and $H/R(G)$ is residually finite so it follows that $H/R(G)$ is a p'-group. Hence $S(G) \supseteq H$ and $[G: S(G)] < \infty$.

Define $G_0 = S(G)^p$, the group generated by all p-elements of $S(G)$, or equivalently $G_0 = \mathbf{O}^{p'}(S(G))$. Clearly G_0 is characteristic in G and G/G_0 is p' by finite. We show now that G_0 has no proper subgroups of finite index. Let $S = S(G_0)$. Then S is a characteristic subgroup of

G_0 of finite index and hence $S \lhd G$. We consider the group $\bar{G} = G/S$. If $\bar{C}$ is the centralizer of $\bar{G}_0$ in $\overline{S(G)}$ then certainly $\overline{S(G)}/\bar{C}$ is finite. Also $\bar{C}/(\bar{C} \cap \bar{G}_0) \subseteq \overline{S(G)}/\bar{G}_0$ is a p'-group so $\bar{C}$ has a finite central Sylow p-subgroup and we conclude that $[\bar{C}: \mathbf{O}_{p'}(\bar{C})] < \infty$. Therefore $\mathbf{O}_{p'}(\bar{C})$ is a normal subgroup of $\bar{G}$ of finite index contained in $\overline{S(G)}$ so clearly $\overline{S(G)} \supseteq \mathbf{O}_{p'}(\bar{C}) \supseteq \overline{R(G)}$ since $R(G) \supseteq G_0 \supseteq S$. Hence by definition of $S(G)$ we have that $\overline{S(G)}/\mathbf{O}_{p'}(\bar{C})$ is a p'-group so $\overline{S(G)}$ is a p'-group and by definition of $S = S(G_0)$ we have $\bar{G}_0 = \langle 1 \rangle$ and $G_0 = S(G_0)$. Thus $G_0/R(G_0)$ is a p'-group. Since $G_0 = S(G)^p$ is generated by p-elements this yields $G_0 = R(G_0)$ and G_0 has no proper subgroups of finite index.

Suppose now that $JK[G_0]$ is nilpotent. Since $S(G)/G_0$ is a p'-group G_0 carries the radical of $S(G)$ and hence $JK[S(G)]$ is nilpotent. Thus by Lemma 16.8 of **[2]**, $JK[G]$ is nilpotent. Finally suppose $\mathcal{S}(G_0)$ carries the radical of G_0. Again G_0 carries $JK[S(G)]$ so $\mathcal{S}(G_0)$ carries the radical of $S(G)$. Since $\mathcal{S}(H)$ is generated by p-elements for any group H it follows easily that $\mathcal{S}(S(G)) = \mathcal{S}(G_0)$. Corollary 5.5 of **[5]** now yields the result.

4. Finite Sylow p-subgroups. Our linear group techniques differ sharply accordingly as the Sylow p-subgroup of G is finite or infinite. In this section we consider the finite case. The following lemma is proved in **[3]** in a slightly different form. It also follows easily from topological considerations.

LEMMA 4.1. *Let $G \subseteq GL_n(F)$ and let $T_1, T_2, \cdots, T_r$ be a finite number of affine subspaces of F_n with $G \subseteq \bigcup T_i$. Then G has a subgroup H of finite index with $H \subseteq T_i$ for some i.*

Proof. We proceed as in Lemma 2.1 of **[3]** with S deleted and with the T_i's affine subspaces. The latter causes no difficulty. At the end of that proof we deduce that G permutes transitively by right multiplication certain affine subspaces $M_1, M_2, \cdots, M_m$. Since $M_1 \cap G \neq \phi$ some M_i contains the identity. If H is the stabilizer of this M_i then $[G: H] < \infty$ and $M_i H \subseteq M_i$ yields $H \subseteq M_i \subseteq T_{i'}$.

LEMMA 4.2. *Let $G \subseteq GL_n(F)$, let $y_1, y_2, \cdots, y_r \in F_n$ be a finite number of matrices and let $\{T_{ij}\}$ for $i = 1, 2, \cdots, r$; $j = 1, 2, \cdots, s$ be a finite number of affine subspaces of F_n. Suppose that for each $x \in G$ there exists i, j with $x^{-1} y_i x \in T_{ij}$. Then G has a subgroup H of finite index such that for some fixed i, j and all $h \in H$, $h^{-1} y_i h \in T_{ij}$.*

Proof. Observe that G acts on F_n by conjugation and that this yields a homomorphism of G into $E = \mathrm{End}_F(F_n) \cong F_{n^2}$. Let the image of G be denoted by $\bar{G}$ so that $\bar{G}$ is contained in the appropriate general linear group. Furthermore for $y \in F_n$ and $e \in E$ we let y^e denote the image of y under e. Thus clearly for $x \in G$, $y \in F_n$ we have $y^{\bar{x}} = x^{-1}yx = y^x$.

For each i, j let

$$M_{ij} = \{e \in E \mid y_i^e \in T_{ij}\}.$$

Since T_{ij} is an affine subspace of F_n it follows easily that M_{ij} is an affine subspace of E. Moreover by assumption $\bar{G} \subseteq \bigcup_{ij} M_{ij}$. Hence by Lemma 4.1 $\bar{G}$ has a subgroup $\bar{H}$ of finite index with $\bar{H} \subseteq M_{ij}$ for some i, j. If H is the complete inverse image of $\bar{H}$ in G then H has the required properties.

If $G \subseteq GL_n(F)$ we let $P_0 = P_0(G)$ be the Sylow p-subgroup of the set of scalar matrices contained in G. Thus P_0 is isomorphic to a subgroup of the multiplicative group $F - \{0\} = F^\circ$. Observe that P_0 is independent of the choice of basis which gives rise to $GL_n(F)$. In other words if $s \in GL_n(F)$ and if G is replaced by $s^{-1}Gs$ then $P_0(s^{-1}Gs) = P_0(G)$. As usual we let π_{P_0}: $K[G] \to K[P_0]$ denote the natural projection and tr: $F_n \to F$ the ordinary matrix trace. The main result of this section is as follows.

PROPOSITION 4.3. *Let $G \subseteq GL_n(F)$ and let*

$$\alpha = 1 + \sum a_i x_i \in JK[G]$$

with $x_i \neq 1$ and with $\pi_{P_0}(\alpha) \notin JK[P_0]$. Suppose that the Sylow p-subgroups of G are finite and that Q is a Sylow q-subgroup of G. Then there exist $x_i \in \operatorname{Supp} \alpha$, a nonscalar group element, $H \subseteq G$ a subgroup of finite index and $\tilde{Q} \subseteq Q$ a subgroup of finite index such that

$$\operatorname{tr} x_i^h (1 - y) = 0$$

for all $h \in H$, $y \in \tilde{Q}$.

Proof. Since G has only finitely many conjugacy classes of p-elements it follows that there are only finitely many possibilities for $\operatorname{tr} x$ if x is a p-element. Say these values are $\mu_1, \mu_2, \cdots, \mu_t \in F$. Furthermore if x is a $\{p, q\}$-element of G then writing $x = x_p x_q$ as a product of its p and q parts with $(x_q)^{q^m} = 1$ we have since char $F = q$

$$(\operatorname{tr} x)^{q^m} = \operatorname{tr} x^{q^m} = \operatorname{tr} x_p^{q^m} = (\operatorname{tr} x_p)^{q^m}.$$

Thus $\operatorname{tr} x = \operatorname{tr} x_p = \mu_i$ for some i.

Let $x \in G$. Then

$$\alpha^x = 1 + \sum a_i x_i^x \in JK[G]$$

and $\pi_{P_0}(\alpha^x) = \pi_{P_0}(\alpha) \not\in JK[P_0]$. If $y \in Q$ then y is of course a q-element so by Lemma 1.2 we deduce that for some $i, x_i^x \not\in P_0$ and $x_i^x y$ is a $\{p, q\}$-element. Observe that this implies that x_i is not a scalar matrix since the scalars contain no elements of order q. Hence we have shown that given $x \in G$, $y \in Q$ there exist i, j with $x_i \in \operatorname{Supp} \alpha$ a nonscalar matrix and with $\operatorname{tr} x_i^x y = \mu_j$.

For fixed x and for those nonscalar x_i's let

$$M_{ij} = \{\gamma \in F_n \mid \operatorname{tr} x_i^x \gamma = \mu_j\}.$$

Then M_{ij} is clearly an affine subspace of F_n and we have $Q \subseteq \bigcup M_{ij}$. Thus by Lemma 4.1 Q has a subgroup Q_x of finite index such that for some subscript $i = f(x)$ we have $Q_x \subseteq M_{ij}$ for some j. That is, $\operatorname{tr} x_i^x y = \mu_j$ for all $y \in Q_x$. Note that $1 \in Q_x$ so $\mu_j = \operatorname{tr} x_i^x$ and the above becomes

$$\operatorname{tr} x_i^x (1 - y) = 0$$

for all $y \in Q_x$. Note also that by choice x_i is not a scalar matrix.

Now for each nonscalar x_i define S_i to be the subspace of F_n given by

$$S_i = \langle x_i^x \mid f(x) = i \rangle.$$

Observe then that for each $x \in G$ there exists i, namely $i = f(x)$, with $x_i^x \in S_i$. Thus by Lemma 4.2 G has a subgroup H of finite index such that for some $i, x_i^h \in S_i$ for all $h \in H$. Say this occurs for the nonscalar matrix x_1.

Let $\{x_1^{w_k}\}$ be a finite spanning set for S_1 with $w_k \in G$ and with $f(w_k) = 1$. If $\tilde{Q} = \bigcap_k Q_{w_k}$ then $[Q : \tilde{Q}] < \infty$ and for all $y \in \tilde{Q}$

$$\operatorname{tr} x_1^{w_k} (1 - y) = 0$$

for all k. Thus for all $s_1 \in S_1$ we have $\operatorname{tr} s_1(1 - y) = 0$ and since S_1 contains all H-conjugates of x_1 the result follows.

Observe that in the above if Q is finite then the conclusion is decidedly uninteresting. Namely we could then have $\tilde{Q} = \langle 1 \rangle$ so certainly $\operatorname{tr} x_i^h(1-y) = 0$ for $y \in \tilde{Q}$. Fortunately in this case we can apply the following well known theorem of Brauer and Feit (see **[6]** Corollary 9.7).

PROPOSITION 4.4. (Brauer-Feit). *Let G be a locally finite subgroup of $GL_n(F)$ with F a field of characteristic $q > 0$. If the Sylow q-subgroups of G are finite then G has a normal abelian subgroup of finite index.*

This is of course a modular analog of Jordan's theorem for complex linear groups. Furthermore there is a bound for the index depending upon n and the size of the Sylow q-subgroups.

5. Infinite Sylow p-subgroups.

We now consider the case of infinite Sylow p-subgroups. This will require a close look at p^nth roots of unity.

LEMMA 5.1. *Let $f > 1$ be an integer and assume that $p \mid f-1$ and that $4 \mid f-1$ if $p = 2$. Then for all integers $a \geqq 1$*

$$|f^a - 1|_p = |a|_p \cdot |f-1|_p.$$

Proof. This is standard. We first consider some special cases. Suppose $p \nmid a$. Since $f \equiv 1\,(p)$ we have

$$\frac{f^a-1}{f-1} = 1 + f + \cdots + f^{a-1} \equiv a\,(p)$$

so $|f^a - 1|_p = |f-1|_p = |a|_p\,|f-1|_p$.

Now let $a = p$. If $p = 2$ then $4 \mid f-1$ so $f = 1 + 4k$ and

$$\frac{f^a-1}{f-1} = 1 + f = 2 + 4k.$$

Thus $|f^a - 1|_2 = 2|f-1|_2$. On the other hand for $p > 2$ we have $p \mid f-1$ so $f \equiv 1 + pk(p^2)$. Thus $f^i \equiv 1 + ipk(p^2)$ and

$$\frac{f^a-1}{f-1} = 1 + f + \cdots + f^{p-1} \equiv p + p^2\,\frac{(p-1)}{2}\,k(p^2)$$

$$\equiv p\,(p^2).$$

Therefore we have $|f^a - 1|_p = p\,|f-1|_p = |a|_p \cdot |f-1|_p$.

The result follows easily by induction on a. Namely if $a = bc$ is a proper factorization then

$$\begin{aligned} |f^a - 1|_p &= |(f^b)^c - 1|_p = |c|_p\,|f^b - 1|_p \\ &= |c|_p\,|b|_p\,|f-1|_p = |a|_p \cdot |f-1|_p. \end{aligned}$$

LEMMA 5.2. *Let $GF(f)$ be a finite field and suppose that $p \mid f-1$ and that $4 \mid f-1$ if $p = 2$. Let η generate the Sylow p-subgroup of the multiplicative group $GF(f)^\circ$. Then for all n the polynomial $x^{p^n} - \eta \in GF(f)[x]$ is irreducible.*

Proof. Note that $o(\eta) = p^m = |f-1|_p$. Thus if δ is a root of $x^{p^n} - \eta$ then $o(\delta) = p^{m+n}$. If $\delta \in GF(f^a)$ then $p^{m+n} \leqq |f^a - 1|_p = |a|_p \cdot |f-1|_p$ by Lemma 5.1 so $p^n \leqq |a|_p$ and $p^n \leqq a$. Thus $x^{p^n} - \eta$ must be irreducible.

We now assume that $G \subseteq GL_n(q^\infty)$ and that G has an infinite Sylow p-subgroup P as described in Lemma 3.4. Furthermore by considering a conjugate of G in $GL_n(q^\infty)$ if necessary we may assume that the maximal abelian divisible subgroup A is diagonalized. If $[P\colon A] = p^a$ then we define the field $F_0 = F_0(G)$ by $F_0 = GF(q)[\mathscr{E}]$ where $o(\mathscr{E}) = p^{a+2}$. Observe that if F is any finite field containing F_0 then clearly $|F^\circ|_p \geqq p^2$ and hence Lemma 5.2 with $F = GF(f)$ will always apply. We fix the choice of F_0.

Let $k = k(A)$ denote the maximal number of distinct eigenvalues of any element of A. Clearly $1 \leqq k(A) \leqq n$. If $k(A) = 1$ then A consists of scalar matrices and is essentially trivial for our purposes. Thus our interest is in $k(A) \geqq 2$.

Let F be a finite subfield of $GF(q^\infty)$. By an F-functional $l\colon GF(q^\infty)_n \to GF(q^\infty)$ we mean a linear functional of the form

$$l([x_{ij}]) = \sum_i^n f_i x_{ii}$$

with $f_i \in F$, some $f_i = 0$ so that not all diagonal entries occur and some $f_i \neq 0$ so that this is not the zero form. The following lemma is the crux of our argument.

LEMMA 5.3. *Let $G \subseteq GL_n(q^\infty)$ and let P be a Sylow p-subgroup of G as described above and with A diagonal. Let*

$$\alpha = 1 + \sum k_i x_i \in JK[G]$$

with $x_i \neq 1$ and $\pi_{P_0}(\alpha) \notin JK[P_0]$. Define

$$F = F(\alpha) = F_0[\operatorname{tr} x_i \mid x_i \in \operatorname{Supp} \alpha].$$

(i) *If $k(A) \geqq 3$ then there exists an F-functional l and a nonscalar x_i with $l(x_i) \in F$.*

(ii) *If $k(A) = 2$ and $G \subseteq SL_n(q^\infty)$ then there exists an F-functional l and a nonscalar x_i such that $l(x_i)^n \in F$.*

Proof. Since $k(A) \geqq 2$ we have $A \neq \langle 1 \rangle$ and we can choose $y \in A$, $y \neq 1$ to have the maximal number $k = k(A)$ of distinct eigenvalues. Since any root of y in A has at least as many distinct eigenvalues as y does, by taking a suitable root if necessary, we may assume that $o(y) > n^2$. This will only be needed for (ii).

Let L be the finite subfield of $GF(q^\infty)$ generated by F_0 and all the entries of all the matrices x_i. Clearly $L \supseteq F$. Let $|L^\circ|_p = p^h$ and choose $x \in A$ with $x^{p^h} = y$. Since A is diagonal we have $x = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Now x is a p-element so by Lemma 1.2 there exists $x_i \notin P_0$ such that $x_i x$ is a p-element, say $i = 1$. This clearly implies that x_1 is not scalar. Write

$$x_1 = \begin{pmatrix} w_1 & & * \\ & w_2 & \\ & & \ddots \\ * & & w_n \end{pmatrix}$$

so all $w_i \in L$ and since $x_1 x$ is a p-element it is conjugate to some element $z \in P$. Note that z may not be diagonal but let its eigenvalues be $\mu_1, \mu_2, \cdots, \mu_n$. Thus we have

$$\sum w_i \lambda_i = \operatorname{tr} x_1 x = \operatorname{tr} z = \sum \mu_i. \tag{1}$$

Choose p-element $\lambda \in GF(q^\infty)$ of sufficiently large order so that $\lambda_i, \mu_i \in \langle \lambda \rangle$ and $o(\lambda) \geqq p^h$. Say $o(\lambda) = p^{h+m}$ and write

$$\lambda_i = \lambda^{b_i'} = \lambda^{p^m b_i + c_i} \qquad 0 \leqq c_i < p^m$$

$$\mu_i = \lambda^{d_i'} = \lambda^{p^m d_i + e_i} \qquad 0 \leqq e_i < p^m.$$

Then (1) yields

$$\sum_1^n (w_i\lambda^{p^m b_i})\lambda^{c_i} = \sum_1^n (\lambda^{p^m d_i})\lambda^{e_i} \tag{2}$$

Note that $o(\lambda^{p^m}) = p^h$ so $\lambda^{p^m} \in L$ and thus λ is a root of the polynomial in $L[t]$ given by

$$\sum_1^n (w_i\lambda^{p^m b_i})t^{c_i} - \sum_1^n (\lambda^{p^m d_i})t^{e_i}.$$

Furthermore since λ^{p^m} in fact generates the p-part of L° Lemma 5.2 implies that the minimal equation for λ over L has degree p^m. Since all c_i and e_i satisfy $0 \leqq c_i,\ e_i < p^m$ we deduce therefore that this polynomial must vanish identically. Hence

$$\sum_1^n (w_i\lambda^{p^m b_i})t^{c_i} = \sum_1^n (\lambda^{p^m d_i})t^{e_i}. \tag{3}$$

We first consider the left hand side (lhs) of (3). If $c_i = c_j$ then $\lambda_i/\lambda_j = \lambda^{p^m(b_i - b_j)}$ so $(\lambda_i/\lambda_j)^{p^h} = 1$ and $\lambda_i^{p^h} = \lambda_j^{p^h}$. Note that by definition of x

$$y = x^{p^h} = \operatorname{diag}(\lambda_1^{p^h}, \lambda_2^{p^h}, \cdots, \lambda_n^{p^h})$$

so we see that x has at least as many distinct c_i's as y has distinct eigenvalues. Now certainly x has at least as many distinct eigenvalues as it has distinct c_i's. Finally $y \in A$ was chosen to have the maximal number k of distinct eigenvalues. All this implies that x has precisely k distinct eigenvalues and that these have distinct c_i's. For convenience let us assume that the rows and columns are so labeled that $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct. Then the lhs of (3) looks like

$$\text{lhs} = \sum_1^k (\sigma_i\lambda^{p^m b_i})t^{c_i}$$

where each σ_i is the appropriate sum of those w_j's such that $\lambda_j = \lambda_i$.

Now if some $\sigma_i = 0$ then

$$l(x_1) = \sum_{\lambda_j = \lambda_i} w_j = 0$$

is an appropriate F-functional with value $0 \in F$ since $k \geqq 2$. Thus we may assume that $\sigma_i \neq 0$ for all i. This implies that the lhs of (3) contains precisely k terms.

We now consider the right hand side (rhs) of (3). By (3) and the above there must be at least k distinct e_i's. Since $o(\lambda) = p^{m+h}$ we have $(\mu_i/\mu_j)^{p^h} = \lambda^{(e_i - e_j)p^h}$ and hence $\mu_i^{p^h} = \mu_j^{p^h}$ if and only if $e_i = e_j$. Now $L \supseteq F_0$ so $h \geqq a$ by definition of F_0. Hence $z^{p^a} \in A$ implies $z^{p^h} \in A$. Observe that z^{p^h} has eigenvalues $\mu_1^{p^h}, \mu_2^{p^h}, \cdots, \mu_n^{p^h}$ so by definition of $k(A)$ there are at most k distinct $\mu_i^{p^h}$'s and hence at most k distinct e_i's. This therefore implies that there are precisely k distinct e_i's say $e_1, e_2, \cdots, e_k$.

Now observe that since $h \geqq a$, $\mu_i^{p^a} = \mu_j^{p^a}$ implies $\mu_i^{p^h} = \mu_j^{p^h}$ and hence $e_i = e_j$. Thus there are at least k distinct $\mu_i^{p^a}$'s. But $z^{p^a} \in A$ so by definition of k there are at most k distinct $\mu_i^{p^a}$'s. Therefore we deduce that $e_i = e_j$ implies that $\mu_i^{p^a} = \mu_j^{p^a}$ so $(\mu_i/\mu_j)^{p^a} = 1$ and $\mu_i/\mu_j \in F_0$ by definition of F_0. Finally by grouping together all the terms of the rhs with the same e_i we have

$$\text{rhs} = \sum_1^k (\tau_i \lambda^{p^m d_i}) t^{e_i}$$

with $\tau_i \in F_0$ since τ_i is the sum of all those terms μ_j/μ_i with $e_j = e_i$.

We now have the equal polynomials in t

$$\sum_1^k (\sigma_i \lambda^{p^m b_i}) t^{c_i} = \sum_1^k (\tau_i \lambda^{p^m d_i}) t^{e_i}$$

with the c_i's distinct, the e_i's distinct and all $\sigma_i \neq 0$. Thus the terms on the right and left sides must match one for one and by renumbering the right side if necessary we deduce that for $i = 1, 2, \cdots, k$

$$\text{(4)} \qquad \begin{aligned} c_i &= e_i \\ \sigma_i &= \tau_i \lambda^{p^m(d_i - b_i)} \qquad \tau_i \in F_0. \end{aligned}$$

Let $\mu = \lambda^{p^m}$ so that $o(\mu) = p^h$ and μ generates the p-part of L°. Then $\sigma_i = \tau_i \mu^{d_i - b_i}$. Now clearly $L \supseteq F \supseteq F_0$ so if $|F^\circ|_p = p^r$ we have $h \geqq r$ and say $h = r + s$. Write

$$\sigma_i = \tau_i \mu^{d_i - b_i} = \tau_i \mu^{p^s u_i + v_i} \qquad 0 \leqq v_i < p^s.$$

Then $\tau_i \in F_0 \subseteq F$, $\mu^{p^s} \in F$, $\operatorname{tr} x_1 \in F$ so

$$\operatorname{tr} x_1 = \sigma_1 + \sigma_2 + \cdots + \sigma_k = \sum_1^k (\tau_i \mu^{p^s u_i}) \mu^{v_i}$$

and μ is a root of the polynomial in $F[t]$ given by

$$\operatorname{tr} x_1 - \sum_1^k (\tau_i \mu^{p^s u_i}) t^{v_i}.$$

But $o(\mu^{p^s}) = p^r = |F^\circ|_p$ so Lemma 5.2 implies that the minimal polynomial of μ over F has degree p^s. Since $0 \leqq v_i < p^s$ we deduce that

$$\sum_1^k (\tau_i \mu^{p^s u_i}) t^{v_i} = \operatorname{tr} x_1. \tag{5}$$

Suppose two distinct v_i's occur. Then let v be one such nonzero v_i. It follows that

$$\sum_{v_i = v} \tau_i \mu^{p^s u_i} = 0$$

so

$$l(x_1) = \sum_{v_i = v} \sigma_i = \left(\sum_{v_i = v} \tau_i \mu^{p^s u_i} \right) \mu^v = 0$$

is an appropriate F-functional for x_1 with value $0 \in F$.

Thus we may suppose that all $v_i = v$. If $\operatorname{tr} x_1 \neq 0$ then by (5) we must have $v = 0$ and hence since $k \geqq 2$

$$l(x_1) = \sigma_1 = \tau_1 \mu^{p^s u_1} \in F$$

is an appropriate functional. Also if $k > 2$ then

$$l(x_1) = \sigma_1 / \tau_1 \mu^{p^s u_1} - \sigma_2 / \tau_2 \mu^{p^s u_2} = 0$$

is an appropriate F-functional with value $0 \in F$.

There remains the case $k(A) = 2$, $\operatorname{tr} x_1 = 0$ and here we can assume $G \subseteq SL_n(q^\infty)$. Note that $\sigma_1 + \sigma_2 = \operatorname{tr} x_1 = 0$ and by (4) $\sigma_1 = \tau_1 \mu_1 / \lambda_1$ and $\sigma_2 = \tau_2 \mu_2 / \lambda_2$. Suppose that λ_1 occurs in x with multiplicity b so that λ_2 occurs with multiplicity $n - b$. Then

$$1 = \det x = \lambda_1^b \lambda_2^{n-b}.$$

Now suppose that $e_j = e_1$ for c values of j so $e_j = e_2$ for $n - c$ values of j. Since $e_i = e_j$ implies that μ_i/μ_j is a p^ath root of unity we then have

$$1 = \det z = \mu_1^c \mu_2^{n-c} \rho$$

where $\rho^{p^a} = 1$ and hence $\rho \in F_0 \subseteq F$.

By renumbering if necessary we may suppose that $o(\lambda_1) \geqq o(\lambda_2)$ so that $o(\lambda_1) = o(x)$. Since $\sigma_2 = -\sigma_1$ we have using $\mu_1^c\mu_2^{n-c} \in F$

$$\sigma_1^n = \sigma_1^c(-\sigma_2)^{n-c} = (\tau_1/\lambda_1)^c(-\tau_2/\lambda_2)^{n-c}\mu_1^c\mu_2^{n-c}$$

so

$$\sigma_1^n = \eta\lambda_1^{-c}\lambda_2^{c-n} \tag{6}$$

for some $\eta \in F$. Thus using $\lambda_2^{n-b} = \lambda_1^{-b}$ we have

$$\begin{aligned}\sigma_1^{n(n-b)} &= \eta^{n-b}\lambda_1^{-c(n-b)}\lambda_2^{(c-n)(n-b)}\\ &= \eta^{n-b}\lambda_1^{-c(n-b)}\lambda_1^{-b(c-n)}\\ &= \eta^{n-b}\lambda_1^{n(b-c)}.\end{aligned}$$

Note that $\sigma_1 \neq 0$ so $\eta \neq 0$ and $\sigma_1, \eta \in L$. Thus $\lambda_1^{n(b-c)}$ is a p-element of L so by definition of h, $o(\lambda_1^{n(b-c)}) \leqq p^h$ and $\lambda_1^{p^h n(b-c)} = 1$. Now $o(x) = o(\lambda_1)$ and $x^{p^h} = y$ so $y^{n(b-c)} = x^{p^h n(b-c)} = 1$. On the other hand y was chosen to have order larger than n^2. Since $1 \leqq b, c \leqq n-1$ this implies easily that $b = c$. Therefore (6) yields

$$\sigma_1^n = \eta\lambda_1^{-b}\lambda_2^{b-n} = \eta \in F$$

and $l(x_1) = \sigma_1$ is an appropriate F-functional with value an nth root of an element of F. This completes the proof.

The main result of this section is now an easy consequence.

PROPOSITION 5.4. *Given the assumptions of Lemma* 5.3, *there exists a subgroup H of finite index in G, an F-functional l and a nonscalar $x_i \in \operatorname{Supp}\alpha$ with*

$$l(x_i^h) = l(x_i)$$

for all $h \in H$.

Proof. We use the notation of Lemma 5.3. For each F-functional l and constant c with $c^n \in F$ let

$$M(l, c) = \{\gamma \in GF(q^\infty)_n \mid l(\gamma) = c\}.$$

Then $M(l,c)$ is an affine subspace of the matrix ring $GF(q^\infty)_n$. Furthermore since F is finite there are only finitely many of these.

Let $x \in G$. Then

$$\alpha^x = 1 + \sum k_i x_i^x \in JK[G]$$

with $x_i^x \neq 1$ and since P_0 is central, $\pi_{P_0}(\alpha^x) = \pi_{P_0}(\alpha) \notin JK[P_0]$. Moreover since $\operatorname{tr} x_i^x = \operatorname{tr} x_i$ we have $F(\alpha^x) = F(\alpha)$. Thus by Lemma 5.3 applied to α^x, there exists an F-functional l and nonscalar x_i^x (and hence x_i is nonscalar) so that $l(x_i^x) = c$ for some c with $c^n \in F$. In other words we have shown that for each $x \in G$ there exists a nonscalar $x_i \in \operatorname{Supp} \alpha$ with $x_i^x \in M(l, c)$ for some l, c. Thus by Lemma 4.2 G has a subgroup H of finite index such that for some fixed i, l, c we have $x_i^h \in M(l, c)$ for all $h \in H$. Finally since $1 \in H$, the definition of $M(l, c)$ yields

$$l(x_i^h) = c = l(x_i^1)$$

and the result follows.

6. Some linear groups. The results of the preceding two sections lead us fairly naturally to the following definition. Let $G \subseteq GL_n(F)$. We say that G is a *large* subgroup of $GL_n(F)$ if for all nonscalar matrices $x \in G$ and all subgroups H of finite index in G, the F-linear span of the matrices $x^{h_1} - x^{h_2}$ for all $h_1, h_2 \in H$ consists precisely of all the matrices in F_n of trace 0. Let us write $S(x, H)$ for the above linear span of $x^{h_1} - x^{h_2}$ and $T(F_n)$ for the set of all matrices of trace 0. We have clearly

LEMMA 6.1. *Let $G \subseteq GL_n(F)$.*

(i) *If L is a field extension of F then G is large in $GL_n(L)$ if and only if it is large in $GL_n(F)$.*

(ii) *If $s \in GL_n(F)$ then G is large in $GL_n(F)$ if and only if $s^{-1}Gs$ is large.*

(iii) *Suppose G is large in $GL_n(F)$, $N \lhd G$ and $[G\colon H] < \infty$. Then H is large in $GL_n(F)$. Moreover either N consists of scalar matrices or N acts irreducibly.*

Our main result is as follows.

THEOREM 6.2. *Let K be a field of characteristic $p>0$ and let G be a locally finite group. Suppose that $G \subseteq SL_n(F)$ is a large subgroup of $GL_n(F)$ where F is a field of finite characteristic $q \neq p$. If P_0 denotes the Sylow p-subgroup of the group of scalar matrices in G, then*

$$JK[G] = JK[P_0] \cdot K[G].$$

Proof. By Lemma 6.1 (i) we may assume that F is algebraically closed. If G consists of scalar matrices then, since $G \subseteq SL_n(F)$, G is in fact a finite abelian group so the result is clearly true here. Thus we may assume by Lemma 6.1 (iii) that G acts irreducibly. Now according to Lemma 3.1 G is conjugate to a subgroup of $GL_n(q^\infty)$. Therefore finally by Lemma 6.1 (i) (ii) we may assume that $F = GF(q^\infty)$ and clearly also that $n \geqq 2$.

Now we have

$$\pi_{P_0}(JK[G]) \cdot K[G] \supseteq JK[G] \supseteq JK[P_0] \cdot K[G].$$

Thus we need only show that $\pi_{P_0}(JK[G]) \subseteq JK[P_0]$. Suppose by way of contradiction that there exists $\beta \in JK[G]$ with $\pi_{P_0}(\beta) \not\in JK[P_0]$. Then certainly $\pi_{P_0}(\beta) \neq 0$ so we can choose $w \in \operatorname{Supp}\beta$, $w \in P_0$. If $\beta = aw + \cdots$ then clearly $\alpha = a^{-1}w^{-1}\beta \in JK[G]$, $\pi_{P_0}(\alpha) = a^{-1}w^{-1}\pi_{P_0}(\beta) \not\in JK[P_0]$ and

$$\alpha = 1 + \sum k_i x_i$$

with $x_i \neq 1$. There are now three cases to consider.

Suppose first that the Sylow p-subgroups of G are infinite and use the notation of Proposition 5.4. By replacing G by a conjugate if necessary we may assume that the divisible subgroup A of P is diagonal. Since P is infinite and $[P:A]<\infty$ we have that A is infinite. Furthermore $G \subseteq SL_n(q^\infty)$ so $k(A) \geqq 2$ since otherwise A would consist of scalars and have order at most n. Thus Proposition 5.4 applies and there exists a subgroup H of G of finite index, a functional l for some subfield of $GF(q^\infty)$ and a nonscalar $x_i \in \operatorname{Supp}\alpha$ with $l(x_i^h) = l(x_i)$ for all $h \in H$. Then for $h_1, h_2 \in H$ we have $l(x_i^{h_1} - x_i^{h_2}) = 0$ so l annihilates $S(x_i, H)$. Since G is large, l therefore annihilates $T(GF(q^\infty)_n)$ certainly a contradiction since $n \geqq 2$.

Now suppose that the Sylow p-subgroups of G are finite but the Sylow q-subgroups of G are infinite and use the notation of Proposition

4.3. Then there exists a nonscalar $x_i \in \operatorname{Supp} \alpha$, $H \subseteq G$ a subgroup of finite index and $\tilde{Q} \subseteq Q$ a subgroup of finite index such that

$$\operatorname{tr} x_i^h(1-y) = 0$$

for all $h \in H$, $y \in \tilde{Q}$. Thus for a fixed $y \in \tilde{Q}$ we have $\operatorname{tr} S(x_i, h) \times (1-y) = 0$ so since G is large $\operatorname{tr} T(GF(q^\infty)_n)(1-y) = 0$ and hence $1-y$ is a scalar matrix. But then y is a scalar q-element so $y = 1$ and $\tilde{Q} = \langle 1 \rangle$, a contradiction since we assumed Q is infinite.

Finally suppose that the Sylow q-subgroups of G are finite. Then by the Brauer-Feit result, Proposition 4.4, G has a normal abelian subgroup B of finite index. Since $n > 1$ B cannot be irreducible and hence by Lemma 6.1 (iii) B consists of scalar matrices and is central in G. Now for any $x \in G$ we have $S(x, B) = 0 \neq T(GF(q^\infty)_n)$ so G must consist of scalar matrices, a contradiction since $n > 1$ and G is irreducible. Thus $\pi_{P_0}(JK[G]) \subseteq JK[P_0]$ and the theorem is proved.

We remark that the assumption of largeness is not as restrictive as it might seem. For example if one wished to study linear groups inductively on the dimension of FG as in Lemma 3.3 then the limiting groups in which induction does not work might be expected to be large. We will see this below at least when $n = 2$.

In addition the assumption $G \subseteq SL_n(F)$ in the above is not very restrictive in view of Lemma 3.2. Finally we could of course neaten the definition of large by assuming that G has no proper subgroups of finite index. We could safely do this in view of Lemma 3.5. We now consider subgroups of $GL_2(F)$.

LEMMA 6.3. *Let $G \subseteq GL_2(F)$ with F algebraically closed and let M be a subspace of F_2. Suppose $T(F_2) > M > 0$ and $g^{-1}Mg = M$ for all $g \in G$. Then either M consists of scalar matrices* (which can only occur for $q = 2$) *or G has a subgroup of finite index which is reducible.*

Proof. Let $u_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then it is easy to see that $g^{-1}u_1g \in Fu_1$ implies that g is upper triangular and $g^{-1}u_2g \in Fu_2$ implies that g is diagonal. Hence if G normalizes either Fu_1 or Fu_2 then G is reducible. We observe in general that if $s \in GL_2(F)$ then G normalizes M if and only if $s^{-1}Gs$ normalizes $s^{-1}Ms$. Thus we can freely modify M by conjugation. Since $\dim T(F_2) = 3$ we have $\dim M = 1$ or 2.

Suppose first that $\dim M = 2$. Then it follows immediately that for some nonscalar matrix τ

$$M = \{\alpha \in T(F_2) \mid \operatorname{tr} \alpha\tau = 0\}.$$

If τ has distinct eigenvalues then by conjugating we may assume τ is diagonal and then M is the set of all matrices of the form $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$. Observe now that M contains precisely two 1-dimensional subspaces of singular matrices namely Fu_1 and Fu_3 where $u_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Since conjugation preserves rank G must permute these two and G has a subgroup of index $\leqq 2$ which normalizes Fu_1 and is therefore reducible. If τ has distinct eigenvalues then by conjugating we may assume that $\tau = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$ and then M is the set of all matrices of the form $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$. Since Fu_1 is then the unique subspace of M of singular matrices we see that G normalizes Fu_1 and is reducible.

Now let $\dim M = 1$ so that $M = F\tau$ with τ nonscalar. By conjugating we may assume that τ is diagonal or $\begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$. Observe that G normalizes $M + S$ where S is the set of scalar matrices. If τ is diagonal then $M + S$ consists of all the diagonal matrices. Thus $M + S$ contains precisely two subspaces of singular matrices one of which is Fu_2. It follows that a subgroup of G of index $\leqq 2$ normalizes Fu_2 and is therefore reducible. Finally if $\tau = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$ then $M + S$ consists of all matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ so Fu_1 is its unique subspace of singular matrices and the lemma is proved.

LEMMA 6.4. *Let $G \subseteq GL_2(F)$ with F algebraically closed. Then either G is large or G has a subgroup of finite index which is reducible.*

Proof. Suppose G is not large and choose $x \in G$ a nonscalar matrix and $H \subseteq G$ a subgroup of finite index with $S(x, H) \neq T(F_2)$. Then $T(F_2) > S(x, H) \supseteq 0$ and $S(x, H)$ is clearly normalized by H. Thus if $S(x, H)$ is not contained in the scalar matrices then by Lemma 6.3 applied to H we see that H has a reducible subgroup of finite index. Finally if $S(x, H)$ consists of scalar matrices then for all $h \in H$, $x^h = x + \lambda I$ for some $\lambda \in F$. Since $\det x^h = \det x$ there are at most two possible values for λ. Therefore $[H: \mathbf{C}_H(x)] \leqq 2$ and $\mathbf{C}_H(x)$ is reducible since its centralizer contains the nonscalar matrix x.

As an application we have for example

PROPOSITION 6.5. *Let G be a locally finite subgroup of $GL_2(F)$ with* char $F = q > 0$. *Let K be a field of characteristic $p \neq q$ and suppose* $\mathbf{O}_p(G) = \langle 1 \rangle$. *Then $JK[G]$ is nilpotent.*

Proof. We may clearly assume that F is algebraically closed and by Lemma 3.2 we may assume that $G \subseteq SL_2(F)$. Hence if G is large in $GL_2(F)$ then Theorem 6.2 yields the result. On the other hand if G is not large then by Lemma 6.4 G has a normal subgroup H of finite index which is reducible. Then H has a normal Sylow q-subgroup Q with abelian quotient. If Q is finite then $\mathbf{C}_H(Q)$ is a normal nilpotent subgroup of G of finite index. Since $\mathbf{O}_p(G) = \langle 1 \rangle$, $\mathbf{C}_H(Q)$ is a p'-group so its group ring is semisimple. On the other hand if Q is infinite then we have easily here $D = \mathbf{D}_H(Q)$ centralizes Q. Thus again D is a p'-group and by Lemma 1.7 $JK[H] = JK[D] \cdot K[H] = 0$. Therefore in either case G has a subgroup of finite index with a semisimple group ring and Lemma 16.8 of [2] yields the result.

We remark that there is no real difficulty in dropping the $\mathbf{O}_p(G) = \langle 1 \rangle$ assumption in the above. The following is certainly not surprising.

LEMMA 6.6. *Let F be an infinite field. Then $SL_n(F)$ is large in $GL_n(F)$.*

Proof. First $PSL_n(F)$ is simple and infinite so $SL_n(F)$ has no proper subgroups of finite index. Let $G = SL_n(F)$ and let $x \in G$ be any nonscalar matrix. Now all vectors cannot be eigenvectors for x so choose v_1 so that $v_2 = xv_1 \notin Fv_1$. If we then extend v_1, v_2 to a basis of the space V being acted on, then by a conjugation in G we may assume x has the form

$$x = \begin{pmatrix} 0 \ \ r \ \ 0 \ \cdots \ 0 \\ \hline * \end{pmatrix}$$

with $r \neq 0$. Since F is infinite choose $\alpha \in F$, $\alpha \neq 0$ with $\alpha^{2n} \neq 1$ and set

$$d_1 = \operatorname{diag}(\alpha^{-(n-1)}, \alpha, \alpha, \cdots, \alpha)$$

$$d_2 = \operatorname{diag}(\alpha, \alpha^{-(n-1)}, \alpha, \cdots, \alpha).$$

Then $d_1, d_2 \in G$ and $\alpha^{-(n-1)} \neq \alpha$ since $\alpha^n \neq 1$.

Now it is easy to see that $y = d_1^{-1}xd_1 - x$ looks like

$$y = \left(\begin{array}{c|c} 0 & s \ \ 0 \ \cdots \ 0 \\ \hline * & 0 \end{array}\right)$$

with $s \neq 0$ and then $z = d_2^{-1} y d_2 - y$ looks like

$$z = \left(\begin{array}{cc|c} 0 & a & \\ b & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)$$

with $a \neq 0$. Moreover since $\alpha^{2n} \neq 1$

$$\alpha^n d_1^{-1} z d_1 - z = a(\alpha^{2n} - 1) e_{12}$$

so the matrix unit e_{12} is contained in $S(x, G)$.

Finally by an appropriate permutation of the basis effected by conjugation in G, e_{12} is conjugate to any ce_{ij} for some $c \neq 0$ and any $i \neq j$. So all e_{ij} with $i \neq j$ are in $S(x, G)$. Since e_{ij} is conjugate via $1 + e_{ji}$ to $(e_{ij} - e_{ji}) + (e_{ii} - e_{jj})$ we conclude that $e_{ii} - e_{jj} \in S(x, G)$. Thus $S(x, G) \supseteq T(F_n)$ and the lemma is proved.

In our last result we drop our assumption that groups are locally finite or that char $F = q$. However the only new results here concern those particular cases.

PROPOSITION 6.7. *Let K be a field of characteristic $p > 0$ and let F be an infinite field of any characteristic.*

(i) *If $G = SL_n(F)$ or $GL_n(F)$ and if $P_0 = P_0(G)$ denotes the Sylow p-subgroup of the scalar matrices in G then*

$$JK[G] = JK[P_0] \cdot K[G].$$

(ii) *If $G = PSL_n(F)$ then $JK[G] = 0$.*

Proof. Suppose first that $F \not\subseteq GF(q^\infty)$ for some prime q possibly equal to p. Then the groups $G = SL_n(F)$, $GL_n(F)$ and $PSL_n(F)$ are not locally finite. By Theorems 4.4, 4.8 and 1.6 of **[4]**, $JK[G] = JK[\Lambda^+(G)] \cdot K[G]$ where $\Lambda^+(G)$ is a certain locally finite characteristic subgroup of G. The result now follows easily in this case.

Now let $F \subseteq GF(p^\infty)$ and apply Theorem 4.4 of **[4]** and Theorem 20.3 of **[2]**. It then follows immediately that for $G = PSL_n(F)$ we have $JK[G] = 0$. On the other hand if $G = SL_n(F)$ or $GL_n(F)$ then clearly $P_0 = \mathbf{O}_p(G)$ and G/P_0 has no finite normal subgroup whose order is divisible by p. Thus in this case we have easily $JK[G] = JK[P_0] \cdot K[G]$.

Finally let $F \subseteq GF(q^\infty)$ with $q \neq p$. By Lemma 6.6, $G = SL_n(F)$ is large in $GL_n(F)$ and hence by Theorem 6.2, $JK[G] = JK[P_0] \cdot K[G]$. In particular $JK[G]$ is nilpotent. Now let $G =$

$GL_n(G)$ and set $D = \mathbf{D}_G(SL_n(F))$. Then by the above and Lemma 1.7, D carries the radical of G. Since $SL_n(F)$ has no proper subgroups of finite index we have clearly $D = \mathbf{C}_G(SL_n(F))$ is the set of scalar matrices in G so the result follows here. In addition since the center of $SL_n(F)$ is finite we see that $K[PSL_n(F)]$ is a direct summand of the semisimple algebra $K[SL_n(F)/P_0]$. Thus $K[PSL_n(F)]$ is semisimple and the proposition is proved.

On the other hand, as was pointed out by A. E. Zalesskii, other types of classical groups are not large in general. For example let G be the orthogonal group with respect to transpose t so that

$$G = \{x \in GL_n(F) \mid x^t x = 1\}.$$

If $n \geqq 2$ then G contains the nonscalar symmetric matrix

$$x = \left(\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{0} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right)$$

where I is the $(n-2)\times(n-2)$ identity matrix. Since G normalizes the set of symmetric matrices we have clearly $S(x, G) \neq T(F_n)$ here and thus G is not large.

Added in proof. A. E. Zalesskii has suggested the following nice paraphrase of Lemma 1.5. The proof is essentially the same.

LEMMA 1.5′. *Let H be a finite subgroup of G and suppose that for all $x \in G$, $H \cap H^x$ contains an element of order p. Then $JK[G] \cap K[H] \neq 0$.*

REFERENCES

1. D. S. Passman, *Groups with normal, solvable Hall p′-subgroups*, Trans. Amer. Math. Soc., **123** (1966), 99–111.
2. ———, *Infinite Group Rings*, Marcel Dekker, New York, 1971.
3. ———, *On the semisimplicity of group rings of linear groups II*, Pacific J. Math., **48** (1973), 215–234.
4. ———, *A new radical for group rings?*, J. Algebra, **28** (1974), 556–572.
5. ———, *Radical ideals in group rings of locally finite groups*, J. Algebra.
6. B. A. F. Wehrfritz, *Infinite Linear Groups*, Springer-Verlag, Berlin, 1973.
7. A. E. Zalesskii, *The Jacobson radical of the group algebra of a solvable group is locally nilpotent* (in Russian), Izv. Akad. Nauk. SSSR, Ser. Mat.

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VARIETIES OF ORTHODOX BANDS OF GROUPS

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The principal aim of the present work is a determination of the lattice of all varieties of semigroups in the title as a direct product of the lattice of all varieties of bands and the lattice of all varieties of groups. The paper also contains certain information concerning lattice properties of these varieties and their defining identities.

1. Introduction. A considerable amount of literature is devoted to varieties of groups, a systematic study of this subject is the book **[7]** by H. Neumann. Varieties of semigroups have also attracted wide attention, most of the known results are summarized in the survey article **[2]** by T. Evans. The lattice of all varieties of bands was determined by Birjukov **[1]**, Gerhard **[4]** and Fennemore **[3]**; some preliminary work in this direction was first performed by Kimura **[6]** and the author **[8]**.

A semigroup S is *completely regular* if for any $a \in S$ there exists $x \in S$ such that $a = axa$, $ax = xa$. It follows at once that then there exists a unique $y \in S$ such that $a = aya$, $y = yay$, $ay = ya$; we write $a^{-1} = y$ and observe that S is a union of its (pairwise disjoint) maximal subgroups G_α and that for $a \in G_\alpha$, a^{-1} is the group inverse of a in G_α. We consider S as a universal algebra with two operations, viz., the binary operation of multiplication, and the unary operation of inversion, $a \to a^{-1}$, satisfying the identities

$$a = aa^{-1}a, \quad a^{-1} = a^{-1}aa^{-1}, \quad aa^{-1} = a^{-1}a. \tag{1}$$

The class $\mathscr{R}$ of all such universal algebras forms a variety. A semigroup S in $\mathscr{R}$ is *orthodox* if the set E_S of all its idempotents forms a subsemigroup. The class $\mathscr{E}$ of all orthodox semigroups in $\mathscr{R}$ is a subvariety of $\mathscr{R}$ and as such can be characterized by the identity

$$ab = abb^{-1}a^{-1}ab, \tag{2}$$

as follows easily from (**[9]**, IV.3.1). A semigroup S in $\mathscr{R}$ in which Green's relation $\mathscr{H}$ is a congruence is a *band of groups* and conversely. The class $\mathscr{F}$ of all bands of groups is a subvariety of $\mathscr{R}$ and as such can be characterized by the identity

$$(a^2bc^2)(a^2bc^2)^{-1} = (abc)(abc)^{-1}, \tag{3}$$

as follows easily from ([9], IV.1.7).

Let $\mathscr{C} = \mathscr{E} \cap \mathscr{F}$ so that $\mathscr{C}$ is the variety of universal algebras with an associative multiplication and an inversion satisfying the identities (1), (2), (3). In fact, we can define $\mathscr{C}$, within $\mathscr{R}$, by a single identity as follows.

PROPOSITION 1. *For any $S \in \mathscr{R}$, we have that $S \in \mathscr{C}$ if and only if S satisfies the identity*

$$(aa^{-1})(bb^{-1}) = (ab)(ab)^{-1}.$$

Proof. Necessity. Let $a, b \in S$, $S \in \mathscr{C}$. Then $a\mathscr{H}aa^{-1}$, $b\mathscr{H}bb^{-1}$ and hence $ab\mathscr{H}a^{-1}bb^{-1}$ since Green's relation $\mathscr{H}$ is a congruence. Now $ab\mathscr{H}(ab)(ab)^{-1}$ so that $(aa^{-1})(bb^{-1})\mathscr{H}(ab)(ab)^{-1}$. But E_S is a subsemigroup of S and $aa^{-1}, bb^{-1}, (ab)(ab)^{-1} \in E_S$ and thus $aa^{-1}bb^{-1} = (ab)(ab)^{-1}$.

Sufficiency. Let $a, b, c \in S$, $S \in \mathscr{R}$ and $a\mathscr{H}b$. Then $aa^{-1} = bb^{-1}$ and hence

$$(ac)(ac)^{-1} = (aa^{-1})(cc^{-1}) = (bb^{-1})(cc^{-1}) = (bc)(bc)^{-1},$$

i.e., $ac\mathscr{H}bc$. This shows that $\mathscr{H}$ is a right congruence. The proof that $\mathscr{H}$ is a left congruence is similar. If $e, f \in E_S$, then $ef = ee^{-1}ff^{-1} = (ef)(ef)^{-1}$, so that $(ef)^2 = ef$. Thus E_S is a subsemigroup of S.

The class $\mathscr{B}$ of all bands is evidently a subvariety of $\mathscr{C}$ and as such can be characterized by the identity $a = a^2$. The class $\mathscr{G}$ of all groups is another subvariety of $\mathscr{C}$ and as such can be characterized by the identity $aa^{-1} = bb^{-1}$. If $\mathscr{V}$ is any variety of universal algebras, $\mathscr{L}(\mathscr{V})$ *denotes the lattice, under inclusion, of all subvarieties of* $\mathscr{V}$. One of the principal results of this paper states that

$$\mathscr{L}(\mathscr{C}) \cong \mathscr{L}(\mathscr{B}) \times \mathscr{L}(\mathscr{G}).$$

We will also establish certain properties of some subvarieties of $\mathscr{C}$. In addition to the notation established above, we will use the notation, terminology and results from [9]. The meet in all our lattices will be the set theoretical intersection, the join will vary and will be denoted by $\vee$. For any semigroup S, we denote by E_S the set of all idempotents of S with the partial multiplication induced by S. For $e \in E_S$, G_e denotes the maximal subgroup of S having e as its identity.

2. Main result. The following lemma is crucial for a large portion of this paper.

LEMMA 1. *For any $\mathcal{V} \in \mathcal{L}(\mathcal{C})$, we have*

$$\mathcal{V} = (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G}).$$

Proof. Let $S \in \mathcal{V}$. According to **[10]**, S is a subdirect product of a band B and a semilattice of groups $T = \bigcup_{\alpha \in Y} G_\alpha$. Since B is then a homomorphic image of S, we have $B \in \mathcal{V} \cap \mathcal{B}$. The conjunction of (**[9]**, IV.4.3) and (**[9]**, III.7.2) yields that T is a subdirect product of semigroups $\{T_\alpha\}_{\alpha \in Y}$, where either $T_\alpha \cong G_\alpha$ or $T_\alpha \cong G_\alpha^0$, the group G_α with a zero adjoined. Since T is a homomorphic image of S, we have $T \in \mathcal{V}$ and thus also $G_\alpha \in \mathcal{V}$ for all $\alpha \in Y$.

Assume that S is completely simple. Then $S \cong L \times G \times R$ where L is a left zero semigroup, G is a group and R is a right zero semigroup, according to (**[9]**, IV.3.3). Clearly $L \times R \in \mathcal{V} \cap \mathcal{B}$ and $G \in \mathcal{V} \cap \mathcal{G}$ and thus $S \in (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G})$.

Suppose next that S is not completely simple. It is easy to see that in S we can find two comparable idempotents, say $e > f$. But then $Y_2 = \{0, 1\}$, the two-element chain, must be contained in $\mathcal{V}$. Now let $G \in \mathcal{V} \cap \mathcal{G}$, and let ρ be the Rees congruence on $Y_2 \times G$ associated with the kernel $\{0\} \times G$ of $Y_2 \times G$. It follows that

$$G^0 \cong (Y_2 \times G)/\rho \in (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G}).$$

We have seen above that T is a subdirect product of semigroups T_α where either $T_\alpha \cong G_\alpha$ or $T_\alpha \cong G_\alpha^0$. Consequently $T \in (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G})$. Finally S is a subdirect product of B and T and thus $S \in (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G})$.

Therefore $\mathcal{V} \subseteq (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G})$, the opposite inclusion is trivial.

LEMMA 2. *Let T be a completely regular semigroup which is a subdirect product of a band B and a group G, and let S be a band and a homomorphic image of T. Then S is a homomorphic image of B.*

Proof. Let φ be a homomorphism of T onto S. We may suppose that $T \subseteq B \times G$. Let $(b, g), (b, h) \in T$ and let $\bar{g} = (b, g)\varphi$, $\bar{h} = (b, h)\varphi$. Since T is completely regular, we have $(b, h^{-1}) \in T$, and thus

$$\begin{aligned}\bar{g} = (b, g)\varphi &= [(b, h)(b, h^{-1})(b, g)]\varphi = (b, h)\varphi(b, h^{-1})\varphi(b, g)\varphi \\ &= (b, h)\varphi(b, g)\varphi = \bar{h}\bar{g}.\end{aligned}$$

A similar argument shows that $\bar{h} = \bar{h}\bar{g}$ and thus $\bar{g} = \bar{h}$. It follows that the mapping ψ defined by

$$\psi: b \to (b, g)\varphi \quad \text{if} \quad (b, g) \in T \quad (b \in B)$$

is single-valued, and thus evidently a homomorphism of B onto S.

LEMMA 3. *Let T be a semigroup which is a subdirect product of a band B and a semigroup C, and let S be a left cancellative semigroup and a homomorphic image of T. Then S is a homomorphic image of C.*

Proof. Let φ be a homomorphism of T onto S, and suppose that $T \subseteq B \times C$. Let $(a, c), (b, c) \in T$. Then

$$(a, c^2)(b, c) = (ab, c^3) = ((ab)b, c^3) = (ab, c^2)(b, c)$$

where $(a, c^2), (ab, c^2) \in T$ and thus

$$[(a, c)\varphi]^2 = (a, c^2)\varphi = (ab, c^2)\varphi = (a, c)\varphi(b, c)\varphi.$$

Left cancellation in S now implies that $(a, c)\varphi = (b, c)\varphi$. It follows that the mapping ψ defined by

$$\psi: c \to (a, c) \quad \text{if} \quad (a, c) \in T \quad (c \in C)$$

is single-valued, and thus evidently a homomorphism of C onto S.

THEOREM. *The mapping χ defined by*

$$\chi: \mathcal{V} \to (\mathcal{V} \cap \mathcal{B}, \mathcal{V} \cap \mathcal{G}) \qquad (\mathcal{V} \in \mathcal{C})$$

is an isomorphism of $\mathcal{L}(\mathcal{C})$ onto $\mathcal{L}(\mathcal{B}) \times \mathcal{L}(\mathcal{G})$.

Proof. It is obvious that χ is inclusion preserving. Let $\mathcal{V}' \in \mathcal{L}(\mathcal{B})$ and $\mathcal{V}'' \in \mathcal{L}(\mathcal{G})$, and let $\mathcal{V} = \mathcal{V}' \vee \mathcal{V}''$. Then

$$\mathcal{V} \cap \mathcal{B} = (\mathcal{V}' \vee \mathcal{V}'') \cap \mathcal{B} \supseteq \mathcal{V}' \cap \mathcal{V} = \mathcal{V}''.$$

In order to establish the opposite inclusion, we let $S \in \mathcal{V} \cap \mathcal{B}$. In view of ([**5**], §23, Theorem 3), there exist $B \in \mathcal{V}'$, $G \in \mathcal{V}''$, a completely regular semigroup T which is a subdirect product of B and G, and a homomorphism φ of T onto S. Hence by Lemma 2, S is a homomorphic image of B and thus $S \in \mathcal{V}'$. Consequently $\mathcal{V} \cap \mathcal{B} = \mathcal{V}'$. A similar argument, using Lemma 3, shows that $\mathcal{V} \cap \mathcal{G} = \mathcal{V}''$. It follows that $\mathcal{V} \in \mathcal{C}$ and that $\mathcal{V}\chi = (\mathcal{V}', \mathcal{V}'')$, proving that χ maps $\mathcal{L}(\mathcal{C})$ onto $\mathcal{L}(\mathcal{B}) \times \mathcal{L}(\mathcal{G})$. Now Lemma 1 easily implies that χ is one-to-one and that χ^{-1} is inclusion preserving. Therefore χ is a lattice isomorphism.

COROLLARY. *For any* $S \in \mathcal{C}$, $\mathcal{V}' \in \mathcal{L}(\mathcal{B})$, $\mathcal{V}'' \in \mathcal{L}(\mathcal{G})$, *we have* $S \in \mathcal{V}' \vee \mathcal{V}''$ *if and only if* $E_S \in \mathcal{V}'$ *and* $G_e \in \mathcal{V}''$ *for all* $e \in E_S$.

Proof. This follows without difficulty from the proof of the theorem and the proof of Lemma 1.

3. Further results.

We consider first the following problem: if $\mathcal{V}' \in \mathcal{L}(\mathcal{B})$ and $\mathcal{V}'' \in \mathcal{L}(\mathcal{G})$ are given by their defining identities, can we set up a system of defining identities for $\mathcal{V}' \vee \mathcal{V}''$? We now proceed to describe such a system.

Let $u = v$ be an identity on $\mathcal{B}$. Substitute every variable x that occurs in $u = v$ by xx^{-1}. We then obtain an identity on $\mathcal{C}$, to be denoted by $\bar{u} = \bar{v}$.

Let $w = z$ be an identity on $\mathcal{G}$. We may suppose that both w and z contain the same set $\{x_1, x_2, \cdots, x_n\}$ of variables. Consider $w = z$ as an identity on $\mathcal{C}$, and let $e = (x_1, x_2 \cdots x_n)(x_1x_2 \cdots x_n)^{-1}$. Substitute each occurrence of x_i in $w = z$ by ex_ie. We then obtain an identity on $\mathcal{C}$, to be denoted by $\hat{w} = \hat{z}$. Note that e depends on the choice of writing the variables, but any single choice will do.

PROPOSITION 2. *Let* $\mathcal{V}'$ (resp. $\mathcal{V}''$) *be the variety of bands* (resp. groups) *defined by a system of identities* $\{u_\alpha = v_\alpha\}$ (resp. $\{w_\beta = z_\beta\}$). *Then* $\mathcal{V}' \vee \mathcal{V}''$ *can be defined by the system* $\{\bar{u}_\alpha = \bar{v}_\alpha, \hat{w}_\beta = \hat{z}_\beta\}$.

Proof. By the above corollary, $\mathcal{V} = \mathcal{V}' \vee \mathcal{V}''$ consists of all $S \in \mathcal{C}$ for which $E_S \in \mathcal{V}'$ and $G_e \in \mathcal{V}''$ for all $e \in E_S$. Let $S \in \mathcal{V}$. Then E_S satisfies $u_\alpha = v_\alpha$ and hence S satisfies $\bar{u}_\alpha = \bar{v}_\alpha$. Next consider $w_\beta = z_\beta$. Let $\{x_1, x_2, \cdots, x_n\}$ be the set of variables occurring in $w_\beta = z_\beta$. For any $a \in S$, we denote by N_a the class of the least semilattice congruence on S containing a. Let $a_1, a_2, \cdots, a_n \in S$ and $e = (a_1a_2 \cdots a_n)(a_1a_2 \cdots a_n)^{-1}$. Then for any $1 \leqq i \leqq n$, $ea_ie \in G_e$ since

$$ea_ie \in N_{ea_ie} = N_eN_{a_i}N_e = N_e$$

and N_e is completely simple. Observing that each G_e satisfies the identity $w_\beta = z_\beta$, we deduce that S satisfies the identity $\hat{w}_\beta = \hat{z}_\beta$. Consequently each $S \in \mathcal{V}$ satisfies all the identities $\bar{u}_\alpha = \bar{v}_\alpha$, $\hat{w}_\beta = \hat{z}_\beta$.

Conversely, let $S \in \mathscr{C}$ satisfy all the identities $\bar{u}_\alpha = \bar{v}_\alpha$, $\hat{w}_\beta = \hat{z}_\beta$. Then E_S satisfies each $u_\alpha = v_\alpha$ so that $E_S \in \mathscr{V}'$. Further, for every $e \in E_S$, G_e satisfies each $\hat{w}_\beta = \hat{z}_\beta$, and hence also $w_\beta = z_\beta$ since G_e has only one idempotent. Thus $G_e \in \mathscr{V}''$. By the above corollary we conclude that $S \in \mathscr{V}' \vee \mathscr{V}'' = \mathscr{V}$.

For example, if $\mathscr{V}'$ is the variety of all rectangular bands and $\mathscr{V}''$ the variety of all groups, then $\mathscr{V} = \mathscr{V}' \vee \mathscr{V}''$ can be defined by the identity $xx^{-1} = xx^{-1}yy^{-1}xx^{-1}$, which is evidently equivalent to $x^2 = xyy^{-1}x$. This identity defines the subvariety of rectangular groups.

As another example, we may take $\mathscr{V}'$ to be the variety of all bands and $\mathscr{V}''$ the variety of all abelian groups. Then $\mathscr{V} = \mathscr{V}' \vee \mathscr{V}''$ can be defined by the identity $xx^{-1}xyxx^{-1} = xx^{-1}yxxx^{-1}$, which is evidently equivalent to $x^2yx = xyx^2$. This identity defines the subvariety of orthodox bands of abelian groups.

We consider next the following question: which subvarieties of $\mathscr{C}$ are simultaneously subvarieties of the variety $\mathscr{S}$ of all semigroups? For an identity $u = v$ on $\mathscr{S}$, we denote by $[u = v]$ the variety of semigroups defined by $u = v$. If x is an element of a semigroup S, $\langle x \rangle$ denotes the cyclic subsemigroup of S generated by x.

PROPOSITION 3. *The following conditions on a subvariety $\mathscr{V}$ of $\mathscr{C}$ are equivalent.*

(i) $\mathscr{V} \in \mathscr{L}(\mathscr{S})$.

(ii) $\mathscr{V} \subseteq [x = x^n]$ *for some integer* $n > 1$.

(iii) $\mathscr{V} \cap \mathscr{G} \subseteq [x = x^n]$ *for some integer* $n > 1$.

(iv) $\mathscr{V} \cap \mathscr{G} \in \mathscr{L}(\mathscr{S})$.

Proof. (i) *implies* (ii). Let $x \in S$ and $S \in \mathscr{V}$. Then $\langle x \rangle \in \mathscr{V}$ since $\mathscr{V} \in \mathscr{L}(\mathscr{S})$. But then $\mathscr{V} \in \mathscr{L}(\mathscr{C})$ implies that $\langle x \rangle \in \mathscr{C}$ which is possible only if $\langle x \rangle$ is a finite group. Hence $x = x^n$ for some $n > 1$. Assume that the set

$$\{n(x) \mid x^{n(x)} = x, \quad x \in S, \quad S \in \mathscr{V}\}$$

is unbounded. Hence there exists an infinite sequence $\langle x_1 \rangle, \langle x_2 \rangle, \cdots$ of cyclic semigroups such that $n(x_1) < n(x_2) < \cdots$. The element (x_i) of the direct product $S = \Pi_{i=1}^{\infty} \langle x_i \rangle$ is clearly of infinite order. Since $S \in \mathscr{V}$, this contradicts to what we have proved above. Thus there exists $n > 1$ such that $\mathscr{V} \subseteq [x = x^n]$.

Items (ii) and (iii) are obviously equivalent. Item (ii) implies item (i) since with $x = x^n$ in any semigroup S, x^{n-1} is the identity of the cyclic group generated by x. The equivalence of (iii) and (iv) follows similarly as the equivalence of (i) and (ii).

We now elucidate another relationship between $\mathscr{L}(\mathscr{C})$ and $\mathscr{L}(\mathscr{S})$.

PROPOSITION 4. *Let $\mathscr{V} \in \mathscr{L}(\mathscr{C})$ and $\mathscr{U} \in \mathscr{L}(\mathscr{S})$. Then $\mathscr{U} \cap \mathscr{C} = \mathscr{V}$ if and only if $\mathscr{U} \cap \mathscr{G} = \mathscr{V} \cap \mathscr{G}$ and $\mathscr{U} \cap \mathscr{B} = \mathscr{V} \cap \mathscr{B}$.*

Proof. If $\mathscr{U} \cap \mathscr{C} = \mathscr{V}$, then

$$\mathscr{U} \cap \mathscr{G} = \mathscr{U} \cap (\mathscr{G} \cap \mathscr{C}) = (\mathscr{U} \cap \mathscr{C}) \cap \mathscr{G} = \mathscr{V} \cap \mathscr{G}$$

and analogously $\mathscr{U} \cap \mathscr{B} = \mathscr{V} \cap \mathscr{B}$.

Conversely, suppose that $\mathscr{U} \cap \mathscr{G} = \mathscr{V} \cap \mathscr{G}$ and $\mathscr{U} \cap \mathscr{B} = \mathscr{V} \cap \mathscr{B}$. The join in $\mathscr{C}$ will be now denoted by $\overset{c}{\vee}$ and the join in $\mathscr{S}$ by $\overset{s}{\vee}$. Using Lemma 1, we obtain

$$(\mathscr{V} \cap \mathscr{B}) \overset{c}{\vee} (\mathscr{U} \cap \mathscr{G}) = (\mathscr{V} \cap \mathscr{B}) \overset{c}{\vee} (\mathscr{V} \cap \mathscr{G}) = \mathscr{V}$$

so that

$$\mathscr{V} \subseteq ((\mathscr{V} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U}) \cap \mathscr{C}. \tag{4}$$

In order to establish the opposite inclusion, we first let $G \in ((\mathscr{V} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U}) \cap \mathscr{G}$. In view of (**[5]**, §23, Theorem 3), there exist $B \in \mathscr{V} \cap \mathscr{B}$, $C \in \mathscr{U}$, a subdirect product T of B and C and a homomorphism of T onto G. By Lemma 3, G is a homomorphic image of C and thus $G \in \mathscr{U}$. Consequently $G \in \mathscr{U} \cap \mathscr{G} = \mathscr{V} \cap \mathscr{G}$. Next let $B \in ((\mathscr{V} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U}) \cap \mathscr{B}$. Then

$$B \in ((\mathscr{U} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U}) \cap \mathscr{B} = \mathscr{U} \cap \mathscr{B} = \mathscr{V} \cap \mathscr{B}.$$

It follows that

$$[((\mathscr{V} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U}) \cap \mathscr{B}] \overset{c}{\vee} [((\mathscr{V} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U}) \cap \mathscr{G}] \subseteq \mathscr{V}$$

and thus by Lemma 1, we have

(5)
$$((\mathscr{V} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U}) \cap \mathscr{C} \subseteq \mathscr{V}.$$

The conjunction of (4) and (5) yields

(6)
$$((\mathscr{V} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U}) \cap \mathscr{C} = \mathscr{V},$$

where $(\mathscr{V} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U} \in \mathscr{L}(\mathscr{S})$. But $\mathscr{V} \cap \mathscr{B} = \mathscr{U} \cap \mathscr{B}$ implies that $(\mathscr{V} \cap \mathscr{B}) \overset{s}{\vee} \mathscr{U} = \mathscr{U}$ which by (6) gives $\mathscr{U} \cap \mathscr{C} = \mathscr{V}$, as required.

Note that Proposition 4 implies the following statement: if $\mathscr{V} \in \mathscr{L}(\mathscr{C})$ and $\mathscr{V} = \mathscr{U} \cap \mathscr{C}$ for some $\mathscr{U} \in \mathscr{L}(\mathscr{S})$, then $\mathscr{V} \cap \mathscr{G} = \mathscr{U} \cap \mathscr{G}$. A converse of this statement can be phrased thus: If $\mathscr{V}' \in \mathscr{L}(\mathscr{B})$ and $\mathscr{U} \in \mathscr{L}(\mathscr{S})$, does there exist $\mathscr{V} \in \mathscr{L}(\mathscr{S})$ such that

$$\mathscr{V}' \overset{c}{\vee} (\mathscr{U} \cap \mathscr{G}) = \mathscr{V} \cap \mathscr{C}?$$

An answer to this question is open. However, we have the following simple result. For any class $\mathscr{D}$ of semigroups, let $\mathscr{D}_{\mathscr{S}}$ denote the variety of semigroups generated by $\mathscr{D}$.

PROPOSITION 5. *Let $\mathscr{V} \in \mathscr{L}(\mathscr{C})$. Then there exists $\mathscr{U} \in \mathscr{L}(\mathscr{S})$ such that $\mathscr{V} = \mathscr{U} \cap \mathscr{C}$ if and only if $\mathscr{V}_{\mathscr{S}} \cap \mathscr{C} = \mathscr{V}$.*

Proof. Necessity. Let $S \in \mathscr{V}_{\mathscr{S}} \cap \mathscr{C}$. According to (**[5]**, §23, Theorem 3) S is a homomorphic image of a subsemigroup T of some semigroup H in $\mathscr{V}$. It follows that $H \in \mathscr{U} \cap \mathscr{C}$ and thus $T \in \mathscr{U}$ and hence also $S \in \mathscr{U}$. Consequently $S \in \mathscr{U} \cap \mathscr{C} = \mathscr{V}$. This proves that $\mathscr{V}_{\mathscr{S}} \cap \mathscr{C} \subseteq \mathscr{V}$, the opposite inclusion is trivial.

Sufficiency. Take $\mathscr{U} = \mathscr{V}_{\mathscr{S}}$.

For example, for $\mathscr{V} = \mathscr{G}$ or the varieties of all left, right or rectangular groups, we have the inequality $\mathscr{V}_{\mathscr{S}} \cap \mathscr{C} \neq \mathscr{V}$. This shows, in particular, that these subvarieties of $\mathscr{C}$ cannot be defined, within $\mathscr{C}$, by semigroup identities alone. To see this, let G be the additive group of all integers, T the subsemigroup of G consisting of all nonnegative integers, S the multiplicative semigroup $\{0, 1\}$, and φ be the mapping defined by: $0\varphi = 1$, $n\varphi = 0$ for all $n \in T$, $n \neq 0$. Then $S \notin \mathscr{V}$ and $S \in \mathscr{V}_{\mathscr{S}} \cap \mathscr{C}$.

REFERENCES

1. A. P. Birjukov, *Varieties of idempotent semigroups*, Algebra i Logika, **9** (1970), 255–273 (in Russian); Transl. Algebra and Logic, Consult. Bureau, **9** (1970), 153–164.
2. T. Evans, *The lattice of semigroup varieties*, Semigroup Forum, **2** (1971), 1–43.
3. C. F. Fennemore, *All varieties of bands*, Math. Nachr., **48** (1971); I:237–252, II:253–262.
4. J. A. Gerhard, *The lattice of equational classes of idempotent semigroups*, J. Algebra, **15** (1970), 195–224.
5. G. Grätzer, *Universal Algebra*, Van Nostrand, Princeton, 1968.
6. N. Kimura, *Note on idempotent semigroups. IV. Identities of three variables*, Proc. Japan Acad., **24** (1958), 121–123.
7. H. Neumann, *Varieties of groups*, Erg. Math. u.i. Grenzg., Vol. **37**, Springer, Berlin, 1967.
8. M. Petrich, *A construction and a classification of bands*, Math. Nachr., **48** (1971), 263–274.
9. ———, *Introduction to Semigroups*, Merill, Columbus, 1973.
10. B. M. Schein, *A note on radicals in regular semigroups*, Semigroup Forum, **3** (1971), 84–85.

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THE GENERALIZED INTERVAL TOPOLOGY ON DISTRIBUTIVE LATTICES

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The author has recently introduced the generalized interval topology on a partially ordered set as an alternative to the standard interval topology. In this paper, the structure of generalized segments in lattices is investigated, and sufficient conditions are given for the generalized interval topology on a distributive lattice to be a lattice topology; adding another condition ensures that the topology is Hausdorff. Similar results are obtained for a slight modification of the generalized interval topology, the generalized star-interval topology, and examples are constructed which illustrate less restrictive situations.

1. Introduction; terminology and notation. In [6], we introduced the concept of generalized intervals in a partially ordered set and showed that they could be used in a natural way to define a topology, called the generalized interval topology, on the set. The definition we used was based on one for intervals, which was given by Frink in [4], and which formally extended the "closed set" definition of the usual interval topology on a totally ordered set to an arbitrary partially ordered set. The use of generalized intervals in place of intervals in Frink's definition did not change the topology on unbounded, totally ordered sets; however, on cardinal products of dually (i.e. both upwards and downwards) directed sets, the generalized interval topology turned out to be not only different from Frink's interval topology but in fact precisely the product of the generalized interval topologies on the factors.

In this paper, we investigate the possible continuity of the lattice operations with respect to the generalized interval topology on a distributive lattice, and give conditions which ensure that the topology is Hausdorff. The definition of generalized intervals adds to the corresponding standard interval certain "relatively perpendicular" elements. The motivation for the definition stems from the plane, where one may consider the set

$$\{(x, y) \mid 1 \leqq y \leqq -1\}$$

to be an interval rather than the usual set

$$\{(0, y) \mid 1 \leqq y \leqq -1\}.$$

The standard polar of an l-group was used to describe these "relatively perpendicular" elements in **[5]**, and thus pointed the way to the definition in **[6]** of upper and lower polars in any partially ordered set. To obtain the necessary machinary to ensure that a distributive lattice has a Hausdorff generalized interval topology which is also a lattice topology, we investigate upper and lower polars (§2) and generalized segments (§3), in some detail. The major results (Theorems 4.3 and 5.2) are proven for certain distributive lattices which, whenever possible, have nontrivial polars that are "minimal" in a natural sense.

Intervals may not be closed with respect to the generalized interval topology. Thus, in **[6]**, we considered the generalized star-interval topology, which for a directed set is just the topology generated by the interval topology and the generalized interval topology. Most of the machinary developed here is valid for star-polars and generalized star-intervals as well as for polars and generalized intervals, and thus only a slight change of hypotheses might be needed to ensure that the main results for the generalized interval topology could be proved directly for the generalized star-interval topology. However, we prefer to use connections, established here and in **[6]**, between the generalized interval and star-interval topologies, to obtain the results for the generalized star-interval topology as corollaries of the results for the generalized interval topology.

Terminology left undefined here may be found in **[1]**, **[2]**, and **[9]**.

Let $(P, \leqq)$ 1029 noted a partially ordered set. We use $\vee$ to indicate the least upper bound of two elements, if it exists. A statement of the form $a \vee b = c$ means that $a \vee b$ exists and equals c. We use $\wedge$ (greatest lower bound) similarly. Let $A, B \subseteq P, x, y \in P$. Then

$$u(A) = \{p \in P \mid p \geqq a \quad \text{for all} \quad a \in A\},$$
$$l(A) = \{p \in P \mid p \leqq a \quad \text{for all} \quad a \in A\},$$

$A \wedge B = \{a \wedge b \mid a \in A,\ b \in B\}$, $A \vee B = \{a \vee b \mid a \in A,\ b \in B\}$, $x \wedge B = \{x\} \wedge B$, $x \vee B = \{x\} \vee B$, $u(x, y) = u(\{x, y\})$, and $l(x, y) = l(\{x, y\})$.

We denote an *open interval* in P by

$$]a, b[= \{x \in P \mid a < x < b\},$$

and an *interval* (or *closed interval*) by

$$[a, b] = \{x \in P \mid a \leqq x \leqq b\}.$$

We may combine the notations, as in

$$]a, b] = \{x \mid a < x \leqq b\}; \quad [a, b[= \{x \mid a \leqq x < b\}.$$

An *initial segment* of P is a set of the form $(-\infty, r] = l(r)$ for some $r \in P$; a *final segment* of P is a set of the form $[r, \infty) = u(r)$ for some $r \in P$. Frink's *interval topology* **[4]** on P takes P and ϕ, together with all final and initial segments as a subbase for its closed sets. We denote the interval topology on P by $\mathcal{I}(P)$.

Let $\{P_\alpha \mid \alpha \in A\}$ be a collection of partially ordered sets. The *cardinal product* of the P_α, denoted by $|\Pi|\ \{P_\alpha \mid \alpha \in A\}$, is the Cartesian product of the P with order defined pointwise, i.e. by: $f \leqq g$ if and only if $\alpha f \leqq \alpha g$ for all $\alpha \in A$. If A is finite, say $A = \{1, 2, \cdots, n\}$, then we usually denote the cardinal product by $P_1 |\times| P_2 |\times| \cdots |\times| P_n$.

We consistently use *totally ordered set* to refer to a partially ordered set in which every two elements are comparable. If P is a partially ordered set, and if T is a totally ordered set, then the *lexicographic product* of P and T, denoted by $P \overleftarrow{\times} T$, is the product of P and T ordered by: $(a, b) \leqq (p, t)$ if and only if $b < t$, or $b = t$ and $a \leqq p$.

If G is an l-group, then for all $A \subseteq G$, $A^+ = \{a \in A \mid a \geqq 0\}$ and $A^- = \{a \in A \mid a \leqq 0\}$.

We let N be the natural numbers, Z the integers, and R the real numbers. Unless otherwise noted, N, Z, and R have their usual orders. By the *plane*, we mean $R |\times| R$.

If $\mathcal{T}$ is a topology on a set X, we use T_0, T_1, and *Hausdorff* to refer to the corresponding separation axioms in sense of **[9]**. If L is a lattice with topology $\mathcal{T}$, then $(L, \mathcal{T})$ is a *topological lattice* if both

$$\vee : (L \times L,\ \mathcal{T} \times \mathcal{T}) \rightarrow (L, \mathcal{T}),$$

$$\wedge : (L \times L,\ \mathcal{T} \times \mathcal{T}) \rightarrow (L, \mathcal{T}),$$

are continuous. Note that $(L, \mathcal{T})$ may be a topological lattice even if $\mathcal{T}$ is not Hausdorff.

2. Upper and lower polars. Upper and lower polars for a partially ordered set were introduced in **[6]** as a generalization of polars for an l-group. For the results of this paper, we need to look into the structure of these new polars more deeply than we did in **[6]**.

Let $(P, \leqq)$ be a partially ordered set. Suppose that $r, s, t \in P$ are such that $r \leqq s \leqq t$. The set

$$(s, t)_{\top} = \{p \in P \mid p \wedge t = s\}$$

is called the *upper polar of t with respect to s* (or the *s, t upper polar*). The set

$$(s,t)_{\top} = \{p \in P \mid p \vee r = s\}$$

is called the *lower polar of r with respect to s* (or the *s, r-lower polar*).

We noted in **[6]** that in an *l*-group G, for all $g > 0$,

$$(0,g)^{\perp} = (g^{\perp})^{+},$$
$$(0,-g)_{\top} = (g^{\perp})^{-},$$

where $g^{\perp} = \{h \in G \mid |h| \wedge g = 0\}$ is the standard polar for an *l*-group (see **[3]**, **[8]**, and **[5]**). Thus, in the plane,

$$((0,0),\ (0,1))^{\perp} = \{(x,0) \mid x \geqq 0\}.$$

For $r, s, t \in P$ with $r \leqq s \leqq t$, we define the *upper star-polar of t with respect to s*, denoted by $^*(s,t)^{\perp}$, to be $(s,t)^{\perp}$ if $s < t$, and $\{s\}$ if $s = t$. Similarly, the *lower star-polar of r with respect to s*, denoted by $^*(s,r)_{\top}$, is defined as $(s,r)_{\top}$ if $r < s$, and $\{s\}$ if $r = s$. All the results of this section will remain true if polars are replaced by star-polars.

PROPOSITION 2.1. *Let $(P, \leqq)$ be a partially ordered set, and let $r, s, t \in P$ be such that $r < s < t$. Then*

(i) $(r,t)^{\perp} \subseteq (r,s)^{\perp}$,
(ii) $(t,r)_{\top} \subseteq (t,s)_{\top}$.

If $(P, \leqq)$ is a modular lattice, then

(iii) $s \vee (r,t)^{\perp} \subseteq (s,t)^{\perp}$,
(iv) $s \wedge (t,r)_{\top} \subseteq (s,r)_{\top}$.

Proof. (i) Let $b \in (r,t)^{\perp}$. Then $b \wedge t = r$. Clearly $r \leqq b$ and $s \leqq t$. If $u \leqq b$ and $u \leqq s$, then $u \leqq b$ and $u \leqq t$, i.e. $u \leqq b \wedge t = r$. Thus $b \wedge s = r$, i.e. $b \in (r,s)^{\perp}$. Statement (ii) is the dual of (i). (iii) Let $b \in (r,t)^{\perp}$. Then $b \wedge t = r$, and hence

$$(s \vee b) \wedge t = s \vee (b \wedge t) = s \vee r = s,$$

i.e. $b \vee s \in (s,t)^{\perp}$. Statement (iv) is the dual of (iii).

PROPOSITION 2.2. *Let $(L, \leqq)$ be a distributive lattice. Let $r, s, t \in L$ be such that $r < s \leqq t$. Then the following statements are equivalent:*

(i) $r \wedge (t,r)_{\top} = r \wedge (s,r)_{\top}$,
(ii) $r \wedge (t,r)_{\top} \supseteq r \wedge (s,r)_{\top}$,
(iii) $s \wedge (t,r)_{\top} = (s,r)_{\top}$,
(iv) $s \wedge (t,r)_{\top} \supseteq (s,r)_{\top}$.

Proof. Clearly, if $s = t$, then statements (i) – (iv) are equivalent. Suppose that $s < t$. By Proposition 2.1, (iii) is equivalent to (iv). Clearly by Proposition 2.1,

$$r \wedge (t, r)_{\top} = r \wedge (s \wedge (t, r)_{\top}) \subseteq r \wedge (s, r)_{\top}.$$

Thus (i) is equivalent to (ii). Similarly, one may see that (iv) implies (ii). It remains to show that (ii) implies (iv). Suppose that (ii) holds, and let $x \in (s, r)_{\top}$. By (ii), there exists $b \in (t, r)_{\top}$ such that $r \wedge b = r \wedge x$. Then $x \vee r = s$, $b \vee r = t$, and since $t \geqq s \geqq x$, $t \wedge x = x$. Thus

$$\begin{aligned} s \wedge b &= (x \vee r) \wedge b \\ &= (x \wedge b) \vee (r \wedge b) \\ &= (x \wedge b) \vee (r \wedge x) \\ &= x \wedge (b \vee r) \\ &= x \wedge t = x, \end{aligned}$$

i.e., $x \in s \wedge (t, r)_{\top}$. Therefore, (iv) holds.

Since Proposition 2.2 holds, its dual also holds. We usually will not state the dual of any result explicitly, even though we may use it later on. As an example, however, we will write out the dual of Proposition 2.2:

PROPOSITION 2.3. *Let $(L, \leqq)$ be a distributive lattice. Let $r, s, t \in L$ be such that $r \leqq s < t$. Then the following statments are equivalent.*

(i) $t \vee (r, t)^{\perp} = t \vee (s, t)^{\perp}$,
(ii) $t \vee (r, t)^{\perp} \supseteq t \vee (s, t)^{\perp}$,
(iii) $s \vee (r, t)^{\perp} = (s, t)^{\perp}$,
(iv) $s \vee (r, t)^{\perp} \supseteq (s, t)^{\perp}$.

Let $(L, \leqq)$ be a lattice. Let $r, t \in L$ be such that $r < t$. The interval $[r, t]$ *has equivalent lower polars* if for all $r < s < t$,

$$s \wedge (t, s)_{\top} = (s, r)_{\top}.$$

Similarly, $[r, t]$ *has equivalent upper polars* if for all $r < s < t$,

$$s \vee (r, s)^{\perp} = (s, t)^{\perp}.$$

If $[r, s]$ has both equivalent lower polars and equivalent upper polars, then $[r, s]$ is said to *have equivalent polars.*

We note that in the plane $[r, t]$ has equivalent polars if and only if $[r, t]$ is totally ordered. However, if we let R^* be R with $-\infty$ and ∞

adjoined, and if $L = (R^*|x|R^*) \overleftarrow{\times} R$, then $[(\infty, \infty, -1), (-\infty, -\infty, 1)]$ has equivalent polars but is not totally ordered. Thus, non-totally ordered intervals may have equivalent polars, even in distributive lattices.

We note that replacing polars by star-polars does not change the above definitions.

Our next result, which will be very useful in the sequal, provides an alternate characterization of intervals which have equivalent lower polars.

PROPOSITION 2.4. *Let $(L, \leqq)$ be a distributive lattice. Let $r, t \in L$ be such that $r < t$. Then $[r, t]$ has equivalent lower polars if and only if for all $r \leqq s < t$, $(t, s)_{\top} = (t, r)_{\top}$, and for all $r < s \leqq t$, $r \wedge (t, r)_{\top} = r \wedge (s, r)_{\top}$.*

Proof. Suppose the conditions hold, and let $r < s < t$. Since $r \wedge (t, r)_{\top} = r \wedge (s, r)_{\top}$, then $s \wedge (t, r)_{\top} = (s, r)_{\top}$ by Proposition 2.2. Since $(t, s)_{\top} = (t, r)_{\top}$, this implies that $s \wedge (t, s)_{\top} = (s, r)_{\top}$. Therefore, $[r, t]$ has equivalent lower polars. Conversely, suppose that $[r, t]$ has equivalent lower polars. Clearly, it suffices to show that both conditions hold for $r < s < t$. By Proposition 2.1, $(t, s)_{\top} \supseteq (t, r)_{\top}$. Let $b \in (t, s)_{\top}$. Then $s \wedge b \in (s, r)_{\top}$, since $[r, t]$ has equivalent lower polars, and hence $(s \wedge b) \vee r = s$. Since $(L, \leqq)$ is $s \wedge (b \vee r) = s$, $s \wedge (b \vee r) = s$, i.e. $b \vee r \geqq s$. Since $b \in (t, s)_{\top}$, $b \vee s = t$. Then

$$t = b \vee s \leqq b \vee b \vee r = b \vee r \leqq t \vee t = t.$$

Hence, $b \vee r = t$, i.e. $b \in (t, r)_{\top}$, and therefore, $(t, s)_{\top} = (t, r)_{\top}$. For the other condition, we note that, since $[r, t]$ has equivalent lower polars, and since $(t, s)_{\top} = (t, r)_{\top}$ by the above,

$$s \wedge (t, r)_{\top} = s \wedge (t, s)_{\top} = (s, r)_{\top}$$

By Proposition 2.2, $r \wedge (t, r)_{\top} = r \wedge (s, r)_{\top}$.

The last three results of this section will be needed in the sequel.

PROPOSITION 2.5. *Let $(L, \leqq)$ be a distributive lattice. Suppose that $r, z, t, d \in L$ are such that $r \leqq z < t \leqq d$ and for all $z < \alpha \leqq d$, $d \in \alpha \vee (z, \alpha)^{\perp}$. If $[r, t]$ has equivalent lower polars, then $[z, t] = \{z, t\}$.*

Proof. Suppose that $z < \alpha \leqq t$. Then $z < \alpha \leqq d$, and hence $d \in \alpha \vee (z, \alpha)^{\perp}$. Let $b \in (z, \alpha)^{\perp}$ be such that $d = \alpha \vee b$. Then

$$t = d \wedge t = (\alpha \vee b) \wedge t = \alpha \vee (b \wedge t).$$

If $b \wedge t < t$, then $\alpha \in (t, b \wedge t)_{\top}$. Since $b \in (z, \alpha)^{\perp}$, $b \geqq z \geqq r$, and hence $b \wedge t \geqq r$. Since $[r, t]$ has equivalent lower polars, this implies (by Proposition 2.4) that $(t, b \wedge t)_{\top} = (t, r)_{\top}$. Thus $\alpha \in (t, r)_{\top}$, i.e. $\alpha = \alpha \vee r = t$. If $b \wedge t = t$, then

$$z = t \wedge z = t \wedge (b \wedge x) = t \wedge x = \alpha,$$

which contradicts our choice of α. Thus $[z, t] = \{z, t\}$.

PROPOSITION 2.6. *Let $(L, \leqq)$ be a lattice. Let $k, r, l, t \in L$ be such that $k \leqq r < l < t$ and $[r, t]$ has equivalent lower polars. Then $t \notin l \vee (k, l)^{\perp}$.*

Proof. Suppose that $t \in l \vee (k, l)^{\perp}$. Then $t = l \vee b$ for some $b \in (k, l)^{\perp}$, and hence $b \in (t, l)_{\top}$ Since $[r, t]$ has equivalent lower polars, $l \wedge (t, l)_{\top} = (l, r)_{\top}$, and hence

$$k = l \wedge b \in l \wedge (t, l)_{\top} = (l, r)_{\top}.$$

Thus $r = k \vee r = l$, which contradicts our choice of r. Therefore, $t \notin l \vee (k, l)^{\perp}$.

PROPOSITION 2.7. *Let $(L, \leqq)$ be a distributive lattice. Let $r, u, w, t \in L$ be such that $r \leqq u < w < t$ and $[r, t]$ has equivalent lower polars. Then $t \notin w \vee (u, w)^{\perp}$.*

Proof. Suppose $t \in w \vee (u, w)^{\perp}$. Then $t = w \vee b$ for some $b \in (u, w)^{\perp}$. Thus $b \in (t, w)_{\top}$ and hence by Proposition 2.4, $b \in (t, r)_{\top}$. Since $b \wedge w = u$,

$$r = r \wedge u = r \wedge b \wedge w = b \wedge r,$$

i.e., $b \geqq r$. Thus $t = r \vee b = b$. But this imples

$$u = b \wedge w = t \wedge w = w,$$

which contradicts our choice of u and w. Therefore, $t \notin w \vee (u, w)^{\perp}$.

3. Generalized intervals and segments. Let $(P, \leqq)$ be a partially ordered set. Let $r, s, t \in P$ be such that $r \leqq s \leqq t$. Let $[r, s, \infty)$ be the set of points $x \in P$ such that there exists $a \in (s, r)_{\top}$ satisfying

(a) $l(a, r) \neq \phi$

(b) $l(a, r) \subseteq l(x)$.

Let $(-\infty, s, t]$ be the set of points $y \in P$ such that there exists $b \in (s,t)^{\perp}$ satisfying

(c) $u(b,t) \neq \phi$,

(d) $u(b,t) \subseteq u(y)$.

We note that if P is a lattice, then

$$(-\infty, s, t] = \{y \in P \mid y \leqq b \vee t \text{ for some } b \in (s,t)^{\perp}\},$$
$$[r, s, \infty) = \{x \in P \mid x \geqq b \wedge r \text{ for some } b \in (s,r)_{\top}\}.$$

Let $[r,s,t] = [r,s,\infty) \cap (-\infty, s, t]$. A *generalized final segment* of P is a set of the form $[r,s,\infty)$ for $r, s \in P$ with $r \leqq s$; a *generalized initial segment* of P is a set of the form $(-\infty, s, t]$ for $s, t \in P$ with $s \leqq t$; and a *generalized interval* of P is a set of the form $[r,s,t]$ for $r,s,t \in P$ with $r \leqq s \leqq t$.

In the plane, the interval $[(0,-1),(0,1)]$ is not a generalized interval; a corresponding generalized interval is

$$[(0,-1),(0,0),(0,1)] = \{(x,y) \mid -1 \leqq y \leqq 1\}.$$

Let $r,s,t \in P$ be such that $r \leqq s \leqq t$. The sets $^*[r,s,\infty)$ and $^*(-\infty,s,t]$ are defined in the same way as $[r,s,\infty)$ and $(-\infty,s,t]$ above, except that when polars appear in the definition, they are replaced by the corresponding star-polars. *Generalized star-segments* and *generalized star-intervals* are defined accordingly. All the results of §3 remain true if polars and generalized segments are replaced by the corresponding star-polars and generalzied star-segments.

The following two results are essentially corollaries of Proposition 2.4.

PROPOSITION 3.1. *Let $(L, \leqq)$ be a distributive lattice. Let $r, t \in L$ be such that $r < t$ and $[r,t]$ has equivalent lower polars. If $r \leqq s < t$, then $[s,t,\infty) \subseteq [r,t,\infty)$.*

Proof. Let $z \in [s,t,\infty)$. Then $z \geqq s \wedge b$ for some $b \in (t,s)_{\top}$. Since $[r,t]$ has equivalent lower polars, $b \in (t,r)_{\top}$ by Proposition 2.4. But $z \geqq s \wedge b \geqq r \wedge b$, and hence $z \in [r,t,\infty)$.

PROPOSITION 3.2. *Let $(L, \leqq)$ be a distributive lattice. Let $r,s,t \in L$ be such that $r < s \leqq t$ and $[r,t]$ has equivalent lower polars. Then $[r,t,\infty) = [r,s,\infty)$.*

Proof. Since $[r,t]$ has equivalent lower polars, by Proposition 2.4

$$s \wedge (t,r)_{\top} = s \wedge (t,s)_{\top} = (s,r)_{\top}.$$

Thus

$$\begin{aligned}[r,t,\infty) &= \{y \in L \mid y \geqq b \wedge r \text{ for some } b \in (t,r)^{\top}\} \\ &= \{y \in L \mid y \geqq b \wedge r \wedge s \text{ for some } b \in (t,r)^{\top}\} \\ &= \{y \in L \mid y \geqq c \wedge r \text{ for some } c \in (s,r)^{\top}\} \\ &= [r,s,\infty)\end{aligned}$$

The next result was proven in **[7]**. Although it will be used only for the main theorem in §5, we include it here to enable us to compare Propositions 2.6 and 2.7 with Lemma 3.4.

PROPOSITION 3.3. *Let $(L, \leqq)$ be a lattice. Let $r, t \in L$ be such that $r < t$. If L is modular, then*

(i) *for all $x \in (-\infty, r, t] \cap [t, \infty)$, there exists $b \in (r,t)^{\perp}$ such that $x = t \vee b$.*

If L is distributive, then

(ii) *for all $x \in (-\infty, r, t] \cap [r, \infty)$, there exists $b \in (r,t)^{\perp}$ such that $x = (x \wedge t) \vee b$.*

In view of Proposition 3.3, the following lemma says that if $k = r$ in Proposition 2.6, or if $u = r$ in Proposition 2.7, then we could have assumed that $[r,t]$ had equivalent upper polars instead of equivalent lower polars.

LEMMA 3.4. *Let $(L, \leqq)$ be a distributive lattice. Let $r < s < t$ be such that $[r,t]$ has equivalent upper polars. Then $t \notin (-\infty, r, s]$.*

Proof. Suppose that $t \in (-\infty, r, s,]$. Then $t \leqq s \vee b$ for some $b \in (r,s)^{\perp}$. Since $r < s < t$, then by the dual of Proposition 2.4, $b \in (r,t)^{\perp}$, and hence

$$t = (s \vee b) \wedge t = s \vee (b \wedge t) = s \vee r = s,$$

This contradicts our choice of s, and thus $t \notin (-\infty, r, s]$.

The next result is the main one of this section, and will be extremely useful in the sequel.

PROPOSITION 3.5. *Let $(L, \leqq)$ be a distributive lattice. Let $r, t \in L$ be such that $r < t$, $[r,t]$ has equivalent upper polars, and $[r,t] \neq \{r,t\}$. Then*

$$\cap \{L \setminus (-\infty, r, s] \mid r < s < t\}$$

is a dual ideal of L.

Proof. Let S denote $\cap\{L \setminus (-\infty, r, s] \mid r < s < t\}$. If $x \in S$ and $x \leqq y$, then clearly $y \in S$. Suppose that $x, y \in S$, but that $x \wedge y \in (-\infty, r, s]$ for some $s \in L$ with $r < s < t$. Let $x' = x \vee r$ and $y' = y \vee r$. If $d \in (r, s)^{\perp}$ is such that $x \wedge y \leqq s \vee d$, then

$$x' \wedge y' = (x \wedge y) \vee r \leqq s \vee d \vee r = s \vee d.$$

Thus $x' \wedge y' \in (-\infty, r, s]$.

If $x' \wedge y' \geqq t$, then clearly $t \in (-\infty, r, s]$, which contradicts Lemma 3.4. Thus $x' \wedge y' \ngeqq t$, and hence, without loss of generality, we may assume that $x' \ngeqq t$. Then $r \leqq x' \wedge t < t$. If $r = x' \wedge t$, then $x' \in (r, t)^{\perp}$. By the dual of Proposition 2.4, $(r, t)^{\perp} = (r, s)^{\perp}$, and thus $x' \in (r, s)^{\perp} \subseteq (-\infty, r, s]$. Since $x \leqq x'$, this implies that $x \in (-\infty, r, s]$, which contradicts our choice of x. Therefore, $r < x' \wedge t < t$, and since $[r, t]$ has equivalent upper polars,

$$(x' \wedge t) \vee (r, x' \wedge t)^{\perp} = (x' \wedge t, t)^{\perp}.$$

Clearly, $x' \in (x' \wedge t, t)^{\perp}$, and hence

$$x' \in (x' \wedge t) \vee (r, x' \wedge t)^{\perp} \subseteq (-\infty, r, x' \wedge t].$$

Since $x \leqq x', x \in (-\infty, r, x' \wedge t]$, and since $x' \wedge t < t$, this contradicts our choice of x. We conclude that $x \wedge y \notin (-\infty, r, s]$, and hence that $x \wedge y \in S$. Thus S is a dual ideal of L.

The next result, which we will need when we consider the Hausdorff separation axiom, indicates how useful Proposition 3.5 can be.

PROPOSITION 3.6. *Let $(L, \leqq)$ be a distributive lattice. Let $r, t \in L$ be such that $r < t$ and $[r, t]$ has equivalent polars. Then*

$$L = (-\infty, r, t] \cup [r, t, \infty).$$

Proof. Let $z \in L$. If $(z \wedge t) \vee r = t$, then

$$z \wedge t \in (t, r)_{|} \subseteq [r, t, \infty),$$

and hence clearly, $z \in [r, t, \infty)$. If $(z \wedge t) \vee r = r$, then $(z \vee r) \wedge t = r$, hence

$$z \vee r \in (r, t)^{\perp} \subseteq (-\infty, r, t],$$

and thus clearly, $z \in (-\infty, r, t]$. Otherwise, $r < (z \wedge t) \vee r < t$. Then, in particular, $[r, t] \neq \{r, t\}$, and we may apply Proposition 3.5 to $[r, t]$. Let $T = \cup\{(-\infty, r, s] \mid r < s < t\}$. Since $r < (z \wedge t) \vee r < t$,

$$(z \vee r) \wedge t = (z \wedge t) \vee r \in T.$$

Since $t \notin T$ by Lemma 3.4, and since $L \setminus T$ is a dual ideal by Proposition 3.5, $z \vee r \in T$. Since, for all $r < s < t$, $(-\infty, r, s] \subseteq (-\infty, r, t]$ by the dual of Proposition 3.1, $z \vee r \in (-\infty, r, t]$, and hence $z \in (-\infty, r, t]$.

In some cases, Proposition 3.6 will not give a good enough "separation" of points. Therefore, we must refine it in certain cases to obtain the "separation" described by Proposition 3.8.

LEMMA 3.7. *Let $(L, \leqq)$ be a modular lattice. Let $r, s, t, u \in L$ be such that $r < s < t < u$, $[s, t] = \{s, t\}$, and both $[r, t]$ and $[s, u]$ have equivalent polars. Then*

$$(-\infty, r, t] \setminus (-\infty, r, s] \subseteq [t, u, \infty)$$

Proof. Let $z \in (-\infty, r, t] \setminus (-\infty, r, s]$. If $(z \wedge t) \vee s = s$, then $(z \vee s) \wedge t = s$ and hence $z \vee s \in (s, t)^{\perp}$. Since $[r, t]$ has equivalent polars,

$$z \vee s \in s \vee (r, s)^{\perp} \subseteq (-\infty, r, s],$$

and hence $z \in (-\infty, r, s]$. This contradicts our choice of z, and hence $(z \wedge t) \vee s > s$. Since $(z \wedge t) \vee s \leqq t$ and $[s, t] = \{s, t\}$, we must have $(z \wedge t) \vee s = t$, i.e. $z \wedge t \in (t, s)_{\top}$. Since $[s, u]$ has equivalent polars,

$$z \wedge t \in t \wedge (u, t)_{\top} \subseteq [t, u, \infty),$$

and therefore, $z \in [t, u, \infty)$.

PROPOSITION 3.8. *Let $(L, \leqq)$ be a distributive lattice. Let $r, s, t, u \in L$ be such that $r < s < t < u$, $[s, t] = \{s, t\}$, and both $[r, t]$ and $[s, u]$ have equivalent polars. Then*

$$L = (-\infty, r, s] \cup [t, u, \infty).$$

Proof. Since $[s, t] = \{s, t\}$, clearly $[s, t]$ has equivalent polars. Thus, by Proposition 3.6,

$$L = (-\infty, s, t) \cup [s, t, \infty).$$

By Proposition 3.2 and its dual,

$$L = (-\infty, r, t] \cup [s, u, \infty).$$

Therefore, by Lemma 3.7 and its dual,

$$L = (-\infty, r, s] \cup [t, u, \infty).$$

The last result of this section is the "discrete" analog of Proposition 3.5.

PROPOSITION 3.9. *Let* $(L, \leqq)$ *be a distributive lattice. Let* $r, s, t \in L$ *be such that* $r < s < t$, $[r, t]$ *has equivalent upper polars, and for all* $e \in L$ *with* $s \leqq e < t$, $e \in s \vee (r, s)^{\perp}$. *Then* $L \setminus (-\infty, r, s]$ *is a dual ideal of* L.

Proof. Clearly, if $x \in L \setminus (-\infty, r, s]$ and $x \leqq y$, then $y \in L \setminus (-\infty, r, s]$. Suppose that $x, y \in L \setminus (-\infty, r, s]$, but that $x \wedge y \in (-\infty, r, s]$. Then, similarly to the beginning of the proof of Proposition 3.5, we have that $(x \wedge y) \vee s \in (-\infty, r, s]$, and hence by Lemma 3.4, that

$$(x \vee s) \wedge (y \vee s) = (x \wedge y) \vee s \not\geqq t.$$

Thus we may assume that $x \vee s \not\geqq t$, i.e. that $(x \vee s) \wedge t < t$. Then, by hypothesis, $(x \vee s) \wedge t = s \vee b$ for some $b \in (r, s)^{\perp}$. Since $[r, t]$ has equivalent upper polars, this implies that

$$x \vee s \in ((x \vee s) \wedge t, t)^{\perp} = (s \vee b, t)^{\perp} = s \vee b \vee (r, s \vee b)^{\perp}.$$

If $d \in (r, s \vee b)^{\perp}$, then

$$r = (s \vee b) \wedge d = (s \wedge d) \vee (b \wedge d),$$

and hence $s \wedge d \leqq r$. Thus, since $s \wedge b = r$,

$$s \wedge (b \vee d) = (s \wedge b) \vee (s \wedge d) = r \vee (s \wedge d) = r,$$

i.e. $b \vee d \in (r, s)^{\perp}$. Therefore, $b \vee (r, s \vee b)^{\perp} \subseteq (r, s)^{\perp}$, and thus $x \vee s \in s \vee (r, s)^{\perp}$. But this implies that $x \vee s \in (-\infty, r, s]$, and hence that $x \in (-\infty, r, s]$, which contradicts our choice of x. We conclude that $x \wedge y \notin (-\infty, r, s]$ and hence that $L \setminus (-\infty, r, s]$ is a dual ideal of L.

4. Continuity of the lattice operations. Let $(P, \leqq)$ be a partially ordered set. The *generalized interval topology* (or *gi-topology*) on P, denoted by $\mathcal{G}(P)$, takes as a subbase for its closed sets, P and ϕ, together with all the final generalized segments and all the initial generalized segments. The *generalized star-interval topology* (or *gi-*topology*) on P, denoted by $\mathcal{G}^*(P)$ takes as a subbase for its closed sets P and ϕ, together with all the generalized star-segments. In **[6]** we proved that $\mathcal{G}(P)$ is an intrinsic topology on P, which is preserved by cardinal products of dually directed sets, and that $\mathcal{G}^*(P)$ is an intrinsic topology which always contains the interval topology, $\mathcal{I}(P)$.

In this section, we show that if intervals with equivalent polars occur throughout a distributive lattice L, and if $\mathcal{G}(L)$ is T_1, then $(L, \mathcal{G}(L))$ is a topological lattice. We first state precisely what is meant by the occurence of intervals with equivalent polars throughout a lattice.

Let $(L, \leqq)$ be a lattice. We say that $r, t \in L$ *provide equivalent polars* for $x, y, z \in L$ in case $x \leqq r < y < t \leqq z$, and $[r, t]$ has equivalent polars. We say that L *has minimal polars* if for all $x, y, z \in L$ with $x < y < z$ and $z \notin y \vee (x, y)^{\perp}$, there exist $r, t \in L$ which provide equivalent polars for x, y, z. Clearly, replacing polars by star-polars does not change the above definitions. Proposition 5.4 will provide a large class of lattices which have minimal polars.

The first result of this section shows that, for modular lattices, having minimal polars means that whenever $x < y < z$ and there can exist r, t which provide minimal polars for x, y, z, then such r, t do in fact exist.

PROPOSITION 4.1. *Let $(L, \leqq)$ be a modular lattice. Let $x, y, z, r, t \in L$ be such that $x \leqq r < y < t \leqq z$, and $z \in y \vee (x, y)^{\perp}$. Then $[r, t]$ does not have equivalent lower polars.*

Proof. Since $z \in y \vee (x, y)^{\perp}$, $z = y \vee b$ for some $b \in (x, y)^{\perp}$. Then

$$t = z \wedge t = (y \vee b) \wedge t = y \vee (b \wedge t),$$
$$x = x \wedge t = (b \wedge y) \wedge t = y \wedge (b \wedge t).$$

Thus $t \in y \vee (x, y)^{\perp}$. If $[r, t]$ has equivalent lower polars, this contradicts Proposition 2.6.

The following result was noted in **[7]**. We include it here to indicate that having minimal polars is a self-dual property, i.e. that a lattice has minimal polars if and only if its dual does.

PROPOSITION 4.2. *Let $(L, \leqq)$ be a lattice and suppose that $x, y, z \in L$ are such that $x < y < z$. Then the following statments are equivalent:*

(i) $z \in y \vee (x, y)^{\perp}$,

(ii) $x \in y \wedge (z, y)_{\top}$.

We are now in a position to prove the main result of this section.

THEOREM 4.3. *Let $(L, \leqq)$ be a distributive lattice. If $\mathcal{G}(L)$ is T_1, and if L has minimal polars, the $(L, \mathcal{G}(L))$ is a topological lattice.*

Proof. Our method is to isolate the difficult part of the proof, and then to prove it separately as Lemma 4.4. We will consider only the continuity of

$$\wedge : (L \times L, \mathcal{G}(L) \times \mathcal{G}(L)) \rightarrow (L, \mathcal{G}(L));$$

the continuity of $\vee$ may be proved dually. Since complements of generalized segments form a subbasis for $\mathcal{G}(L)$, it clearly suffices to show that if $x, y \in L$ and $x \wedge y \in L \setminus X$ for some generalized segment X, then there exist closed sets Y and W such that $x \in L \setminus W$, $y \in L \setminus Y$, and $(L \setminus W) \wedge (L \setminus Y) \subseteq L \setminus X$.

If X is a generalized final segment, than we may choose Y and W in the following manner. Suppose that $x \wedge y \in L \setminus [k, l, \infty)$ for some $k, l \in L$ with $k \leqq l$. If $\alpha, \beta \in L \setminus [k, l, \infty)$, then $\alpha \wedge \beta \leqq \beta$ and hence $\alpha \wedge \beta \in L \setminus [k, l, \infty)$. Thus, if $x = y$, then

$$x = x \wedge y = y \in L \setminus [k, l, \infty),$$

and we may choose $W = [k, l, \infty) = Y$. Suppose $x \neq y$. Clearly either $x \in L \setminus [k, l, \infty)$ or $y \in L \setminus [k, l, \infty)$, and thus, without loss of generality, we may assume that $x \in L \setminus [l, l, \infty)$. Let $Y = \{y\}$ and $W = [k, l, \infty)$. Since $\mathcal{G}(L)$ is T_1, Y is closed. If $\alpha \in L \setminus Y$ and $\beta \in L \setminus W$, then $\alpha \wedge \beta \leqq \beta$ and hence $\alpha \wedge \beta \in L \setminus [k, l, \infty)$.

It remains to show that such Y and W exist when X is a generalized initial segment. The problem is more difficult here than in the case where X is a generalized final segment, and requires the hypothesis that L has minimal polars. Suppose that $x \wedge y \in L \setminus (-\infty, k, l]$. Then, since L is a lattice, $k < l$, and hence the proof of Theorem 4.3 will be complete when we prove Lemma 4.4.

LEMMA 4.4. *Let $(L, \leqq)$ be a distributive lattice which has minimal polars. If $x, y, k, l \in L$ are such that $k < l$ and $x \wedge y \notin (-\infty, k, l]$, then there exist $u, w \in L$ such that*

(i) $x, y \in L \setminus (-\infty, u, w]$;

(ii) *for all* $\alpha, \beta \in L \setminus (-\infty, u, w]$, $\alpha \wedge \beta \in L \setminus (-\infty, k, l]$.

Proof. We note that since $x \wedge y \notin (-\infty, k, l]$, then $(x \wedge y) \vee l > l$. Furthermore, if $(x \wedge y) \vee l \in l \vee (k, l)^{\perp}$, then $(x \wedge y) \vee l \in (-\infty, k, l]$, and hence $x \wedge y \in (-\infty, k, l]$. Thus

(1) $$(x \wedge y) \vee l \notin l \vee (k, l)^{\perp}.$$

Also, if $k \leqq r < l$, then

(2) $$(-\infty, k, l] \subseteq (-\infty, r, l]:$$

if $z \in (-\infty, k, l]$, then $z \leqq l \vee b$ for some $b \in (k, l)^{\perp}$; by Proposition 2.1 (iii), $r \vee b \in (r, l)^{\perp}$, and since $z \leqq l \vee b = l \vee r \vee b$, this implies that $z \in (-\infty, r, l]$.

(A) Suppose that there exist $c, d \in L$ such that

$$k \leqq c < l < d \leqq (x \wedge y) \vee l,$$

$d \notin l \vee (c, l)^{\perp}$, and for all $e \in L$ with $l \leqq e < d$, $e \in l \vee (c, l)^{\perp}$. Since L has minimal polars, there exist $r, t \in L$ such that $c \leqq r < l < t \leqq d$, and $[r, t]$ has equivalent polars. By Proposition 2.6, $t \notin l \vee (c, l)^{\perp}$, and hence $t = d$. If $l \leqq e < t = d$, then by Proposition 2.1 (iii),

$$e \in l \vee (c, l)^{\perp} = l \vee (r \vee (c, l)^{\perp}) \subseteq l \vee (r, l)^{\perp},$$

and thus by Proposition 3.9, $L \setminus (-\infty, r, l]$ is a dual ideal of L.

If $x \wedge y \in (-\infty, r, l]$, then $x \wedge y \leqq l \vee b$ for some $b \in (r, l)^{\perp}$. Thus

$$t = d \leqq (x \wedge y) \vee l \leqq l \vee b.$$

Since $b \in (r, l)^{\perp}$ and $[r, t]$ has equivalent polars, $b \in (r, t)^{\perp}$ by the dual of Proposition 2.4. Thus $b \wedge t = r$, and hence

$$t = (l \vee b) \wedge t = l \vee (b \wedge t) = l \vee r = l.$$

This contradicts our choice of t, and thus $x \wedge y \notin (-\infty, r, l]$. Then clearly, $x, y \in L \setminus (-\infty, r, l]$. Let $\alpha, \beta \in L \setminus (-\infty, r, l]$. Since $L \setminus (-\infty, r, l]$ is a dual ideal, and since $k \leqq c \leqq r < l$, we have by (2) that

$$\alpha \wedge \beta \in L \setminus (-\infty, r, l] \subseteq L \setminus (-\infty, k, l].$$

Therefore, if $u = r$ and $w = l$, then conditions (i) and (ii) above are satisfied.

(B) Suppose that for all $c, d \in L$ such that

$$k \leqq c < l < d \leqq (x \wedge y) \vee l$$

and $d \notin l \vee (c, l)^{\perp}$, there exists $e \in L$ such that $l < e < d$ and $e \notin l \vee (c, l)^{\perp}$. Since L has minimal polars, by (1) there exist $r, t \in L$ such that

$$k \leqq r < l < t \leqq (x \wedge y) \vee l$$

and $[r, t]$ has equivalent polars. By Proposition 2.6, $t \notin l \vee (r, l)^{\perp}$, and thus, by hypothesis, there exists $t' \in L$ such that $l < t' < t$ and $t' \notin l \vee (r, l)^{\perp}$. Thus, we may find $u, w \in L$ such that $r \leqq u < l < w \leqq t'$ and $[u, w]$ has equivalent polars.

Let $r < s < t$. If $s \vee l = t$, then $s \in (t, l)_{|}$, and hence by Proposition 2.4, $s \in (t, r)_{\top}$, i.e. $s = s \vee r = t$. This contradicts our choice of s, and thus $s \vee l < t$. Hence, by Lemma 3.4, $t \notin (-\infty, r, s \vee l]$. Suppose that $x \wedge y \in (-\infty, r, s]$. Then by the dual of Proposition 3.1, $x \wedge y \in (-\infty, r, s \vee l]$, and thus, clearly $(x \wedge y) \vee l \in (-\infty, r, s \vee l]$. Since $t \leqq (x \wedge y) \vee l$, this implies that $t \in (-\infty, r, s \vee l]$, which is a contradiction. Thus, $x \wedge y \notin (-\infty, r, s]$. By Lemma 3.4, $t \notin (-\infty, r, s]$. Since $r < s < t$ was arbitrary, we may conclude from Proposition 3.5 that for all $r < s < t$,

$$x \wedge y \wedge t \notin (-\infty, r, s]. \tag{3}$$

Now suppose that $x \wedge y \wedge t \in (-\infty, u, w]$. Then $x \wedge y \wedge t \leqq w \vee b$ for some $b \in (u, w)^{\perp}$. Hence $x \wedge y \wedge t \in (-\infty, r, (w \vee b) \wedge t]$. If $(w \vee b) \wedge t < t$, this contradicts (3) above. Thus $(w \vee b) \wedge t = t$, i.e. $w \vee (b \wedge t) = t$. Hence $b \wedge t \in (t, w)_{\top}$, and by Proposition 2.4, $b \wedge t \in (t, r)_{\top}$, i.e. $(b \wedge t) \vee r = t$. Since $b \in (u, w)^{\perp}$, $b \wedge w = u$. Thus

$$b \wedge t \geqq b \wedge w = u \geqq r,$$

and hence $(b \wedge t) \vee r = b \wedge t$. But then $b \wedge t = t$, i.e. $b \geqq t$, and we have

$$u = b \wedge w \geqq t \wedge w = w.$$

This contradicts our choice of u and w, and we conclude $x \wedge y \wedge t \notin (-\infty, u, w]$. Clearly, this implies that $x, y \in L \setminus (-\infty, u, w]$. Finally, we note that if $\alpha, \beta \in L \setminus (-\infty, u, w]$, then $\alpha, \beta \in L \setminus (-\infty, u, s]$ for all $u < s < w$ by the dual of Proposition 3.1, and hence $\alpha \wedge \beta \in L \setminus (-\infty, u, s]$ for all $u < s < w$ by Proposition 3.5. Thus, in particular, $\alpha \wedge \beta \in L \setminus (-\infty, u, l]$, and hence by (2), $\alpha \wedge \beta \in L \setminus (-\infty, k, l]$.

Since (A) and (B) exhaust the possibilities, we conclude that Lemma 4.4, and hence Theorem 4.3, hold.

COROLLARY 4.5. *Let* $(L, \leqq)$ *be a distributive lattice. If* $\mathcal{G}(L) \supseteq \mathcal{I}(L)$, *and if* L *has minimal polars, then* $(L, \mathcal{G}^*(L))$ *is a topological lattice.*

Proof. Since $\mathcal{I}(L)$ is T_1, $\mathcal{G}(L)$ is T_1; by [**6**; Proposition 3.6], $\mathcal{G}(L) = \mathcal{G}^*(L)$. The corollary then follows from Theorem 4.3.

We note that proving the results of §3 for generalized star-segments, and considering the case of $(-\infty, l]$ in Lemma 4.4 would allow us to drop the hypothesis that $\mathcal{G}(L) \supseteq \mathcal{I}(L)$ in Corollary 4.5.

5. The Hausdorff separation axiom. This section is devoted primarily to establishing that distributive lattices which have minimal polars, and which satisfy the additional requirement that there be enough polars to "separate" points, have Hausdorff generalized interval topologies.

In [**7**], we introduced the following condition: A lattice $(L, \leqq)$ is said to be *almost polar-dense* if, whenever $x, y \in L$ are such that $x < y$ and for all $x < d < y$, $y \in d \vee (x, d)^{\perp}$, then there exist $c, e \in L$ such that $c < x < y < e$, $y \notin x \vee (c, x)^{\perp}$, and $x \notin y \wedge (e, y)_{\perp}$. We proved in [**7**] that a totally ordered set is almost polar-dense if and only if its gi-topology is equivalent to its interval topology (and hence to its gi-*topology). For modular lattices, we have the following [**7**; Proposition 2.5].

PROPOSITION 5.1. *Let $(L, \leqq)$ be a modular lattice. If L is almost polar-dense, then $\mathcal{G}(L) = \mathcal{G}^*(L)$.*

The main result of this section is the following.

THEOREM 5.2. *Let $(L, \leqq)$ be an almost polar-dense, distributive lattice. If L has minimal polars, then $\mathcal{G}(L)$ is Hausdorff.*

Proof. Let $x, y \in L$ be distinct. Without loss of generality, we may assume that $x < x \vee y$.

(A) Suppose that for all $b, c \in L$ with $x \leqq b < c \leqq x \vee y$, there exists $d \in L$ such that $b < d < c$ and $c \notin d \vee (b, d)^{\perp}$. We first prove the following: (α) If $x \leqq b < c \leqq x \vee y$, then there exist $r, t \in L$ such that $b < r < t < c$ and $[r, t]$ has equivalent polars. To see this, let $x \leqq b < c \leqq x \vee y$. By hypothesis, there exist $d, e, f \in L$ such that $b < d < e < f < c$, $c \notin f \vee (b, f)^{\perp}$, $f \notin d \vee (b, d)^{\perp}$, and $f \notin e \vee (d, e)^{\perp}$. Since L has minimal polars, there exist $r, t \in L$ such that

$$b < d \leqq r < e < t \leqq f < c,$$

and $[r, t]$ has equivalent polars. This proves (α).

We first apply (α) to obtain $r, t \in L$ such that $x < r < t < x \vee y$ and $[r, t]$ has equivalent polars. We then apply (α) to $r < t$ to obtain $u, w \in L$ such that $r < u < w < t$ and $[u, w]$ has equivalent polars. By Proposition 3.6,

$$L = (-\infty, u, w] \cup [u, w, \infty),$$

and hence

$$(L \setminus (-\infty, u, w]) \cap (L \setminus [u, w, \infty)) = \phi.$$

By the dual of Proposition 2.7, $r \notin u \wedge (w, u)_{|}$. Since L is distributive, this implies that $r \notin [u, w, \infty)$ by the dual of Proposition 3.3. Therefore, $x \in L \setminus [u, w, \infty)$. Dually, $x \vee y \notin (-\infty, u, w]$, and since $x \in (-\infty, u, w]$, this implies that $y \in L \setminus (-\infty, u, w]$.

(B) Suppose that there exist $b, c \in L$ such that $x \leqq b < c \leqq x \vee y$ and for all $b < d < c$, $c \in d \vee (d, b)^{\perp}$. Since L is almost polar-dense, there exists $k \in L$ such that $k < b$ and $c \notin b \vee (k, b)^{\perp}$. Since L has minimal polars, there exist $r, s \in L$ such that $k \leqq r < b < s \leqq c$ and $[r, s]$ has equivalent polars. By Proposition 2.5, $[b, s] = \{b, s\}$, and hence, since L is almost polar-dense, there exists $f \in L$ such that $s < f$ and $f \notin s \vee (b, s)^{\perp}$ (Proposition 4.2). Since L has minimal polars, there exist $u, t \in L$ such that $b \leqq u < s < t \leqq f$ and $[u, t]$ has equivalent polars. Since $[b, s] = \{b, s\}$, $u = b$, i.e. $[b, t]$ has equivalent polars. By Proposition 3.8,

$$L = (-\infty, r, b] \cup [s, t, \infty),$$

and hence

$$(L \setminus (-\infty, r, b]) \cap (L \setminus [s, t, \infty)) = \phi.$$

By the dual of Lemma 3.4, $b \notin [s, t, \infty)$, and therefore, $x \in L \setminus [s, t, \infty)$. Dually, $x \vee y \in L \setminus (-\infty, r, b]$. Since $x \leqq b$, $x \in (-\infty, r, b]$ and thus if $y \in (-\infty, r, b]$, $x \vee y \in (-\infty, r, b]$. Therefore, $y \in L \setminus (-\infty, r, b]$.

We conclude that there exist $\alpha, \beta, \gamma, \delta \in L$ such that $\alpha < \beta$, $\gamma < \delta$, $y \in L \setminus (-\infty, \alpha, \beta]$, $x \in L \setminus [\gamma, \delta, \infty)$, and

$$(L \setminus (-\infty, \alpha, \beta]) \cap (L \setminus [\gamma, \delta, \infty)) = \phi.$$

By definition of $\mathscr{G}(L)$,

$$L \setminus (-\infty, \alpha, \beta],\ L \setminus (\gamma, \delta, \infty) \in \mathscr{G}(L),$$

and hence $\mathscr{G}(L)$ is Hausdorff.

COROLLARY 5.3. *Let $(L, \leqq)$ be an almost polar-dense, distributive lattice. If L has minimal polars, then $\mathscr{G}^*(L) = \mathscr{G}(L)$ is a Hausdorff lattice topology on L.*

Proof. This result follows from Proposition 5.1, Corollary 4.5, and Theorem 5.2.

We conclude this section by showing how to construct many natural examples of almost polar-dense, distributive lattices which have minimal polars.

A partially ordered set $(P, \leqq)$ is said to be *unbounded* if for all $p \in P$, there exist $r, t \in P$ such that $r < p < t$.

PROPOSITION 5.4. *Let $\{T_\alpha \mid \alpha \in A\}$ be a collection of unbounded, totlally ordered sets. Then $|\Pi| \{T_\alpha \mid \alpha \in A\}$ is an almost polar-dense, distributive lattice which has minimal polars.*

Proof. By [7; Corollary 2.8], $|\Pi| \{T_\alpha \mid \alpha \in A\}$ is almost polar-dense. Clearly it is a distributive lattice. That it has minimal polars follows from the fact that if $x, y, z \in |\Pi| \{T_\alpha \mid \alpha \in A\}$ are such that $x < y < z$ and $z \notin y \vee (x, y)^\perp$, then there exists $\alpha \in A$ such that $\alpha x < \alpha y < \alpha z$.

6. Some examples.

In this section, we construct various examples to illuminate Theorems 4.3 and 5.2 and Corollaries 4.5 and 5.3.

EXAMPLE 6.1. Let M_5 be the five-element nondistributive, modular lattice. Since $\mathcal{G}^*(M_5)$ is T_1, $\mathcal{G}^*(M_5)$ is discrete, and hence a Hausdorff lattice topology. It is easy to see, however, that $\mathcal{G}(M_5)$ is indiscrete since $(-\infty, a, b] = [a, b, \infty) = M_5$ for all $a \leqq b$. Thus, $\mathcal{G}(M_5)$ is a lattice topology which is not even T_0. Clearly, M_5 is not almost polar-dense, but since if $a < b < c$, $c \in b \vee (a, b)^\perp$, vacuously L has minimal polars.

EXAMPLE 6.2. Consider the natural numbers, N. Clearly, $\mathcal{G}^*(N)$ is discrete, and hence $(N, \mathcal{G}^*(N))$ is a Hausdorff topological lattice. It was noted in [6] that the closure of $\{1\}$ with respect to $\mathcal{G}(N)$ is $\{1, 2\}$, and that therefore, $\mathcal{G}(N)$ is not T_1. Clearly, $\{n\} \in \mathcal{G}(N)$ for all $n \geqq 3$, and furthermore,

$$\{1, 2\} = L \setminus [3, 4, \infty] \in \mathcal{G}(N),$$

$$\{1\} = L \setminus [2, 3, \infty) \in \mathcal{G}(N).$$

Since thus $2 \notin \{1\} \in \mathcal{G}(N)$, $\mathcal{G}(N)$ is T_0. It is easy to see that $(N, \mathcal{G}(N))$ is a topological lattice, and since N is totally ordered, N is distributive. Clearly, N has minimal polars but is not almost polar-dense.

EXAMPLE 6.3. Let $L = |\Pi| \{N \mid n \in N\}$. Since the generalized interval topology is preserved by cardinal products of dually directed sets [6], $\mathcal{G}(L)$ is a T_0, non-T_1, lattice topology by Example 6.2. We will show first that $(L, \mathcal{G}^*(L))$ is not a topological lattice, and second that $\mathcal{G}^*(L)$ is not Hausdorff.

Let $c_1, f \in L$ be defined by $nc_1 = 1$ for all $n \in N$, and

$$nf = \begin{cases} 2 & \text{if} \quad n = 1 \\ 1 & \text{if} \quad n \neq 1. \end{cases}$$

If $a, b, z \in L$ are such that $a \leqq b$ and $z \notin {}^*[a, b, \infty)$, then there exists $l \in N$ such that $lz < la$ and if $a < b, la < lb$. We denote the minimal such $l \in N$ by $m(z, a, b)$.

Suppose that $a_i, b_i\, c_j, d_j \in L$, $1 \leqq i \leqq \alpha$, $1 \leqq j \leqq \beta$, are such that $f \in P_f$ where $P_f \in \mathcal{G}^*(L)$ is defined by

$$(4)\, P_f = L \setminus [(\cup \{{}^*[a_i, b_i, \infty) \mid 1 \leqq i \leqq \alpha\}) \cup (\cup \{{}^*(-\infty, c_j, d_j] \mid 1 \leqq j \leqq \beta\})]$$

It is easy to see that for all $1 \leqq j \leqq \beta$, $c_j = d_j$ and $1d_j = 1$. Let $\Gamma \in L$ be defined by

$$(5) \qquad n\Gamma = \begin{cases} 1 & \text{if} \quad n = 1 \\ 1 & \text{if} \quad n = m(f, a_i, b_i) \text{ for some } 1 \leqq i \leqq \alpha \\ (\vee \{nd_j \mid 1 \leqq j \leqq \beta\}) + 1 & \qquad \text{otherwise} \end{cases}$$

Since for some $n \in N$, $n\Gamma = (\vee \{nd_j \mid 1 \leqq j \leqq \beta\}) + 1$, and since $c_j = d_j$ for all $1 \leqq j \leqq \beta$, $\Gamma \notin \cup \{{}^*(-\infty, c_j, d_j] \mid 1 \leqq j \leqq \beta\}$. Since

$$(m(f, a_i, b_i))\Gamma = 1 \leqq (m(f, a_i, b_i))f < (m(f, a_i, b_i))a_i$$

for all $1 \leqq i \leqq \alpha$, $\Gamma \notin \cup \{{}^*[a_i, b_i, \infty) \mid 1 \leqq i \leqq \alpha\}$. Thus $\Gamma \in P_f$. Clearly, $\Gamma \wedge f = c_1$.

Clearly, $f \in L \setminus \{c_1\} \in \mathcal{G}^*(L)$, and $f \wedge f = f$. Thus, if $(L, \mathcal{G}^*(L))$ is a topological lattice, there exist $P_1, P_2 \in \mathcal{G}^*(L)$ such that $f \in P_1 \cap P_2$ and $P_1 \wedge P_2 \subseteq L \setminus \{c_1\}$. Since $P_1, P_2 \in \mathcal{G}^*(L)$, there exists a P_f, of the form (4) above, such that $f \in P_f \subseteq P_1 \cap P_2$. Then, if Γ is constructed as in (5) above,

$$c_1 = \Gamma \wedge f \in P_f \wedge P_f \subseteq P_1 \wedge P_2 \subseteq L \setminus \{c_1\}.$$

This is a contradiction, and hence $(L, \mathcal{G}^*(L))$ is not a topological lattice.

We conclude this example by showing that $\mathcal{G}^*(L)$ is not Hausdorff. Suppose that $c_1 \in S \in \mathcal{G}^*(L)$ and $f \in P \in \mathcal{G}^*(L)$. Then, by definition of $\mathcal{G}^*(L)$, there exists P_f, of the form (4) above, such that $f \in P_f \subseteq P$, and there exist $r_k, s_k \in L$, $1 \leqq k \leqq \gamma$ such that

$$c_1 \in S_1 = L \setminus \cup\{{}^*[r_k, s_k, \infty) \mid 1 \leqq k \leqq \gamma\} \subseteq S.$$

Let $\Upsilon \in L$ be defined by

$$n\Upsilon = \begin{cases} 1 & \text{if} \quad n = m(f, a_i, b_i) \text{ for some } 1 \leqq i \leqq \alpha \\ 1 & \text{if} \quad n = m(c_1, r_k, s_k) \text{ for some } 1 \leqq k \leqq \gamma \\ (\vee\{nd_j \mid 1 \leqq j \leqq \beta\}) + 1 & \text{otherwise.} \end{cases}$$

As in the case of Γ above, $\Upsilon \in P_f$, and since $m(c_1, r_k, s_k)\Upsilon = 1$ for all $1 \leqq k \leqq \gamma$, $\Upsilon \in S_1$. Thus

$$\Upsilon \in P_f \cap S_1 \subseteq P \cap S,$$

i.e. $P \cap S \neq \phi$, and hence $\mathscr{G}^*(L)$ is not Hausdorff.

Since $[c_1, f] = \{c_1, f\}$, then for all $c_1 < d < f$, $f \in f \vee (c_1, d)^{\perp}$. Thus, since $(-\infty, c_1] = \{c_1\}$, L is not almost polar-dense. It is easy to see that L has minimal polars.

EXAMPLE 6.4. Let $L \subseteq R \mid \times \mid R$ be defined by

$$L = ([0, 2] \times]0, 2]) \cup ([3, 5] \times [0, 2[) \cup \{(0, 0), (5, 2)\}.$$

Clearly, L is a lattice. Since the lattice

$$\{(0, 0), (0, 1), (1, 1), (3, 1), (3, 0)\}$$

is a sublattice of L, L is not modular.

We will first show that $\mathscr{G}(L)$ is Hausdorff. Let $(a, b), (x, y) \in L$ be such that $(a, b) \neq (x, y)$. Clearly.

$$([0, 2] \times]0, 2]) \cup \{(0, 0)\} = L \setminus [(3, 1), (4, 1), \infty) \in \mathscr{G}(L),$$

$$([3, 5] \times [0, 2[) \cup \{(3, 2)\} = L \setminus (-\infty, (1, 1), (2, 1)] \in \mathscr{G}(L).$$

Thus, we may assume that $0 \leqq a \leqq 2$ and $0 \leqq x \leqq 2$, or $3 \leqq a \leqq 5$ and $3 \leqq x \leqq 5$. These cases are dual, and hence we will consider only the case where $0 \leqq a \leqq 2$ and $0 \leqq x \leqq 2$. Suppose that $b \neq y$. Without loss of generality, we may assume that $y < b$. Let $y < \delta < b$. Then (x, δ), $(a, \delta) \in L$, and

$$(L \setminus (-\infty, (x, y), (x, \delta)]) \cap (L \setminus [(a, \delta), (a, b), \infty)) = \phi,$$

$$(a, b) \in L \setminus (-\infty, (x, y), (x, \delta)] \in \mathscr{G}(L),$$

$$(x, y) \in L \setminus [a, \delta), (a, b), \infty) \in \mathscr{G}(L).$$

If $b = y$, then we may assume that $x < a$. Since $b = y$, $0 < y$. Thus, if $x < \gamma < a$, $(\gamma, b) = (\gamma, y) \in L$, and

$$(L \setminus (-\infty, (x, y), (\gamma, y)]) \cap (L \setminus [(\gamma, b), (a, b), \infty)) = \phi,$$
$$(a, b) \in L \setminus (-\infty, (x, y), (\gamma, y)] \in \mathcal{G}(L),$$
$$(x, y) \in L \setminus [(\gamma, b), (a, b), \infty) \in \mathcal{G}(L).$$

We conclude that $\mathcal{G}(L)$ is Hausdorff.

We will show next that $(L, \mathcal{G}(L))$ is not a topological lattice. Suppose that $(3, 0) \notin (-\infty, x, y]$ for some $x \leqq y$. Then clearly,

$$[(3, 0), (3, 1)] \subseteq L \setminus (-\infty, x, y]. \tag{6}$$

Suppose that $(3, 0) \notin [(a, b), (c, d), \infty)$ for $(a, b) \leqq (c, d)$. If $b = 0$, then $3 < a < c$, and thus,

$$[(3, 0), (3, 1)] \subseteq L \setminus [(a, b), (c, d), \infty). \tag{7}$$

If $b > 0$, then for $0 < \eta < b$,

$$[(3, 0), (3, \eta)] \subseteq L \setminus [(a, b), (c, d), \infty). \tag{8}$$

Let $Y = L \setminus [(1, 1), (2, 1), \infty)$. Then $(0, 0) \in Y \in \mathcal{G}(L)$. We note that $(2, 1) \wedge (3, 0) = (0, 0)$. If $(2, 1) \in A \in \mathcal{G}(L)$ and $(3, 0) \in B \in \mathcal{G}(L)$, then we wish to show that $A \wedge B \not\subseteq Y$. By definition of $\mathcal{G}(L)$,

$$(3, 0) \in \bigcap_{i=1}^{n} (L \setminus X_i) \subseteq B,$$

where the X_i are generalized initial and final segments. By (6), (7), and (8), there exists $0 < \mu < 1$ such that

$$[(3, 0), (3, \mu)] \subseteq \bigcap_{i=1}^{n} (L \setminus X_i) \subseteq B.$$

Thus, we have $(2, 1) \in A$ and $(3, \mu) \in B$, and hence $(2, \mu) \in A \wedge B$. Clearly

$$(2, \mu) \in [(1, 1), (2, 1), \infty),$$

and hence $(2, \mu) \notin Y$. Thus, $A \wedge B \not\subseteq Y$, and since $(2, 1) \wedge (3, 0) = (0, 0)$, we therefore conclude that

$$\wedge : (L \times L, \mathscr{G}(L) \times \mathscr{G}(L)) \rightarrow (L, \mathscr{G}(L))$$

is not continuous. Hence, $(L, \mathscr{G}(L))$ is not a topological lattice. We note that dually one may show that $\vee$ is also not continuous.

We show next that $\mathscr{G}(L) = \mathscr{G}^*(L)$. Clearly, if $(a, b) \in L$ is such that $0 < a$ and $0 < b$, then there exists $(c, d) \in L$ such that

$$(-\infty, (a, b)] = (-\infty, (c, d), (a, b)].$$

Furthermore, it is easy to see that if $x > 0$,

$$(-\infty, (0, x)] = (-\infty, (0,0), (0, x)] \cap \left(\bigcap_{n \in N} (-\infty, ((n+1)^{-1}, 1), (n^{-1}, 1)] \right),$$

$$(-\infty, (x, 0)] = (-\infty, (0,0), (x,0)] \cap \left(\bigcap_{n \in N} (-\infty, (0, (n+1)^{-1}), (0, n^{-1})] \right),$$

$$(-\infty, (0,0)] = \bigcap_{n \in N} (-\infty, ((n+1)^{-1}, (n+1)^{-1}), (n^{-1}, n^{-1})].$$

Similarly, final segments are closed with respect to $\mathscr{G}(L)$, and hence $\mathscr{I}(L) \subseteq \mathscr{G}(L)$. Since L is a lattice, this implies that $\mathscr{G}^*(L) = \mathscr{G}(L)$ by [6; Proposition 3.6]. We conclude that $\mathscr{G}^*(L)$ is Hausdorff, but that $(L, \mathscr{G}^*(L))$ is not a topological lattice.

Since $[(0, 0), (3, 0)] = \{(0, 0), (3, 0)\}$, and since $(-\infty, (0, 0)] = \{(0, 0)\}$, then L is not almost polar-dense. Furthermore, if $3 < t \leqq 4$, then

$$((0, 0), (3, 0))^{\perp} = ([0, 2] \times]0, 2]) \cup \{(0, 0)\},$$

$$((3, 0), (t, 0))^{\perp} = [(3, 0), (3, 2)[.$$

Thus, $(4, 0) \not\in ((0, 0), (3, 0))^{\perp}$, and if

$$(0, 0) \leqq (r, 0) < (3, 0) < (t, 0) \leqq (4, 0),$$

then

$$\begin{aligned}(3, 0) \vee ((r, 0), (3, 0))^{\perp} &= (3, 0) \vee ((0, 0), (3, 0))^{\perp} \\ &= [(3, 0), (3, 2)[\cup \{(5, 2)\} \\ &\neq ((3, 0), (t, 0))^{\perp}.\end{aligned}$$

Therefore, $[(r, 0), (t, 0)]$ does not have equivalent polars, and we conclude that L does not have minimal polars.

Added in Proof. Theorem 4.3 does not require the hypothesis that $\mathcal{G}(L)$ is T_1: Let $y = \varnothing$ instead of $\{y\}$ in the second part of the second paragraph of the proof.

REFERENCES

1. G. Birkhoff, *Lattice Theory*, third edition, Amer. Math. Soc. Coll. Pub. **25**, Providence, 1967.
2. N. Bourbaki, *General Topology*, Addison-Wesley Pub. Co., Don Mills, Ontario, 1966 (translated from the French, *Topologie Générale*, Hermann, Paris).
3. P. Conrad, *Lex-subgroups of lattice-ordered groups* , Czech. Math. J. **18** (1968, 86 - 103.
4. O. Frink, *Topology in lattices* , Trans. Amer. Math. soc., **51** (1942), 569 - 582.
5. R. H. Redfield, *A topology for a lattice-ordered group* , Trans. Amer. Math. Soc. **187** (1974), 103–125.
6. ———, *Generalized intervals and topology*, (to appear).
7. ———, *Almost polar-dense lattices*, Canad. Math. Bull., (to appear).
8. F. Šik, *Zur theorie der holbgeordnete Gruppen*, Czech. Math. J., **6** (1956), 1 - 25.
9. W. Thron, *Topological Structures*, Holt, Rinehart and Winston, New York, 1966.

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ALGEBRAIC MAXIMAL SEMILATTICES

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A topological semigroup S is maximal if it is closed in each topological semigroup that contains it. The semigroup S is called absolutely maximal if each continuous image is maximal. In this paper we are concerned with those discrete semilattices that are absolutely maximal. Thus we are concerned with those algebraic conditions on a semilattice which force it to be topologically closed.

In [9] Stralka studies those semigroups which have the congruence extension property. The semilattices we are concerned with and all their homomorphic images have this property. In fact, every congruence on such a semilattice S is closed. Thus S admits a compact Hausdorff topology $\mathscr{F}(S)$ under which multiplication is continuous. By [5] S admits a unique such topology. Also, since S has the congruence extension property for finite subsemilattices, the topology $\mathscr{F}(S)$ has a base which consists of subsemilattices [3].

In §II we give definitions, and we give necessary and sufficient conditions for a sublattice of a compact lattice to be closed. In §III we characterize those discrete semilattices and lattices which are absolutely maximal. Also, we show $(S, \mathscr{F}(S))$ is stable and 0-dimensional. In §IV we indicate how absolutely maximal discrete semilattices are constructed from a class of simple examples.

II. Definitions. Let S denote a topological semilattice. The Bohr compactification of S is a pair $(B(S), b_s)$ where $B(S)$ is a compact semilattice, $b_s\colon S \to B(S)$ is a continuous homomorphism and if $f\colon S \to T$ is a continuous homomorphism with T a compact semilattice, then there is a unique continuous homomorphism which makes the following diagram commute:

$$\begin{tikzcd}
B(S) \arrow[dr, "\hat{f}"] & \\
S \arrow[u, "b_S"] \arrow[r, "f"'] & T
\end{tikzcd}$$

For the existence of the Bohr Compactification see either [1] or [2].

For each $U \subseteq S$ with $U \neq \emptyset$ let $M(U) = \{y \in S \mid \text{there is an } x \in U \text{ with } xy = x\}$, $L(U) = U \cdot S$ and $CL(U)$ denotes the closure of U. Define $\leqq$ on S by $x \leqq y$ if and only if $xy = x$. Let $\{x_a\}_{a \in A}$ be a net in S. To say $x_\alpha \uparrow x$ means the net converges to x and $x_\alpha \leqq x_\beta$ whenever $\alpha \leqq \beta$. We define $x_\alpha \downarrow x$ is a similar manner. For a topological semilattice T Hom(S, T) denotes the collection of continuous homomorphisms from S to T. Let I denote the unit interval with $xy = \min\{x, y\}$ and let $I_1 = \{0, 1\} \subseteq I$.

PROPOSITION 1. *Let L be a compact topological semilattice with identity element and let A be a sublattice of S. Then A is closed if and only if A is complete.*

Proof. Assume A is complete and let $x \in \mathrm{CL}(A)$. Let $\mathcal{U}$ be the collection of sequences of open sets about x having the following property; $\{U_n\}_{n=1}^{\infty} \in \mathcal{U}$ if and only if $\mathrm{CL}(U_{n+1}) \wedge \mathrm{CL}(U_{n+1}) \subseteq U_n$ for $n = 1, 2, \cdots$. Partially order $\mathcal{U}$ by $\{U_n\}_{n=1}^{\infty} \leqq \{V_n\}_{n=1}^{\infty}$ if $V_n \subseteq U_n$ for all n. Then $\mathcal{U}$ with this partial order is a directed set. Now fix $\alpha = \{U_n\}_{n=1}^{\infty} \in \mathcal{U}$. Note that $\cap_{i=1}^{\infty} U_n = \cap_{i=1}^{\infty} \mathrm{CL}(U_n)$ is a sublattice of L and if $(\cap_{i=1}^{\infty} U_n) \cap A \neq \emptyset$, then $(\cap_{i=1}^{\infty} U_n) \cap A$ is closed under taking inf s and thus has a zero which will be denoted by $z(\alpha)$. Thus we show this intersection exists.

For each n let $b_n \in U_n \cap A$ and let $\{a_p^n\}_{p=1}^{\infty}$ be the sequence given by $a_p^n = \wedge_{j=1}^{p} b_{n+j}$. Then $\{a_p^n\}_{p=1}^{\infty} \subseteq U_n$ and is a decreasing sequence and thus has a limit point t in $\mathrm{CL}(U_n) \cap A$. Clearly, $t \in \mathrm{CL}(U_m)$ for all $m > n$ and thus $(\cap_{i=1}^{\infty} U_n) \cap A \neq \emptyset$. It is clear that if $\alpha, \beta \in \mathcal{U}$ with $\alpha < \beta$, then $z(\alpha) \leqq z(\beta)$. Thus $\{z(\alpha)\}_{\alpha \in \mathcal{U}}$ is an increasing net in A which converges to x. Since A is complete, and l compact, $x \in \mathrm{CL}(A)$.

In [5] Lawson defines B^+ for an ideal in a semilattice S to be $\{x \mid \text{there is a net } \{x_\alpha\}_{\alpha \in \Gamma} \subseteq B \text{ with } x_\alpha \uparrow x\}$. He shows for an ideal B in a compact semilattice S is closed if and only if $B^+ = B$. Thus one has

PROPOSITION 2. *Let B be a subsemilattice of a compact semilattice S. Then B is closed if and only if B contains arbitrary infs and $B = B^+$.*

We also need the following from [5].

PROPOSITION 3. *Let S and T be compact semilattices and lef f be a homomorphism from S to T. Then f is continuous if and only if f has the property that $f(x_\alpha) \uparrow f(x)$ whenever $x_\alpha \uparrow x$ and $f(y_\alpha) \downarrow f(y)$ whenever $y_\alpha \downarrow y$.*

Comment 4. It is not the case that a complete lattice necessarily admits a compact Hausdorff topology for which both operations are continuous. For consider the lattice on the integers with 0 the smallest element, 1 the largest element and each maximal chain having three elements. However, $(Z, \wedge)$ does admit a compact Hausdorff topology with $\wedge$ continuous.

III. Maximal semilattices and lattices. Throughout this section S will denote a discrete semilattice which is absolutely maximal. Since S is a locally compact semilattice with a base for the topology which consist of subsemilattices, $\mathrm{Hom}(S,I)$ separates points **[4]**. Thus there exists a continuous injection α from S into a compact semilattice. Since $\alpha(S)$ is closed it is compact and S therefore admits a compact topology $\mathscr{F}(S)$ with multiplication continuous. By **[5]**, $\mathscr{F}(S)$ is unique, and therefore $(\alpha(S), \alpha)$ is the Bohr compactification of S. Note that $\alpha(S)$ is the Bohr compactification of $\alpha(S)$ with the discrete topology. Therefore, we first characterize those compact semilattices T which are the Bohr compactification of T with the discrete topology.

For a semilattice T we let T_d denote T with the discrete topology.

PROPOSITION 5. *Let T be a compact semilattice with $T = B(T_d)$. Then*

(a) $\mathrm{Hom}(T, I_1)$ *separates points.*

(b) *If U is a subsemilattice of T, then $M(U)$ is both open and closed.*

(b′) *Each prime ideal of T is both open closed*

(c) $\dim S = 0$.

Proof. (a) Let $x, y \in T$ and assume $x \not\in M(y)$. Let $\phi\colon T \to I_1$ be given by $\phi(s) = 1$ if $s \in M(y)$ and 0 otherwise. Since $T = B(T_d)$, $\phi \in \mathrm{Hom}(T, I_1)$, and $\phi(y) = 1 \neq 0 = \phi(x)$. It now follows that Hom (T, I_1) separates points.

(b), (b′) Same as (a).

(c) Since $\mathrm{Hom}(S, I_1)$ separates points and S is compact, S can be embedded in a 0-dimensional semilattice and is therefore 0-dimensional.

LEMMA 6. *Let T be a compact semilattice with $M(U)$ both open and closed for each subsemilattice U of S. If C is a chain in T, then C is finite.*

Proof. Assume T has an infinite chain C. Then $\mathrm{CL}(C)$ is a chain and must have a limit point z. Since $M(z)$ is open, there is net $\{x_\alpha\}_{\alpha\in\Gamma}$ in C with $x_\alpha \downarrow z$ and $x_\alpha \neq z$ for each $\alpha \in \Gamma$. Let $N = \bigcap_{\alpha\in\Gamma} M(x\alpha)$; then

$M(U)$ is closed with $z \notin M(U)$ with is a contradiction. Thus C must be finite.

PROPOSITION 7. *Let T be a compact semilattice. Then the following are equivalent:*

(a) $T = B(T_d)$.

(b) *$M(U)$ is both open and closed for each subsemilattice U of S.*

(c) *Each chain in T is finite.*

(d) $\mathrm{Hom}(T, I_1) = \mathrm{Hom}(T_d, I_1)$.

(e) *There is a compact semilattice R with $|R| > 1$ with* $\mathrm{Hom}(T, R) = \mathrm{Hom}(T_d, R)$.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (e) trivial, (e) $\Rightarrow$ (b) by proof Proposition 5, (b) $\Rightarrow$ (c) by Lemma 6. Thus we show (c) $\Rightarrow$ (a).

Let $f \in \mathrm{Hom}(T_d, R)$ where R is a compact semilattice and each chain in T is finite. Let $\{x_\alpha\}_{\alpha \in \Gamma}$ be a net in T with $x_\alpha \downarrow x$. Since each chain is finite, eventually $x_\alpha = x$ and $f(x_\alpha) \downarrow f(x)$. Similarly, if $y_\alpha \uparrow y$ then $f(y_\alpha) \uparrow f(y)$. By Proposition 4, $f \in \mathrm{Hom}(T, R)$ and thus $T = B(T_d)$.

LEMMA 8. *Let T be a topological semilattice and let R be a subsemilattice of T with each chain finite. Then R is closed.*

Proof. Let $x \in \mathrm{CL}(R)$. Let $\mathcal{U}$ be the collection of sequences of open sets about x satisfying; $\{U_n\}_{n=1}^{\infty} \in \mathcal{U}$ if and only if $U_{n+i} U_{n+1} \leqq U_n$ for all n. Partially order $\mathcal{U}$ by $\{U_n\}_{n=1}^{\infty} \leqq \{V_n\}_{n=1}^{\infty}$ if and only if $V_n \subseteq U_n$ for all n. Then $\mathcal{U}$ with this partial order is a directed set. Fix $\{U_n\}_{n=1}^{\infty} = \alpha \in \mathcal{U}$. Then $\bigcap_{i=1}^{\infty} U_n$ is a subsemilattice of T and if $(\bigcap_{i=1}^{\infty} U_n) \cap R \neq \emptyset$, then $(\bigcap_{i=1}^{\infty} U_n) \cap R$ has a zero. For each positive integer n let $b_n \in U_n \cap R$, and for each positive integer p let $a_p^n = b_{n+1} b_{n+2} \cdots b_{n+p}$. As before, $a_p^n \in U_n$ for all p. Since each chain in R is finite, there is a q such that if $p > q$ then $a_q^n = a_q^n$. Thus $\{a_p^n\}_{p=1}^{\infty}$ converges to a^n in U_n. Clearly, if $m > n$ then $a^m \geqq a^n$. Thus there is a m_0 such that if $n \geqq m_0$ then $a^n = a^{m_0}$. It now follows that $a^{m_0} \in (\bigcap_{i=1}^{\infty} U_n) \cap R$. Let $z(\alpha)$ be the zero of $(\bigcap_{i=1}^{\infty} U_n) \cap R$. Thus $\{z(\alpha)\}_{\alpha \in \mathcal{U}}$ converges to x. Thus $r = x \in R$ and R is closed.

We now summarize our results in the form of a theorem.

THEOREM 9. *Let T be a discrete semilattice. Then T is absolutely maximal if and only if each maximal chain in T is finite.*

It is clear that we also have

COROLLARY 10. *Let L be a discrete lattice with each chain finite. Then each lattice homomorphic image of L is closed.*

We close this section with some additional properties a semilattice T with $T = B(T_d)$ must have. The proofs are all straightforward and will be omitted.

PROPOSITION 11. *Let T be a compact semilattice with $T = B(T_d)$. Then*

(a) *Each semilattice of T is closed.*
(b) *If R is a sublattice of T, then $R = B(R_d)$.*
(c) *If R is a homomorphic image of T, then $R = B(R_d)$.*
(d) *T is stable (that is, there are no dimension raising homomorphisms on T).*

IV. Examples. Throughout this section S_d is assumed to be a discrete absolutely maximal semilattice and S will denote $B(S_d)$. For each $x \in S_d$ let $A(x) = \{y \in S_d \mid x < y$ and $M(x) \cap L(y) = \{x, y\}$.

LEMMA 12. *For each $x \in S_d$ $A(x)$ is infinite if and only if x is a limit point of S. Further, if $A(x)$ is infinite, then* $\mathrm{CL}_S(A(x)) = A(x) \cup \{x\}$.

Proof. Assume $A(x)$ is infinite and let $\{y_\alpha\}_{\alpha \in \Gamma}$ be a net in $A(x)$ which converges (in S) to y. Assume each $y_\alpha \neq y$. Let $z \in A(x)$; then $zy_\alpha = x$ if $y_\alpha \neq z$. Thus $zy = x$. It now follows that $y = y^2 = \lim yy_\alpha = \lim x = x$. Thus $\mathrm{CL}_S(A(x)) = A(x) \cup \{x\}$ and x is a limit point of S.

Now assume x is a limit point of S and let $\{z_\alpha\}_{\alpha \in \Gamma}$ be a net in S which converges to x and $z_\alpha \neq x$. For each $\alpha \in \Gamma$ let $x_\alpha \in A(x) \cap L(Z_\alpha)$. Such x_α's exist since each chain in T is finite. Thus $\{x_\alpha\}_{\alpha \in \Gamma}$ is a net in $A(x)$ which converges to x. It now follows that $A(x)$ is infinite.

EXAMPLE 13. Let X be a compact well-ordered space and let B be the set of limit points of X. Let ρ be defined on X by $x\rho y$ if and only if $x = y$ or $x, y \in B$. Then X/ρ is a compact Hausdorff space. Define multiplication on X/ρ by $[x][y] = [x]$ if $[x] = [y]$ and B otherwise. Then X/ρ with this multiplication is a compact semilattice with each chain finite. Thus $(X/\rho)_d$ is an absolutely maximal semilattice.

EXAMPLE 14. *Let* $T = \{((1/n),(1/p)) \mid n,p$ positive integers, $n \leqq p \leqq 2n\} \cup \{(0,0)\}$ with multiplication defined by

$$\left(\frac{1}{n},\frac{1}{p}\right)\left(\frac{1}{m},\frac{1}{q}\right) = \begin{cases} (0,0) & \text{if} \quad n \neq m \\ \left(\frac{1}{n}, \min\left\{\frac{1}{p},\frac{1}{q}\right\}\right) & \text{if} \quad n = m. \end{cases}$$

Then each chain in T_d is finite and thus T_d is absolutely maximal. Note that although chain in T is finite there is no upper bound on the length of chains.

Observation 15. Let x be a limit point of S. Then $\mathrm{CL}_S(A(x))$ is isomorphic to X/ρ for a suitable compact well-ordered space X (see example 13).

Observation 16. There is a discrete semilattice T_d which is absolutely maximal and the set of limit points of T is S.

Question 17. If S is a maximal semilattice is it absolutely maximal?

Question 18. Are these reasonable conditions one can impose on a locally compact semilattice to guarantee that it be maximal?

REFERENCES

1. J. F. Buglund and K. H. Hofmann, *Compact semitopological semigroups and weakly almost periodic functions*, Lecture Notes; Springer-Verlag, New York (1967).
2. M. Friedberg and J. W. Stepp, *A note on the Bohr Compactification*, Semigroup Forum, **6** (1973), 362–364.
3. A.Y.W. Lau, *Small semilattices and costability*, Doctoral Dissertation, The University of Houston, Houston, Texas 1971.
4. J. D. Lawson, *Topological semilattices with small semilattices*, J. London Math. Soc., (2), (1969), 719–724.
5. ———, *Intrinsic topologies in topological lattices and semi-lattices*, (to appear).
6. J. W. Stepp, *A note on maximal locally compact semigroups*, Proc. Amer. Math. Soc., (20), **1** (1969), 251–253.
7. ———, *On the free compact lattice generated by a topological semilattice*, (to appear).
8. A. R. Stralka, *The congruence extension property for compact topological lattices*, Pacific J. Math., (38), **3** (1971), 795–802.
9. ———, *Extending congruences on semigroups*, Trans. Amer. Math. Soc., **166** (1972), 147–161.

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A SHEAF THEORETIC REPRESENTATION OF RINGS WITH BOOLEAN ORTHOGONALITIES

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It is shown that certain associative rings with Boolean orthogonalities are isomorphic to rings of global sections.

Let A be a ring and $\perp$ a relation on A. For each subset S of A define

$$S^{\perp} = \{x \in A \mid x \perp s \quad \text{for all} \quad s \in S\} \quad \text{and} \quad S^{\perp\perp} = (S^{\perp})^{\perp}.$$

When $S = \{s\}$ we write $s^{\perp}$ and $s^{\perp\perp}$ instead of $\{s\}^{\perp}$ and $\{s\}^{\perp\perp}$. Subsets of A of the form $S^{\perp}$ are *polars*. The relation $\perp$ is a *Boolean orthogonality* if all polars are two-sided ideals and if, for all $x, y \in A$,

1. $x \perp y \to y \perp x$, 2. $x \perp x \to x = 0$, and
3. $x^{\perp\perp} \cap y^{\perp\perp} = (0) \to x \perp y$.

The set of polars is a Boolean algebra (see [3]) with meet and join defined by

$$B \wedge C = B \cap C \quad \text{and} \quad B \vee C = (B^{\perp} \wedge C^{\perp})^{\perp}.$$

Boolean orthogonalities have been studied by Davis [3], Cornish [1] and by Cornish and Stewart [2].

Throughout this paper we shall assume that A is an associative ring with an identity and with a Boolean orthogonality $\perp$. We shall also assume that the following *finiteness condition* is satisfied:

for any two elements $x, y \in A$ there is a finite set $F \subseteq A$ such that $x^{\perp\perp} \wedge y^{\perp\perp} = F^{\perp\perp}$.

Notice that if $F = \{f_1, \cdots, f_n\}$, then $F^{\perp} = f_1^{\perp} \wedge \cdots \wedge f_n^{\perp}$ and $F^{\perp\perp} = f_1^{\perp\perp} \vee \cdots \vee f_n^{\perp\perp}$.

An ideal I of A is a $\perp$-*ideal* if $F^{\perp\perp} \subseteq I$ for every finite set $F \subseteq I$, and I is $\perp$-*prime* if $I \neq A$ and whenever the intersection of two polars B and C is contained in I, either $B \subseteq I$ or $C \subseteq I$.

LEMMA. *Assume that P is either a $\perp$-prime ideal or $P = A$, that I is a $\perp$-ideal and that $x \in A \setminus I$ is such that $x^{\perp\perp} \wedge a^{\perp\perp} \subseteq I$ implies that $a \in P$. Then there is a $\perp$-prin e $\perp$-ideal Q such that $I \subseteq Q \subseteq P$ and $x \notin Q$.*

Proof. Using Zorn's Lemma select a $\perp$-ideal $Q \supseteq I$ maximal with respect to the property "$x \notin Q$ and $x^{\perp\perp} \wedge a^{\perp\perp} \subseteq Q$ implies that $a \in P$". Clearly $I \subseteq Q \subseteq P$.

Suppose that B and C are polars neither of which is contained in Q. Choose $b \in B \backslash Q$ and $c \in C \backslash Q$. Then

$$B' = \cup \{F^{\perp\perp} \mid F \text{ is a finite subset of } \{b\} \cup Q\}$$

and

$$C' = \cup \{G^{\perp\perp} \mid G \text{ is a finite subset of } \{c\} \cup Q\}$$

are $\perp$-ideals which properly contain Q. By the maximality of Q either $x^{\perp\perp} \subseteq B'$ or $x^{\perp\perp} \wedge b_1^{\perp\perp} \subseteq B'$ for some $b_1 \in A \backslash P$, and $x^{\perp\perp} \subseteq C'$ or $x^{\perp\perp} \wedge c_1^{\perp\perp} \subseteq C'$ for some $c_1 \in A \backslash P$. Thus we obtain finite sets $\{b, f_1, \cdots, f_n\} \subseteq \{b\} \cup Q$ and $\{c, g_1, \cdots, g_m\} \subseteq \{c\} \cup Q$ such that one of $x^{\perp\perp}$, $x^{\perp\perp} \wedge b_1^{\perp\perp}$, $x^{\perp\perp} \wedge c_1^{\perp\perp}$ or $x^{\perp\perp} \wedge b_1^{\perp\perp} \wedge c_1^{\perp\perp}$ is contained in

$$\begin{aligned}
&\{b, f_1, \cdots, f_n\}^{\perp\perp} \wedge \{c, g_1, \cdots, g_m\}^{\perp\perp} \\
&\quad = (b^{\perp\perp} \vee f_1^{\perp\perp} \vee \cdots \vee f_n^{\perp\perp}) \wedge (c^{\perp\perp} \vee g_1^{\perp\perp} \vee \cdots \vee g_m^{\perp\perp}) \\
&\quad = (b^{\perp\perp} \wedge c^{\perp\perp}) \vee H^{\perp\perp}
\end{aligned}$$

where H is a finite subset of Q (we have used the distributivity of the Boolean algebra of polars and also the finiteness condition). If $b^{\perp\perp} \wedge c^{\perp\perp} \subseteq Q$, then $x^{\perp\perp} \subseteq Q$ or $x^{\perp\perp} \wedge d^{\perp\perp} \subseteq Q$ for some $d \in A \backslash P$ both of which contradict the choice of Q. Thus $B \cap C \nsubseteq Q$ and we conclude that Q is $\perp$-prime.

For the remainder of this paper $\bar{X}$ will be fixed set of $\perp$-prime ideals which contains all $\perp$-prime $\perp$-ideals and which is *full* (that is, if I is a sum of polars and $I \neq A$, then $I \subseteq P$ for some $P \in \bar{X}$).

PROPOSITION 2. (Cornish [1]). *For each $P \in \bar{X}$,*

$$\{x \in A \mid x^{\perp} \nsubseteq P\} = \cap \{R \in \bar{X} \mid R \subseteq P\} = \cap \{Q \in \bar{X} \mid Q \subseteq P$$

and Q is a $\perp$-prime $\perp$-ideal$\}$.

Proof. Suppose that $x^{\perp} \nsubseteq P$ and R is a $\perp$-prime ideal contained in P. Then $x^{\perp\perp} \wedge x^{\perp} = (0) \subseteq R$ and so $x \in R$.

If $x^{\perp} \subseteq P$, then by Lemma 1 (take $I = x^{\perp}$) there is a $\perp$-prime $\perp$-ideal $Q \subseteq P$ such that $x \notin Q$. This establishes the result.

The set described in the proposition will be denoted by O_P. We note that $O_P = P$ if and only if P is minimal in $\bar{X}$.

Let $P \in \bar{X}$. The set O_P, being an intersection of $\perp$-ideals, is itself a $\perp$-ideal. Define a relation (also denoted by $\perp$) on A/O_P by

$$(x + O_P) \perp (y + O_P) \leftrightarrow x^{\perp\perp} \wedge y^{\perp\perp} \subseteq O_P.$$

This relation is well-defined because if $x_1 = x + a$ and $y_1 = y + b$ where $a, b \in O_P$, then

$$x_1^{\perp\perp} \wedge y_1^{\perp\perp} = (x + a)^{\perp\perp} \wedge (y + b)^{\perp\perp} \subseteq (x^{\perp\perp} \vee a^{\perp\perp}) \wedge (y^{\perp\perp} \vee b^{\perp\perp})$$

and so $x_1^{\perp\perp} \wedge y_1^{\perp\perp} \subseteq (x^{\perp\perp} \wedge y^{\perp\perp}) \vee F^{\perp\perp}$ where F is a finite subset of O_P. It is routine to check that

$$x^{\perp} + O_P \subseteq (x + O_P)^{\perp} \quad \text{and} \quad x^{\perp\perp} + O_P \subseteq (x + O_P)^{\perp\perp}$$

for each $x \in A$, and that the relation $\perp$ is a Boolean orthogonality on A/O_P.

PROPOSITION 3. *For each $P \in \bar{X}$, $\bar{P} = P/O_P$ is a $\perp$-prime ideal of A/O_P which contains all proper polars of A/O_P.*

Proof. Let $\bar{B}$ and $\bar{C}$ be polars in A/O_P such that $\bar{B} \cap \bar{C} \subseteq \bar{P}$. Suppose that $\bar{B} \not\subseteq \bar{P}$. Then there is an element $b \in A$ such that $b + O_P \in \bar{B} \backslash \bar{P}$. Let $c + O_P \in \bar{C}$. Then

$$(b^{\perp\perp} + O_P) \cap (c^{\perp\perp} + O_P) \subseteq (b + O_P)^{\perp\perp} \cap (c + O_P)^{\perp\perp} \subseteq \bar{B} \cap \bar{C} \subseteq \bar{P}$$

and so $b^{\perp\perp} \cap c^{\perp\perp} \subseteq P$. Since $b \notin P$ we conclude that $c \in P$ and so $\bar{C} \subseteq \bar{P}$. Thus $\bar{P}$ is $\perp$-prime.

Suppose that $a^{\perp\perp} \wedge b^{\perp\perp} \subseteq O_P$. Then there is a finite set $\{f_1, \cdots, f_n\} \subseteq O_P$ such that

$$a^{\perp\perp} \wedge b^{\perp\perp} = \{f_1, \cdots, f_n\}^{\perp\perp} = f_1^{\perp\perp} \vee \cdots \vee f_n^{\perp\perp}.$$

For each $i = 1, \cdots, n$, $f_i \in O_P$ and so $f_1^{\perp} \not\subseteq P$. Thus $f_1^{\perp} \wedge \cdots \wedge f_n^{\perp} \not\subseteq P$. Also, $b^{\perp\perp} \wedge f_1^{\perp} \wedge \cdots \wedge f_n^{\perp} \subseteq a^{\perp}$ because $a^{\perp\perp} \wedge b^{\perp\perp} \wedge f_1^{\perp} \wedge \cdots \wedge f_n^{\perp} = (0)$. If $a \notin O_p$, then $a^{\perp} \subseteq P$ and so, since $f_1^{\perp} \wedge \cdots \wedge f_n^{\perp} \not\subseteq P$, $b^{\perp\perp} \subseteq P$. Thus $\bar{P}$ contains $(a + O_P)^{\perp}$ for all $a \notin O_P$. It follows that $\bar{P}$ contains all proper polars of A/O_P.

Let S be the disjoint union of the factor rings A/O_P. The relation (also denoted by $\perp$) on the product

$$\Pi\{A/O_P \mid P \in \bar{X}\}$$
$$= \{f\colon \bar{X} \to S \mid f(P) \in A/O_P \quad \text{for} \quad \text{all} \quad P \in \bar{X}\}$$

defined by

$$f \perp g \leftrightarrow f(P) \perp g(P) \quad \text{in} \quad A/O_P \quad \text{for} \quad \text{all} \quad P \in \bar{X}$$

is a Boolean orthogonality. Each $a \in A$ determines a function $\hat{a} \in \Pi\{A/O_P \mid P \in \bar{X}\}$ defined by $\hat{a}(P) = a + O_P$. It follows from Lemma 1 that $\cap\{P \mid P$ is a $\perp$-prime $\perp$-ideal$\} = (0)$ and so $\cap\{O_P \mid P \in \bar{X}\} = (0)$. Thus we obtain the usual embedding

$$A \hat{\to} \hat{A} \subseteq \Pi\{A/O_P \mid P \in \bar{X}\}.$$

This embedding respects orthogonalities; that is, $a \perp b$ in A if and only if $\hat{a} \perp \hat{b}$ in the product.

We define a topology on $\bar{X}$ by declaring the basic open sets to be the subsets of the form

$$\bar{X}(a) = \{P \in \bar{X} \mid a^{\perp\perp} \not\subseteq P\}.$$

Notice that $\bar{X}(a) \cap \bar{X}(b) \supseteq \bar{X}(c)$ for all $c \in a^{\perp\perp} \wedge b^{\perp\perp}$ and so these sets do qualify as a topological base.

Suppose that $\{\bar{X}(a) \mid a \in C\}$ is a cover of $\bar{X}$ consisting of basic open sets. Then $\Sigma\{a^{\perp\perp} \mid a \in C\} = A$ because $\bar{X}$ is full. Since A has an identity there is a finite set $F \subseteq C$ such that $\Sigma\{a^{\perp\perp} \mid a \in F\} = A$. Thus $\{\bar{X}(a) \mid a \in F\}$ covers $\bar{X}$ and so $\bar{X}$ is quasi-compact.

Give S the topology generated by sets of the form $\hat{a}[U] = \{a + O_P \mid P \in U\}$ where U is open in $\bar{X}$ and $a \in A$. We obtain a sheaf of rings $(S, \pi, \bar{X})$ where $\pi\colon S \to \bar{X}$ is the projection onto $\bar{X}$.

Let $\Gamma = \{f \mid f \in \Pi\{A/O_P \mid P \in \bar{X}\}$ is continuous$\}$ be the ring of global sections. The following observation shows that $\hat{A} \subseteq \Gamma$: for all $x, y \in A$, $\{P \in \bar{X} \mid x - y \in O_P\}$ is open in $\bar{X}$. To see this notice that if $x - y \in O_Q$, then $Q \in \bar{X}(u) \subseteq \{P \in \bar{X} \mid x - y \in O_P\}$ where u is any element in $(x - y)^{\perp} \backslash Q$.

THEOREM 4. $\hat{A} = \Gamma$.

Proof. Let $f \in \Gamma$. Since $\bar{X}$ is quasi-compact there are finite sets $\{a_1, \cdots, a_n\}$ and $\{v_1, \cdots, v_n\}$ such that $\bar{X} = \bar{X}(a_1) \cup \cdots \cup \bar{X}(a_n)$ and $f(P) = v_i + O_P$ for all $P \in \bar{X}(a_i)$.

Notice that $v_i - v_j \in \cap\{O_P \mid P \in \bar{X}(a_i) \cap \bar{X}(a_j)\}$, so $(v_i - v_j)^{\perp} \subseteq$

$Q \in \bar{X}$ implies that $a_i^{\perp\perp} \wedge \imath_j^{\perp\perp} \subseteq Q$. It follows from Lemma 1 (take $P = A$ and $I = (v_i - v_j)^\perp$ for each $x \not\in (v_i - v_j)^\perp$) that

$$(v_i - v_j)^\perp = \cap \{Q \mid (v_i - v_j)^\perp \subseteq Q \in \bar{X}\}$$

and so $a_i^{\perp\perp} \wedge \imath_j^{\perp\perp} \subseteq (v_i - v_j)^\perp$. Thus $(v_i - v_j)^{\perp\perp} \wedge \imath_i^{\perp\perp} \subseteq a_j^\perp$.

Since $\bar{X} = \bar{X}(a_1) \cup \cdots \cup \bar{X}(a_n)$ and $\bar{X}$ is full, $a_1^{\perp\perp} + \cdots + a_n^{\perp\perp} = A$. Choose $u_i \in a_i^{\perp\perp}$ such that $1 = u_1 + \cdots + u_n$ and let $v = u_1 v_1 + \cdots + u_n v_n$. Then

$$v - v_j = u_1(v_1 - v_j) + \cdots + u_n(v_n - v_j) \in a_j^\perp \subseteq O_P$$

for all $P \in \bar{X}(a_j)$. Thus $f(P) = v_j + O_P = v + O_P$ for all $P \in \bar{X}(a_j)$ and so $f = \hat{v} \in \hat{A}$.

f-rings (Keimal [4]). Let A be an f-ring with identity. The relation defined by $x \perp y \leftrightarrow |x| \wedge |y| = 0$ is a Boolean orthogonality and $x^{\perp\perp} \wedge y^{\perp\perp} = (|x| \wedge \ y|)^{\perp\perp}$. Let $\bar{X}$ be the set of irreducible ℓ-ideals. Then $\bar{X}$ is full because polars are ℓ-ideals and sums of ℓ-ideals are again ℓ-ideals. Also, all $\perp$-prime $\perp$-ideals are irreducible ℓ-ideals and so A is isomorphic to the f-ring of all global sections of the sheaf $(S, \pi, \bar{X})$.

Reduced rings (Koh [5]). Let A be a ring with identity and no nonzero nilpotent elements. The relation defined by $x \perp y \leftrightarrow xy = 0$ is a Boolean orthogonality and $x^{\perp\perp} \wedge y^{\perp\perp} = (xy)^{\perp\perp}$. Let $\bar{X}$ be the set of all prime ideals of A. Clearly $\bar{X}$ is full. Also, all $\perp$-prime $\perp$-ideals are completely prime and so A is isomorphic to the ring of global sections of the sheaf $(S, \pi, \bar{X})$. Each stalk A/O_P is reduced (Proposition 2) and the prime ideal P/O_P contains all zero divisors (Proposition 3).

Semiprime rings. Let A be a semiprime ring with identity. The relation defined by $x \perp y \leftrightarrow (x)(y) = (0)$ is a Boolean orthogonality. However, the finiteness condition may not be satisfied as the following example shows.

Let R be a semiprime ring with identity, R' the ring of 3×3 matrices with entries from R,

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Define $\bar{x}$ and $\bar{y}$ in $P = \Pi\{R_n \mid R_n = R' \text{ for } n = 1, 2, \cdots\}$ by

$$\bar{x}(n) = \begin{cases} x & \text{if} \quad n \equiv 1 \pmod 2 \\ 0 & \text{if} \quad n \not\equiv 1 \pmod 2 \end{cases}$$

$$\bar{y}(n) = \begin{cases} y & \text{if} \quad n \equiv 1 \pmod 3 \\ 0 & \text{if} \quad n \not\equiv 1 \pmod 3. \end{cases}$$

Notice that $\bar{x}\bar{y} = \bar{y}\bar{x} = \bar{x}^2 = \bar{y}^2 = 0$. Let E be the subring of P which is generated by the identity of P, $\bar{x}$, $\bar{y}$ and

$$\Sigma\{R_n \mid R_n = R' \quad \text{for} \quad n = 1, 2, \cdots\}. \quad \text{Then}$$

$$\bar{x}^{\perp\perp} = \{f \in E \mid f(n) = 0 \quad \text{for} \quad n \not\equiv 1 \pmod 2\},$$

$$\bar{y}^{\perp\perp} = \{f \in E \mid f(n) = 0 \quad \text{for} \quad n \not\equiv 1 \pmod 3\},$$

and so

$$\bar{x}^{\perp\perp} \wedge \bar{y}^{\perp\perp} = \{f \in E \mid f(n) = 0 \quad \text{for} \quad n \not\equiv 1 \pmod 6\}.$$

If $\bar{x}^{\perp\perp} \wedge \bar{y}^{\perp\perp} = \{f_1, \cdots, f_n\}^{\perp\perp}$, then at least one of the f_i must satisfy $f_i(n) \neq 0$ for infinitely many positive integers n. But then there are integers α, β and γ such that $f_i(n) = (\alpha + \beta\bar{x} + \gamma\bar{y})(n)$ for all but a finite number of positive integers n. This is incompatible with the requirement that $f_i(n) = 0$ for $n \not\equiv 1 \pmod 6$.

When the finiteness condition is satisfied (for instance, when A satisfies the maximum condition on annihilators), A is isomorphic to the ring of all global sections of the sheaf $(S, \pi, \underset{\sim}{X})$ where $\underset{\sim}{X}$ is the set of prime ideals of A. Each stalk A/O_P is semiprime (Proposition 1) and the prime ideal P/O_P contains all two-sided annihilator ideals of A/O_P (Proposition 2).

References

1. W. H. Cornish, *Boolean orthogonalities and minimal primes*, preprint.

2. W. H. Cornish and P. N. Stewart, *Direct and subdirect decompositions of universal algebras with a Boolean orthogonality*, preprint.

3. G. Davis, *Rings with orthogonality relations*, Bull. Austral. Math. Soc., **4** (1971), 163–178.

4. K. Keimal, *The representation of lattice ordered groups and rings by sections in sheaves*, Lecture Notes in Math. 248, 1–98, Springer, Berlin-Heidelberg-New York, 1971.

5. K. Koh, *On functional representations of a ring without nilpotent elements*, Canad. Math. Bull., **14** (1971), 349–352.

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A GENERALIZED JENSEN'S INEQUALITY

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A generalized Jensen's inequality for conditional expectations of Bochner-integrable functions which extends the results of Dubins and Scalora is proved using a different method.

1. Introduction. Let $(\Omega, \mathbf{F}, P)$ be a probability space, $(\mathbf{U}, \|\cdot\|)$ a complex (or real) Banach space and $(\mathbf{V}, \|\cdot\|, \geqq_v)$ an ordered Banach space over the complex (or real) field such that the positive cone $\{v \in \mathbf{V} : v \geqq_v \theta\}$ is closed. Let x be a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$. Let $\mathbf{G}$ be a sub-σ-field of the σ-field $\mathbf{F}$ and let f be a function on $\Omega \times \mathbf{U}$ to $\mathbf{V}$ such that for each $u \in \mathbf{U}$ the function $f(\cdot, u)$ is strongly measurable with respect to $\mathbf{G}$ and such that for each $\omega \in \Omega$ the function $f(\omega, \cdot)$ is continuous and convex in the sense that $tf(\omega, u_1) + (1-t)\ f(\omega, u_2) \geqq_v f(\omega, tu_1 + (1-t)u_2)$ whenever u_1, $u_2 \in \mathbf{U}$ and $0 \leqq t \leqq 1$. For any Bochner-integrable function z on $(\Omega, \mathbf{F}, P)$ to any Banach space $\mathbf{W}$, we define $E[z \mid \mathbf{G}]$ "a conditional expectation of z relative to $\mathbf{G}$" as a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{W}$ such that $E(z \mid \mathbf{G}]$ is strongly measurable with respect to $\mathbf{G}$ and that

$$\int_A E[z \mid \mathbf{G}](\omega) dP = \int_A z(\omega) dP, \qquad A \in \mathbf{G},$$

where the integrals are Bochner-integrals.

The purpose of this note is to prove the following generalized Jensen's inequality:

THEOREM. *If $f(\cdot, x(\cdot))$ is Bochner-integrable, then*

$$E[f(\cdot, x(\cdot)) \mid \mathbf{G}](\omega) \geqq_v f(\omega, E[x \mid \mathbf{G}](\omega)) \quad \text{a.e.} \tag{J}$$

The above theorem extends the results of Dubins **[2]** (cf. Mayer **[5**, p. 79]) and Scalora **[6**, p. 360, Theorem 2.3]. It is proved in **[2]** that the theorem is true for the case in which the spaces $\mathbf{U}$ and $\mathbf{V}$ are both the real numbers $\mathbf{R}$, while in **[6]** Scalora uses the methods of Hille-Phillips **[4]** to prove the theorem when the function $f(\omega, u)$ is replaced by a continuous, subadditive positive-homogeneous function $g(u)$ on $\mathbf{U}$ to $\mathbf{V}$. It should be noted that the method of the prôof used here is different than those used previously, the previous methods appear to be ineffective for a proof of the extension.

2. Preliminaries. We refer to [4] and [6] for the definitions and basic properties of the concepts of Bochner-integrals and the conditional expectation of a Bochner-integrable function. Our proof of the theorem is based on the following lemmas. Unless otherwise specified, functions in Lemma 1–5 are defined on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$.

LEMMA 1. ([4, p. 73, Corollary 1]). *A function is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countably-valued functions.*

LEMMA 2. (Egoroff's theorem, [4, p. 72] or [3, p. 149]). *A sequence $\{z_j\}_{i=1}^{\infty}$ of strongly measurable functions is almost uniformly convergent to a function z if and only if*

$$\|z_i(\omega) - z(\omega)\| \to 0 \text{ a.e. } as\, i \to \infty.$$

The following lemma is an immediate consequence of Lemma 1 and Lemma 2.

LEMMA 3. *If z is a strongly measurable function, then for any positive number M there exists a sequence $\{z_i\}_{i=1}^{\infty}$ of simple functions such that $\|z_i(\omega)\| \leqq \|z(\omega)\| + M$ a.e., $i = 1, 2, \cdots$, and $\|z_i(\omega) - z(\omega)\| \to 0$ a.e. as $i \to \infty$.*

LEMMA 4. ([6, p. 356, Theorem 2.2]).

(a) *If $z(\omega) = u$ on Ω then $E[z \mid \mathbf{G}](\omega) = u$ a.e.*

(b) *If z and z_i, $i = 1, 2, \cdots$, are Bochner-integrable functions such that $z(\omega) = \sum_{i=1}^{n} t_i z_i(\omega)$ a.e. where t_i are scalars then $E[z \mid \mathbf{G}](\omega) = \sum_{i=1}^{n} t_i E[z_i \mid \mathbf{G}](\omega)$ a.e.*

(c) $\|E[z \mid \mathbf{G}](\omega)\| \leqq E[\|z\| \mid \mathbf{G}](\omega)$ a.e., *for any Boxhner-integrable function z.*

(d) *If z is a Bochner-integrable function and $z_i, i = 1, 2, \cdots$, are strongly measurable functions such that $\|z_i(\omega) - z(\omega)\| \to 0$ a.e. as $i \to \infty$, and if there is a real-valued integrable function $y(\omega) \geqq 0$ such that $\|z_i(\omega)\| \leqq y(\omega)$ a.e., $i = 1, 2, \cdots$, then z_i's are Bochner-integrable and $\|E[z_i \mid \mathbf{G}](\omega) - E[z \mid \mathbf{G}](\omega)\| \to 0$ a.e. as $i \to \infty$.*

LEMMA 5. *If z is a Bochner-integrable function and $z_i, i = 1, 2, \cdots$, are strongly measurable functions such that $\|z_i(\omega) - z(\omega)\| \to 0$ uniformly a.e. as $i \to \infty$, then there exists an integer N such that $z_i, i = N, N+1, \cdots$, are Bochner-integrable functions, and*

$$\|E[z_i \mid \mathbf{G}](\omega) - E[z \mid \mathbf{G}](\omega)\| \to 0 \textit{ uniformly}$$

a.e. *as $i \to \infty$.*

Proof. An immediate consequence of Lemma 4 and the fact that $E[\cdot|\mathbf{G}]$ is a positive operator on the space of all real-valued integrable functions.

LEMMA 6. *If z is a strongly measure function on $(\Omega,\mathbf{G},P)$ to a Banach space $\mathbf{W}$, and if on $(\Omega,\mathbf{F},P)$, y is a numerically-valued integrable function such that zy is a Bochner-integrable function with values in $\mathbf{W}$, then*

$$E[zy\,|\,\mathbf{G}](\omega) = zE[y\,|\,\mathbf{G}](\omega) \text{ a.e..}$$

Proof. By using Lemma 3 and Lemma 4, the proof when $\mathbf{W}$ is the real numbers $\mathbf{R}$ as given by Billingsley [**1**, p. 110, Theorem 10.1] can be applied to obtain the general result.

LEMMA 7. *Let g be a convex function on $\mathbf{U}$ to $\mathbf{V}$. If $u_i \in \mathbf{U}$ and $t_i \in \mathbf{R}$, $t_i \geqq 0$, $i = 1,2,\cdots n$, such that*

$$\sum_{i=1}^{n} t_i = 1, \textit{ then} \sum_{i=1}^{n} t_i g(u_i) \geqq_v g\left(\sum_{i=1}^{n} t_i u_i\right).$$

Proof. By induction.

3. Proof of the theorem. We first note that if $F \in \mathbf{F}$ with $P(F) > 0$ and z is a simple function on $(\Omega,\mathbf{F},P)$ to $\mathbf{U}$ such that $\chi_F f(\cdot, z(\cdot))$ is Bochner-integrable, then

$$(1)\quad E[\chi_F f\left(\cdot, z(\cdot)\,|\,\mathbf{G}\right](\omega) \geqq_v E[\chi_F\,|\,\mathbf{G}](\omega) f\left(\omega, \frac{E[\chi_F z\,|\,\mathbf{G}](\omega)}{E[\chi_F\,|\mathbf{G}](\omega)}\right) \text{ a.e. on } F.$$

To see this, let $z = \sum_{i=1}^{n} u_i \chi_{A_i}$, where $u_i \in \mathbf{U}$ and A_i's are disjoint sets of $\mathbf{F}$ such that $\sum_{i=1}^{n} \chi_{A_i} = 1$. It is clear that $F \subset \{\omega : E[\chi_F\,|\mathbf{G}](\omega) > 0\}$ a.e.. Since $f(\cdot, u_i)$ is strongly measurable with respect to $\mathbf{G}$ and $f(\omega, \cdot)$ is convex, by using Lemma 4, (b), Lemma 6 and Lemma 7, we then have

$$\frac{1}{E[\chi_F\,|\,\mathbf{G}](\omega)} E[\chi_F f(\cdot, z(\cdot))\,|\,\mathbf{G}](\omega)$$

$$= \frac{1}{E[\chi_F\,|\,\mathbf{G}](\omega)} \sum_{i=1}^{n} f(\omega, u_i)\, E[\chi_F \chi_{A_i}\,|\,\mathbf{G}](\omega) \text{ a.e. on } F.$$

$$\geqq_v f\left(\omega, \frac{1}{E[\chi_F|\mathbf{G}](\omega)} \sum_{i=1}^{n} u_i E[\chi_F\chi_{A_i}|\mathbf{G}](\omega)\right) \text{a.e. on } F$$

$$= f\left(\omega, \frac{E[\chi_F z|\mathbf{G}](\omega)}{E[\chi_F|\mathbf{G}](\omega)}\right) \text{ a.e. on } F.$$

Nextly, since x is assumed to be a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$, x is strongly measurable, and hence by the definition of strong measurability (or by Lemma 3) there exists a sequence $\{x_i\}_{i=1}^{\infty}$ of simple functions on $(\Omega, \mathbf{F}, p)$ to $\mathbf{U}$ such that $\|x_i(\omega) - x(\omega)\| \to 0$ a.e.. Furthermore, since $f(\omega, \cdot)$ is continuous on $\mathbf{U}$ it follows that $\|f(\omega, x_i(\omega)) - f(\omega, x(\omega))\| \to 0$ a.e..

Therefore, by Lemma 2 we can find an increasing sequence, $\Omega_1 \subset \Omega_2 \subset \cdots$, of sets of $\mathbf{F}$ with $P(\Omega - \Omega_k) < 1/k$, $k = 1, 2, \cdots$, such that

(2) $\|\chi_{\Omega_{k_0}}(\omega)x_i(\omega) - \chi_{\Omega_{k_0}}(\omega)x(\omega)\| \to 0$ uniformly a.e. and

(3) $\|\chi_{\Omega_k}(\omega)f(\omega, x_i(\omega)) - \chi_{\Omega_k}(\omega)f(\omega, x(\omega))\| \to 0$ uniformly a.e., as $i \to \infty$, for each $k = 1, 2, \cdots$.

According to Lemma 5, (2) implies

(2′) $\|E[\chi_{\Omega_k}x_i|\mathbf{G}](\omega) - E[\chi_{\Omega_k}x|\mathbf{G}](\omega)\| \to 0$ uniformly a.e. as $i \to \infty$, for each $k = 1, 2, \cdots$, and (3) implies

(3′) $\|E[\chi_{\Omega_k}f(\cdot, x_i(\cdot))|\mathbf{G}](\omega) - E[\chi_{\Omega_k}f(\cdot, x(\cdot))|\mathbf{G}](\omega)\| \to 0$ uniformly a.e. as $i \to \infty$, for each $k = 1, 2, \cdots$.

Now by using the continuity of $f(\omega, \cdot)$ again, it follows from (2′) that

$$(4) \qquad \left\| f\left(\omega, \frac{E[\chi_{\Omega_k}x_i|\mathbf{G}](\omega)}{E[\chi_{\Omega_k}|\mathbf{G}](\omega)}\right) - f\left(\omega, \frac{E[\chi_{\Omega_k}x|\mathbf{G}](\omega)}{E[\chi_{\Omega_k}|\mathbf{G}](\omega)}\right)\right\| \to 0$$

a.e. on Ω_k as $i \to \infty$.

On the other hand, from (1) we obtain

$$(1') \quad E(\chi_{\Omega_k}f(\cdot, x_i(\cdot))|\mathbf{G}](\omega) \geqq_v E[\chi_{\Omega_k}|\mathbf{G}](\omega) f\left(\omega, \frac{E[\chi_{\Omega_k}x_i|\mathbf{G}](\omega)}{E[\chi_{\Omega_k}|\mathbf{G}](\omega)}\right)$$

a.e. on Ω_k, for each $k = 1, 2, \cdots$, and each $i = 1, 2, 3 \cdots$.

Letting $i \to \infty$ in (1′) and using (3′) and (4), we obtain

$$(1'') \quad E[\chi_{\Omega_k}f(\cdot, x(\cdot))|\mathbf{G}](\omega) \geqq_v E[\chi_{\Omega_k}|\mathbf{G}](\omega) f\left(\omega, \frac{E[\chi_{\Omega_k}x|\mathbf{G}](\omega)}{E[\chi_{\Omega_k}|\mathbf{G}](\omega)}\right),$$

a.e. on Ω_k, since the positive cone of $(\mathbf{V}; \geqq_v)$ is closed.

Finally, since $|\chi_{\Omega_k}(\omega)| \leqq 1$ and $\chi_{\Omega_k}(\omega) \to 1$ a.e., by using Lemma 4, (a) and (d), and the continuity of $f(\omega, \cdot)$, when $k \to \infty$ we have

$$\text{(J)} \qquad E[f(\cdot, x(\cdot))|\mathbf{G}](\omega) \geqq_v f(\omega, E[x|\mathbf{G}](\omega)) \quad \text{a.e.}.$$

4. Remark. In particular, when $\mathbf{G}$ is the trivial sub-σ-field $\mathbf{Z} = \{\Omega, \phi\}$, inequality (J) reduces to

$$\text{(J}'\text{)} \qquad \int_\Omega f(\omega, x(\omega))dP \geqq_v f\left(\omega, \int_\Omega x(\omega)dP\right).$$

When the function $f(\omega, u)$ is replaced by a continuous and convex function g on $\mathbf{U}$ to $\mathbf{V}$, inequalilties (J) and (J′) become

$$\text{(K)} \qquad E[g(x(\cdot))|\mathbf{G}](\omega) \geqq_v g(E[x|\mathbf{G}](\omega)) \quad \text{a.e. and}$$

$$\text{(K}'\text{)} \qquad \int_\Omega g(x(\omega))dP \geqq_v g\left(\int_\Omega x(\omega)dP\right).$$

As we have mentioned in the introduction, this result extends a theorem of Scalora [6] in which the stronger condition that g is subadditive and positive-homogeneous is assumed.

References

1. Patrick Billingsley, *Ergodic Theory and Information, New York, J. Wiley and Sons*, 1965.
2. *L. E. Dubins, Rises and upcrossings of nonnegative martingales*, Illinois J. Math., **6** (1962), 226–241.
3. N. Dunford and J. T. Schwartz, *Linear Operators. Part I* : General Theory, Interscience, New York, 1958.
4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publ., Vol. 31, Providence, 1957.
5. P. A. Meyer, *Probability and Potentials*, London, Blaisdell Publishing Col, 1966.
6. F. S. Scalora, *Abstract martingale convergence theorems*, Pacific J. Math., **11** (1961), 347–374.

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SECOND ORDER DIFFERENTIAL OPERATORS WITH SELF-ADJOINT EXTENSIONS

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Let $\mathscr{H}$ denote the Hilbert space of square summable analytic functions on the unit disk, and consider those formal differential operators

$$L = p_2 \frac{d^2}{dz^2} + p_1 \frac{d}{dz} + p_0$$

which give rise to symmetric operators in $\mathscr{H}$. Examples have been given where the symmetric operators associated with these formal operators have defect indices (0, 0) and (2, 2) and hence are either self-adjoint or have self-adjoint extensions in $\mathscr{H}$. In this note a class of symmetric operators with defect indices (1, 1) is given.

Let $\mathscr{A}$ denote the space of functions analytic on the unit disk and $\mathscr{H}$ the subspace of square summable functions in $\mathscr{A}$ with inner product

$$(f, g) = \iint_{|z|<1} f(z)\overline{g(z)}\, dxdy.$$

A complete orthonormal set for $\mathscr{H}$ is obtained by normalizing the powers of z. From this it follows that $\mathscr{H}$ is identical with the space of power series $\Sigma_{n=0}^{\infty} a_n z^n$ which satisfy

$$\sum_{n=0}^{\infty} |a_n|^2/(n+1) < \infty. \tag{1.1}$$

Let L be such that it maps polynomials into $\mathscr{H}$ and has the property $(Lz^n, z^m) = (z^n, Lz^m)$, $n, m = 0, 1, 2, \cdots$. Let $\mathscr{D}_0$ be the subspace of polynomials and set $T_0 f = Lf$ for f in $\mathscr{D}_0$. Then T_0 is symmetric and the defect indices m^+ and m^- of its closure, S, are just the number of linearly independent solutions of $Lu = iu$ and $Lu = -iu$ respectively which are in $\mathscr{H}$. See [2]. In [2] and [3] examples of such symmetric operators S with defect indices (0, 0) and (2, 2) are provided. We now give a class of operators with defect indices (1, 1).

2. Consider the operator L,

$$L = (c_1 z^3 + \bar{c}_1 z)\frac{d^2}{dz^2} + ((c_2 + 3c_1)z^2 + \bar{c}_2)\frac{d}{dz} + 2c_2 z. \tag{2.1}$$

In [3] it is shown that L gives rise to symmetric T_0. Concerning the defect indices of its closure S, we have the following.

THEOREM 2.1. *Let L be the operator of* (2.1) *then S has defect indices $m^+ = m^- = 1$.*

Proof. The idea of the proof is to show that the equation $L\phi = \pm i\phi$ has precisely one power series solution $\phi(z) = \sum_{j=0}^{\infty} a_j z^j$ and that there exists a $K > 0$ and a positive integer p such that $|a_j| \leqq K j^{-1/p}$ for j sufficiently large. Consequently the series $\sum_{j=0}^{\infty} |a_j|^2/(j+1)$ converges and ϕ belongs to $\mathcal{H}$, and $m^+ = m^- = 1$.

Dividing $L\phi = \pm i\phi$ by c_1 we have the differential equation

$$(z^3 + \omega z)\phi'' + [(3+\alpha)z^2 + \beta]\phi' + 2\alpha z\phi = \lambda\phi, \tag{2.2}$$

where $\omega = \bar{c}_1/c_1$, $\alpha = c_2/c_1$, $\beta = \bar{c}_2/c_1$, and $\lambda = \pm i/c_1$.

Substituting $\sum_{j=0}^{\infty} a_j z^j$ into (2.2) we obtain

$$\beta a_1 + \sum_{j=1}^{\infty} [(j+1)(\omega j + \beta)a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1}]z^j \tag{2.3}$$

$$= \lambda a_0 + \sum_{j=1}^{\infty} \lambda a_j z^j \qquad \lambda \neq 0.$$

If $\beta = 0$ we have $a_0 = 0$ and (2.3) can be solved recursively for $a_2, a_3, \cdots$, in terms of a_1 since $\omega j + \beta$ never vanishes. Thus we have but one analytic solution

$$\phi(z) = z(1 + a_2 z^2 + \cdots).$$

If $\beta \neq 0$, we have $a_1 = \lambda a_0/\beta$ and (2.3) can be solved recursively for a_2, a_3, etc., provided that $(\omega j + \beta)$ never vanishes for $j = 1, 2, \cdots$. Thus we are able to obtain the single formal power series solution $\phi(z) = 1 + a_1 z + a_2 z^2 + \cdots$. The case when $(\omega j + \beta)$ vanishes for some positive integer j presents some complications and will be considered later in the proof. Solving (2.3) for a_{j+1} we have

$$a_{j+1} = \frac{1}{\omega}\left\{\frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + \left(1 + \frac{\beta}{\omega}\right)j + \frac{\beta}{\omega}}\right\}. \tag{2.4}$$

But $\beta/\omega = \bar{c}_2/\bar{c}_1 = \bar{\alpha}$, hence (2.4) becomes

$$a_{j+1} = \frac{1}{\omega}\left\{\frac{-[j^2 + j\alpha + (\alpha - 1)]a_{j-1} + \lambda a_j}{j^2 + (1+\bar{\alpha})j + \bar{\alpha}}\right\}. \tag{2.4}$$

Thus we obtain the estimate

$$|a_{j+1}| \leqq \frac{1}{|\omega|}\left|\frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1+\bar{\alpha})j + \bar{\alpha}}\right| |a_{j-1}| + \frac{|\lambda|}{|\omega|}\frac{1}{|j^2 + (1+\bar{\alpha})j + \bar{\alpha}|}|a_j|. \tag{2.5}$$

Since $|\omega| = 1$ we have

$$|a_{j+1}| \leqq |u_1(j)|\,|a_{j-1}| + |u_2(j)|\,|a_j|, \tag{2.6}$$

where

$$u_1(j) = \frac{j^2 + j\alpha + (\alpha - 1)}{j^2 + (1+\bar{\alpha})j + \bar{\alpha}},$$

and

$$u_2(j) = \frac{\lambda}{j^2 + (1+\bar{\alpha})j + \bar{\alpha}}.$$

We now estimate $|u_1(j)|$ and $|u_2(j)|$ for large j. Since $|u_2(j)|$ tends to zero as j^{-2} it follows that there exists an $M > 0$ such that

$$|u_2(j)| \leqq \frac{M}{j^2}, \qquad \text{for } j \text{ sufficiently large.} \tag{2.7}$$

Concerning $|u_1(j)|$ we obtain, upon dividing,

$$u_1(j) = \left(1 - \frac{1}{j}\right) + \frac{2}{j}\operatorname{Im}(\alpha)i + O(j^{-2}).$$

Thus $|u_1(j)|^2 = 1 - 2/j + O(j^{-2})$, and hence by a direct calculation,

$$|u_1(j)| = 1 - \frac{1}{j} + O(j^{-2}).$$

For $\xi > 0$, we note that $|u_1(j)| \leqq 1 - \xi j^{-1}$ for j sufficiently large if and only if $-1 < -\xi$, or $\xi < 1$. Hence we have

$$|u_1(j)| \leqq 1 - \frac{\xi}{j}, \quad \text{for } j \text{ sufficiently large} \tag{2.8}$$

and $0 < \xi < 1$.

Using (2.6), (2.7), and (2.8) we obtain, for j sufficiently large,

$$\begin{aligned} |a_{j+1}| &\leqq (1 - \xi j^{-1}) |a_{j-1}| + M j^{-2} \| a_j | \\ &\leqq (1 - \xi j^{-1} + M j^{-2}) M(j), \qquad 0 < \xi < 1, \end{aligned}$$

where $M(j) = \max\{|a_{j-1}|, |a_j|\}$.

Thus, for sufficiently large j, we have

$$|a_{j+1}| \leqq (1 - \gamma j^{-1}) M(j), \tag{2.9}$$

where $0 < \gamma = \xi/2 < \frac{1}{2}$.

Now consider the expression $(1 - \gamma j^{-1})(j-1)^{-1/p}$, where p is a positive integer. This is dominated by $(j+1)^{-1/p}$ for j sufficiently large if and only if

$$j^{p+1} + (-p\gamma + 1) j^p + \cdots \leqq j^{p+1} - j^p.$$

Hence, if and only if $-p\gamma + 1 < -1$ or $-p\gamma < -2$. Since $\gamma > 0$, $p > 2/\gamma$. Thus we have

$$(1 - \gamma j^{-1})(j-1)^{-1/p} \leqq (j+1)^{-1/p}, \quad p > \frac{2}{\gamma}. \tag{2.10}$$

We now show that there exists a positive constant K for which $|a_j| \leqq K j^{-1/p}$ for $j \geqq 1$. Let j_1 be such that (2.9) and (2.10) hold for $j > j_1$. Let $K = \max_{j \leqq j_1} |a_j| j^{1/p}$ so that $|a_j| \leqq K j^{-1/p}$ for $j \leqq j_1$. Using (2.9) it follows that

$$|a_{j_1+1}| \leqq (1 - \gamma j_1^{-1}) M(j_1),$$

where

$$\begin{aligned} M(j_1) &= \operatorname{Max}(K j_1^{-1/p}, K(j_1 - 1)^{-1/p}) \\ &= K(j_1 - 1)^{-1/p}. \end{aligned}$$

Hence,

$$|a_{j_1+1}| \leqq (1 - \gamma j_1^{-1}) K (j_1 - 1)^{-1/p},$$

and using (2.10) we have

$$|a_{j_1+1}| \leqq K(j_1+1)^{-1/p}. \tag{2.11}$$

We now proceed inductively to establish

$$|a_{j_1+k}| \leqq K(j_1+k)^{-1/p}, \qquad k = 2, 3, \cdots. \tag{2.12}$$

Let

$$\begin{aligned} K_1 &= \max_{j \leqq j_1+1} |a_j| j^{1/p} \\ &= \max\{K, K(j_1+1)^{-1/p}\} \leqq K, \end{aligned}$$

making use of (2.11). Using (2.9) we have

$$|a_{j_1+2}| \leqq (1 - \gamma(j_1+1)^{-1}M(j_1+1),$$

where,

$$\begin{aligned} M(j_1+1) &= \operatorname{Max}(|a_{j_1+1}|, |a_{j_1}|) \\ &= \operatorname{Max}(K(j_1+1)^{-1/p},\ K(j_1)^{-1/p}) \\ &= K(j_1)^{-1/p}. \end{aligned}$$

It follows from (2.10) that

$$\begin{aligned} |a_{j_1+2}| &\leqq (1 - \gamma(j_1+1)^{-1})K(j_1)^{-1/p} \\ &\leqq K(j_1+2)^{-1/p}. \end{aligned}$$

Continuing on in this manner we establish (2.12). Hence any solution $\Sigma_{j=0}^{\infty} a_j z^j$ whose coefficients satisfy (2.4) is in $\mathcal{H}$. To complete the proof we have only to deal with the case where $j\omega + \beta$ vanishes for some positive integer j.

We now consider the case when $j\omega + \beta$ vanishes for some positive integer n. The analytic solution obtained from (2.3) by taking $a_0 = a_1 = \cdots = a_n = 0$, and solving recursively for $a_{n+2}, a_{n+3}, \cdots$, in terms of a_{n+1} is, as we have seen, in $\mathcal{H}$. If there were a second analytic solution corresponding to $a_0 \neq 0$ it would be in $\mathcal{H}$ as well, and $m^+ (m^-)$ would be 2. We now show that this is not the case, i.e., $m^+ = m^- = 1$. To do this we make use of the following result.

Let μ be such that $\operatorname{Im}(\mu) > 0$ and let $\mathcal{D}_\mu^+$ be the nullspace of the operator $S^* - \mu$. Then the dimension of $\mathcal{D}_\mu^+$ is equal to m^+. Similarly, let $\mathrm{Im}(\mu) < 0$ and let $\mathcal{D}_\mu^-$ be the nullspace of the operator $S^* - \mu$, then the dimension of $\mathcal{D}_\mu^-$ is equal to m^-, [1, p. 1232].

Using this we see that m^+ is just the number of linearly independent solutions of $L\phi = \mu\phi$ in $\mathcal{H}$ for any μ such that $\mathrm{Im}(\mu) > 0$. Similarly, m^- is the number of linearly independent solutions of $L\phi = \mu\phi$ in $\mathcal{H}$ for any μ such that $\mathrm{Im}(\mu) < 0$. Hence, if we can show that there exist μ such that $\mathrm{Im}\,\mu > 0$ ($\mathrm{Im}\,\mu < 0$) for which there is no analytic solution corresponding to $a_0 \neq 0$ we will have shown that $m^+ = m^- = 1$.

Consider (2.3), where λ is now μ/c_2, and suppose that $\beta = -n\omega$. Taking $j = 1, 2, \cdots, n$ we obtain the following set of $n + 1$ linear equations in a_0 thru a_n:

$$-n\omega a_1 = \lambda a_0$$

$$(j+1)(j-n)\omega a_{j+1} + (j^2 + j\alpha + \alpha - 1)a_{j-1} = \lambda a_j, \quad j = 1, 2, \cdots, n-1$$

$$(n^2 + n\alpha + \alpha - 1)a_{n-1} = \lambda a_n.$$

Thus we are led to consider the homogeneous system

$$\begin{aligned} -\lambda a_0 - n\omega a_1 &= 0 \\ 2\alpha a_0 - \lambda a_1 + 2(2-n)\omega a_2 &= 0 \\ (n^2 + n\alpha - 2n)a_{n-2} - \lambda a_{n-1} - n\omega a_n &= 0 \\ (n^2 + n\alpha + \alpha - 1)a_{n-1} - \lambda a_n &= 0 \end{aligned}$$

Since the parameter $\lambda = \mu/c_2$ appears only on the diagonal the system determinant $D_n(\lambda)$ is a polynomial in λ of degree $n + 1$,

$$D_n(\lambda) = (-1)^{n+1}\lambda^{n+1} + \cdots.$$

Thus $D_n(\lambda)$ vanishes at most $n + 1$ points in the complex plane, and we can find μ in the upper half-plane and lower half-plane for which $D_n(\mu/c_2) \neq 0$. Thus $a_0 = a_1 = \cdots = a_n = 0$ and there is only one analytic solution of $L\phi = \mu\phi$.

REFERENCES

1. N. Dunford and J. Schwartz, *Linear Operators*, Part II, Interscience Publishers, 1963.
2. A. L. Villone, *Self-adjoint differential operators*, Pacific J. Math., **35** (1970), 517–531.
3. ———, *Self-Adjoint Extensions of Symmetric Differential Operators*, Pacific J. Math., **49** (1974), 569–577.

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ON THE STRUCTURE OF THE FOURIER-STIELTJES ALGEBRA

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If G is a locally compact group, denote its Fourier-Stieltjes algebra by $B(G)$ and its Fourier algebra by $A(G)$. If G is compact, then $B(G) = A(G)$ and $\sigma(B(G))$, the spectrum of $B(G)$, is G. If G is not compact then $\sigma(B(G))$ contains partial isometries and projections different from e, the identity of G. More generally, $\sigma(B(G))$ is closed under operations that commute with "representing" and the "taking of tensor products". It is shown that $\sigma(B(G))$ contains a smallest positive element, z_F; and that $g \in G \subset \sigma(B(G)) \mapsto z_F g \in \sigma(B(G))z_F$ is an epimorphism of G into $\bar{G}$, the almost periodic compactification of G.

A structure theorem is given for the closed, bi-translation, invariant subspaces of $B(G)$. In so doing we introduce the concepts of inverse Fourier transform localized at π, and the standardization of π, where π is a continuous, unitary representation of G.

Introduction. In this paper we establish some facts about the structure of the Fourier-Stieltjes algebra, $B(G)$, of a locally compact group G, which were inspired by **[7]**, **[11]** and **[14]**. In particular, we apply the characterization of the (nonzero) spectrum of $B(G)$, $\sigma(B(G))$, obtained in Theorem 1 (ii) of **[17]** to investigate further the structure of this spectrum. As one of several applications, we relate the smallest, positive element of $\sigma(B(G))$ to the almost periodic compactification of G. It soon becomes apparent that a deep understanding of closed, bi-translation invariant subspaces (and more specially, sub-algebras and ideals) of $B(G)$ is needed. It is to this end that we introduce a canonical or standard form for any continuous, unitary representation π of G on Hilbert space, and with it the notion of the inverse Fourier transform "localized at π".

We follow the notational conventions of **[17]** and define in the text any new notations introduced.

The spectrum of $B(G)$. If $s \in \sigma(B(G))$ there are naturally associated two (norm-decreasing, algebra) endomorphisms of $B(G)$, viz., $\gamma_s\colon b \in B(G) \mapsto s\,.\,b \in B(G)$ and $\delta_s\colon b \in B(G) \mapsto b\,.\,s \in B(G)$ where, for example, $\langle x, s\,.\,b\rangle = \langle xs, b\rangle$ for all $x \in W^*(G) = B(G)'$. Letting $s\,.\,B(G) = \{s\,.\,b \in B(G)\colon b \in B(G)\} = \gamma_s(B(G))$, similarly for $B(G)\,.\,s$, we observe that these are right, respectively left, translation invariant subalgebras of $B(G)$; where we adopt the convention that the right translate of $b \in B(G)$ by $g \in G$ is $b \,.\, g$ and $\langle x, b \,.\, g\rangle = \langle gx, b\rangle$ for all $x \in W^*(G)$. We also observe that the kernels of γ_s and δ_s are right, respectively left, translation invariant ideals of $B(G)$. In case $s = s^2$ we write, for example, $(e - s) \,.\, B(G) = \ker \gamma_s$, where e is the unit in $W^*(G)$. We should also observe that $s^2 = s$ implies that $s \,.\, B(G)$ and $(e - s) \,.\, B(G)$ are norm-closed. We now have the following:

PROPOSITION 1. *If $s \in W^*(G)$ is an idempotent, i.e., $s^2 = s$, then the following are equivalent*:
(i) $s \in \sigma(B(G))$;
(ii) *$s \,.\, B(G)$ is an algebra and $(e - s) \,.\, B(G)$ is an ideal in $B(G)$*;
(ii)′ *γ_s is an endomorphism*;
(iii) *$B(G) \,.\, s$ is an algebra and $B(G) \,.\, (e - s)$ is an ideal in $B(G)$*;
(iii)′ *δ_s is an endomorphism.*

Proof. That (i) implies (ii) and (ii)′ is immediate. We now show that (ii) $\Rightarrow$ (ii)′ $\Rightarrow$ (i). Consider that for $b_1, b_2 \in B(G)$,

$$
\begin{aligned}
s \,.\, (b_1 b_2) &= s \,.\, ((s \,.\, b_1 + (e - s) \,.\, b_1)(s \,.\, b_2 + (e - s) \,.\, b_2)) \\
&= s \,.\, (s \,.\, b_1 \, s \,.\, b_2) + s \,.\, (((e - s) \,.\, b_1)(b_2)) \\
&\qquad\qquad\qquad\qquad + s \,.\, ((s \,.\, b_1)(e - s) \,.\, b_2) \\
&= s \,.\, (s \,.\, b_1 \, s \,.\, b_2) \text{ (since } (e - s) \,.\, B(G) \text{ is an ideal and } s^2 = s) \\
&= s \,.\, b_1 \, s \,.\, b_2 \text{ (since } s \,.\, B(G) \text{ is an algebra)},
\end{aligned}
$$

hence (ii)′. Evaluation at e shows that $s \in \sigma(B(G))$, thus (ii)′ $\Rightarrow$ (i). The remainder of the proposition follows immediately by symmetry.

If $s^2 = s \in \sigma(B(G))$, we call $(e - s) \,.\, B(G)$ a right-prime or δ-prime ideal; similarly, $B(G) \,.\, (e - s)$ is called a left-prime or γ-prime ideal, where our terminology here is influenced by **[13]**. Note that a δ-prime ideal $I \subset B(G)$ has the property that if $(b_1 \,.\, g_1)(b_2 \,.\, g_2) \in I$ for all $g_1, g_2 \in G$, then either $b_1 \in I$ or $b_2 \in I$.

The following results show that $\sigma(B(G))$ is closed under certain operations, and a basis for generalizing some of the results of **[14]**, **[15]** on the structure of the spectrum of convolution measure algebras is thus obtained. Recall first that any operator s on Hilbert space has a left and right polar decomposition, viz., $s = v_\gamma |s|_\gamma$ where $|s|_\gamma = (s^*s)^{1/2}$ and $s = |s|_\delta v_\delta$ where $|s|_\delta = (ss^*)^{1/2}$. Also for later notational convenience let $\sigma(B(G))_+$ and $\sigma(B(G))_p$ denote the positive, hermitian elements and (self-adjoint) idempotents in $\sigma(B(G))$, respectively. Note that idempotents of norm one are self-adjoint and that $v_\delta = v_\gamma$. We then have:

THEOREM 1. *If $s \in \sigma(B(G))$, then $v_\gamma, v_\delta, |s|_\gamma$, and $|s|_\delta$ are also in $\sigma(B(G))$.*

LEMMA. *If $s \in \sigma(B(G))_+$, then the positive square root, $s^{1/2}$, is also in $\sigma(B(G))_+$.*

Proof of Lemma. Let π_1 and π_2 be any two continuous unitary representations of G on Hilbert space, and by the same letters denote their canonical extensions to $W^*(G)$. Abusing notation again, let $\pi_1 \otimes \pi_2$ denote both the usual tensor product group representation of G and its canonical extension to $W^*(G)$. Now in $W^*(G)$ $s^{1/2}s^{1/2} = s$, and $\pi_1 \otimes \pi_2(s^{1/2}s^{1/2}) = (\pi_1 \otimes \pi_2(s^{1/2}))^2$. But $\pi_1 \otimes \pi_2(s) = \pi_1(s) \otimes \pi_2(s)$ since $s \in \sigma(B(G))$, cf., **[17]**, Theorem 1, (ii); and $\pi_1(s) \otimes \pi_2(s) = (\pi_1(s^{1/2}) \otimes \pi_2(s^{1/2}))^2$. Thus by uniqueness of the positive square root, we have $\pi_1(s^{1/2}) \otimes \pi_2(s^{1/2}) = \pi_1 \otimes \pi_2(s^{1/2})$, hence $s^{1/2} \in \sigma(B(G))$ by **[17]** Theorem 1, (ii) again.

Proof of Theorem 1. We prove v_γ and $|s|_\gamma$ are in $\sigma(B(G))$, the remainder of the theorem follows by symmetry. Note first that $s^*s \in \sigma(B(G))$, since $s^*s \neq 0$, cf., **[17]** Theorem 1 (iii). Thus by the lemma $|s|_\gamma \in \sigma(B(G))$. Now again let π_1, π_2 be representations as above. We have $\pi_1 \otimes \pi_2(s) = \pi_1 \otimes \pi_2(v_\gamma)\pi_1 \otimes \pi_2(|s|_\gamma)$ and $\pi_1(s) \otimes \pi_2(s) = (\pi_1(v_\gamma) \otimes \pi_2(v_\gamma))(\pi_1(|s|_\gamma) \otimes \pi_2(|s|_\gamma))$. Since $s, |s|_\gamma \in \sigma(B(G))$ we have

$$\begin{aligned}&\pi_1 \otimes \pi_2(v_\gamma)\,\pi_1(|s|_\gamma) \otimes \pi_2(|s|_\gamma)\\ &\quad = (\pi_1(v_\gamma) \otimes \pi_2(v_\gamma))(\pi_1(|s|_\gamma) \otimes \pi_2(|s|_\gamma))\end{aligned}$$

Now $v_\gamma^* v_\gamma$ is the support of $|s|_\gamma$, by the definition of the polar decomposition. But it is easy to see that $\pi_1 \otimes \pi_2(v_\gamma)$ and $\pi_1(v_\gamma) \otimes \pi_2(v_\gamma)$ are partial isometries, both with initial projections equal to the support of $\pi_1(|s|_\gamma) \otimes \pi_2(|s|_\gamma)$. Thus again by uniqueness of the polar decomposition and **[17]** Theorem 1 (ii), we have $v_\gamma \in \sigma(B(G))$.

As corollaries of the method of argument in the foregoing proofs we have:

COROLLARY. *If $s \in \sigma(B(G))_+$, then $s^z \in \sigma(B(G))$ for all complex z with* $\operatorname{Re} z > 0$.

REMARK. We understand by s^0 the support projection of s, which is in $\sigma(B(G))$; and the map $z \mapsto s^z$ is analytic for $\operatorname{Re} z > 0$.

REMARK. Speaking loosely, the weakly compact *-semigroup $\sigma(B(G))$, see first corollary of Theorem 2, is closed under any operation that commutes with representing and the taking of tensor products. To see that "raising to the z power" has these properties when defined, application of the spectral theorem for self-adjoint operators will suffice; or alternatively apply a standard analytic function proof.

COROLLARY. *Let s and t be in $\sigma(B(G))_+$ and let $s \leqq t$, then there exists a unique $a \in \sigma(B(G))$ satisfying $s^{1/2} = at^{1/2}$, with support of a majorized by that of t.*

Proof. This follows from **[4]** Chap. 1, §1.6, Lemma 2.

We now show that $\sigma(B(G))_+$ has a smallest element z_F, which is a central idempotent.

THEOREM 2. $z_F = \sup\{z[\pi]: z[\pi] =$ *support in* $W^*(G)$ *of finite dimensional* (unitary) *representation* $\pi\}$. *Then z_F is a central projection in $W^*(G)$, and $z_F \in \sigma(B(G))_+$. Moreover if $s \in \sigma(B(G))_+$, we have* $z_F s = z_F$, i.e., $z_F \leqq s$.

Proof. It is clear that $z_F . B(G)$ is an algebra, since the tensor product of two finite dimensional representations is itself finite dimensional. That $(e - z_F) . B(G)$ is an ideal in $B(G)$ follows from **[13]**. Briefly, if $b_1 \in (e - z_F) . B(G)$ and $b_2 \in B(G)$, let π_{p_1} and π_{p_2} be the cyclic representations arising from, say, the left absolute values p_1 and p_2 which arise from the left polar decompositions of b_1 and b_2, respectively. Then $z[\pi_{p_1}]z_F = 0$, and thus $z[\pi_{p_1} \otimes \pi_{p_2}]z_F = 0$, by **[13]**. But $z[\pi_{p_1}] . b_1 = b_1$; and $z[\pi_{p_2}] . b_2 = b_2$. Hence $z[\pi_{p_1} \otimes \pi_{p_2}] . b_1 b_2 = b_1 b_2$, and thus $z_F . b_1 b_2 = 0$. Thus by Proposition 1, $z_F \in \sigma(B(G))$.

We now show that z_F is the smallest element in $\sigma(B(G))_+$. First consider the case where q is an idempotent in $\sigma(B(G))_+$. Now $z_F q$ is an idempotent in $\sigma(B(G))_+$ satisfying $z_F \geqq z_F q$, or else $z_F q = 0$. In the latter case $(e - q) . B(G)$ is an ideal of $B(G)$ that contains 1, the identity of $B(G)$; hence $q = 0$, which is impossible since $0 \notin \sigma(B(G))$. More generally, if $z_F q \neq z_F$, consider that $(e - z_F q) . B(G)$ is a closed, right translation invariant ideal in $B(G)$ which contains a positive definite function p which is a coefficient of a finite dimensional, irreducible representation π. If π is on Hilbert space H_π, there is an orthonormal basis $\{\xi_i\}_{i=1}^{\dim \pi}$ of H_π so that, supposing $p = \omega_{\xi_1}, \omega_{\xi_1,\xi_i} \in (e - z_F q) . B(G)$ $i = 1, 2, \cdots, \dim \pi$; but then

$$1 = \sum_{i=1}^{\dim \pi} \bar{\omega}_{\xi_1,\xi_i}\, \omega_{\xi_1,\xi_i} \in (e - z_F q) . B(G)$$

where $\bar{\omega}_{\xi_1,\xi_i}(\pi(x)) = \overline{(\pi(x)\xi_1 \mid \xi_i)}$, the bar denoting complex conjugation, $i = 1, 2, \cdots \dim \pi$. Thus again we get $q = 0$, an impossibility. Thus z_F is the smallest idempotent in $\sigma(B(G))_+$. In general, let $s \in \sigma(B(G))_+$; and let $e_s = \text{weak-lim}_{n\to\infty} s^n$, the projection on the eigenspace of s corresponding to eigenvalue 1. Since $\sigma(B(G))$ is weakly compact, $e_s \in \sigma(B(G))_+$ (because $e_s \neq 0$). But then $z_F \leqq e_s \leqq s$, and we are done.

We can now refine [17] Theorem 1, (iii):

Corollary. *$\sigma(B(G))$ is a weakly compact $*$-semigroup.*

Remark. The reader should be careful to note that $\sigma(B(G))$ is not a topological semigroup (in general) in the weak topology. However, $\sigma(B(G))$ is a topological semigroup in the strong topology (see discussion of topology following Proposition 3). Then $\sigma(B(G))$ is not (in general) compact in the strong topology, neither is $*$ strongly continuous, though $*$ is weakly continuous.

Proof. All that remains to be shown is that if $x, y \in \sigma(B(G))$, then $xy \neq 0$. But by Theorem 2 above, we have that z_F is smaller than either the support or range projections of x and y. Thus it is easy to see that $z_F xy \neq 0$, hence $xy \neq 0$.

The following corollary is stated to illustrate in the simplest case, a relationship between the topology of G and the idempotents in $\sigma(B(G))_+$.

Corollary. *G is compact if and only if the only central element in $\sigma(B(G))_p$ is e.*

Proof. If G is compact, $A(G) = B(G)$; and $\sigma(B(G))$ is G. Thus the only idempotent in $\sigma(B(G))$ is e. Conversely, let $s \in \sigma(B(G))$, and let $s = v_\gamma \mid s \mid_\gamma$. Then $z_F \leqq v_\gamma^* v_\gamma$, and $z_F \leqq v_\gamma v_\gamma^*$. But $z_F = e$ by hypothesis, hence v_γ is unitary, and $\mid s \mid_\gamma = e$. Thus $\sigma(B(G))$ is topologically isomorphic with G, e.g., G is compact, cf. [17] Theorem 1.

Example. Consider the group $SL(2,\mathbf{R})$. In this case $z_F = z_0 =$ support of the trivial representation of G. We must always have $z_F \geqq z_0$, and this example shows that equality may be obtained.

Any analysis of the structure of a semigroup should include a discussion of its ideals, idempotents, and groups. To begin with we have:

PROPOSITION 2. *If* $s \in \sigma(B(G))$, *the principal left ideal* $\sigma(B(G))s = \sigma(B(G))|s|_\gamma = \{t \in \sigma(B(G)) : t^*t \leqq s^*s\}$. *Similarly,*

$$s\sigma(B(G)) = |s|_\delta \sigma(B(G)) = \{t \in \sigma(B(G)) : tt^* \leqq ss^*\}.$$

Proof. Clearly $\sigma(B(G))s \subset \{t \in \sigma(B(G)) : t^*t \leqq s^*s\}$, since if $x \in \sigma(B(G))$, $\|x\|_{W^*(G)} = 1$, and $(xs)^*(xs) \leqq s^*s\, \|x\|^2_{W^*(G)}$. Now $s = v_\gamma |s|_\gamma$ implies $\sigma(B(G))s = \sigma(B(G))v_\gamma |s|_\gamma \subset \sigma(B(G))|s|_\gamma$. But $v_\gamma^* s = |s|_\gamma$ yields the opposite inclusion. Finally, suppose $t^*t \leqq s^*s$, $t \in \sigma(B(G))$. Then by the second corollary of Theorem 1, $|t|_\gamma = a|s|_\gamma$ for some $a \in \sigma(B(G))$. But then $t = v'_\gamma |t|_\gamma = v'_\gamma a |s|_\gamma$ is in $\sigma(B(G))|s|_\gamma = \sigma(B(G))s$. To get the corresponding "right-handed" proposition just observe that the * operation on $\sigma(B(G))$ induces a symmetry between right and left.

Letting $|s|_\gamma^\infty$ denote the projection on the eigenspace of $|s|_\gamma$ corresponding to eigenvalue 1, we have the following chain of inclusions:

COROLLARY. *If* $s \in \sigma(B(G))$, *then for* $1 < \alpha < \beta$,

$$\sigma(B(G))|s|_\gamma^\infty \subset \sigma(B(G))|s|_\gamma^\beta \subset \sigma(B(G))|s|_\gamma^\alpha \subset \sigma(B(G))s$$
$$\subset \sigma(B(G))|s|_\gamma^{1/\alpha} \subset \sigma(B(G))|s|_\gamma^{1/\beta} \subset \sigma(B(G))|s|_\gamma^0.$$

A similar statement holds for the corresponding principal, right ideals.

Proof. We have, e.g., $|s|_\gamma^\beta = |s|_\gamma^{\beta-\alpha}\, |s|_\gamma^\alpha$. The rest is clear.

PROPOSITION 3. $I \subset \sigma(B(G))$ *is a left-ideal if and only if* $s \in I$, $t \in \sigma(B(G))$, *and* $t^*t \leqq s^*s$ *imply* $t \in I$. *A corresponding statement holds for right ideals.*

Proof. If s and t satisfy the above conditions, then for some $a \in \sigma(B(G))$,

$$t = v_\gamma |t|_\gamma = v_\gamma a\, |s|_\gamma \in \sigma(B(G))|s|_\gamma = \sigma(B(G))s \subset I.$$

Conversely, given $s \in I \subset \sigma(B(G))$ where I satisfies the above condition, we must show that $xs \in I$ if $x \in \sigma(B(G))$. But $(xs)^*(xs) \leqq s^*s$, hence we are done.

We remark that there is a map from the (weakly) closed, right ideals in $\sigma(B(G))$ to the left translation invariant "radical" ideals in $B(G)$, where if I is such an ideal in $\sigma(B(G))$, the corresponding radical ideal is $\{b \in B(G): b(s) = 0 \text{ for all } s \in I\}$.

Before going further we must discuss the strong and weak topologies on $\sigma(B(G))$, as is done in **[14]** for G abelian. We have that $\sigma(B(G))$ is compact, the involution * is continuous, and multiplication is separately continuous in the weak (or what is the same, weak operator) topology on $\sigma(B(G))$. Also, the weak topology is weaker than any of the following strong topologies. (Note that consequently principal ideals in $\sigma(B(G))$ are weakly hence strongly closed.) Due to the non-abelianess of G, there are four strong topologies on $\sigma(B(G))$: the strong operator topology; the left-strong topology, i.e., $s \to s_0$ in $\sigma(B(G))$ if and only if $\|\gamma_s(b) - \gamma_{s_0}(b)\| \to 0$ for each $b \in B(G)$; the right-strong topology, i.e., $s \to s_0$ in $\sigma(B(G))$ if and only if $\|\delta_s(b) - \delta_{s_0}(b)\| \to 0$ for each $b \in B(G)$; and the *-strong topology, i.e., $s \to s_0$ in $\sigma(B(G))$ provided both $s \to s_0$ and $s^* \to s_0^*$ in the strong operator topology. It is easy to verify that $s \to s_0$ strongly (as operators in $W^*(G)$) if and only if $s \to s_0$ left-strongly, and $s^* \to s_0^*$ strongly (as operators in $W^*(G)$) if and only if $s \to s_0$ right-strongly, and the involution * is a homeomorphism between the left and right strong topologies. Multiplication in $\sigma(B(G))$ is jointly continuous in all the strong topologies, whereas the involution is continuous in the *-strong topology. Finally, it is clear that the map $s \in \sigma(B(G)) \to s^*s \in \sigma(B(G))_+$ (resp., $ss^* \in \sigma(B(G))_+$) is continuous from the left-strong (resp., right-strong) topology to the weak topology.

It is well to note the following for later use.

PROPOSITION 4. (i) *If* $\{s_\alpha\}$ *is a net in* $\sigma(B(G))$, $s \in \sigma(B(G))$, *and* $s_\alpha^* s_\alpha \leqq s^*s$ (resp., $s_\alpha s_\alpha^* \leqq ss^*$) *for all* α, *then* $s_\alpha \to s$ *left-strongly* (resp., right-strongly) *if and only if* $s_\alpha \to s$ *weakly*;

(ii) *the weak and left-strong* (resp., right-strong) *topologies agree on any set of the form* $\{s \in \sigma(B(G)): s^*s = t,\ t \in \sigma(B(G))_+\}$ (resp., $\{s \in \sigma(B(G)): ss^* = t,\ t \in \sigma(B(G))_+\}$);

(iii) *The weak and left-*(or right) *strong topologies agree on any subset of* $\sigma(B(G))_+$ *which is totally ordered.*

Proof. (i) Suppose $s_\alpha^* s_\alpha \leqq s^*s$ and $s_\alpha \to s$ weakly. Then for a positive definite function $p \in B(G)$, and $x \in W^*(G)$,

$$\|s_\alpha . p - s . p\| = \sup_{\|x\| \leqq 1} |\langle p, x\ (s_\alpha - s)\rangle| \leqq$$

$$\sup_{\|x\| \leqq 1} p(xx^*)^{1/2} p((s_\alpha - s)^*(s_\alpha - s))^{1/2} \leqq \|p\|^{1/2}(2p(s^*s) - 2\operatorname{Re} p(s^* s_\alpha))^{1/2}$$

which converges to zero. Since any $b \in B(G)$ is a linear combination of four positive definite functions we are done. The rest of (i) is by symmetry.

(ii) Immediate from (i).

(iii) Let s and net $\{s_\alpha\}$ be in a totally ordered subset of $\sigma(B(G))_+$. Our claim follows from the inequality $p((s_\alpha - s)^2) \leqq 2|p(s_\alpha - s)|$, where p is positive definite, cf., **[4]** Appendice II.

Now any interval, $\{t \in \sigma(B(G))_+ : s_1 \leqq t \leqq s_2\}$ in $\sigma(B(G))_+$, determined by $s_1, s_2 \in \sigma(B(G))_+$ is closed in both the weak and strong topologies. On the other hand,

PROPOSITION 5. *If $S \subset \sigma(B(G))_+$ is strongly closed, then S contains minimal and maximal elements.*

Proof. All strong topologies coincide on the set of self-adjoint elements in $\sigma(B(G))_+$, now apply Proposition 4 (iii), weak compactness of $\sigma(B(G))_+$, and Zorn's lemma.

As in the abelian case, minimal elements of strongly open-closed subsets of $\sigma(B(G))_+$ are especially important in the theory. Before discussing these objects, however, let us have the following notations, $G_{p,\gamma} = \{s \in \sigma(B(G)) : s^*s = p\}$, $G_{p,\delta} = \{s \in \sigma(B(G)) : ss^* = p\}$, $G_p = G_{p,\gamma} \cap G_{p,\delta}$, where $p \in \sigma(B(G))_p$.

PROPOSITION 6. (i) *G_p is a topological group with $*$ for inverse, p for identity, and the right or left-strong topology or the weak topology — all of which coincide on G_p. G_p is $*$-strongly closed in $\sigma(B(G))$.*

(ii) *$G_{p,\gamma} \subset \sigma(B(G))p$, $G_{p,\delta} \subset p\sigma(B(G))$ and the following inclusions hold:* $\sigma(B(G))p\sigma(B(G)) \supset p\sigma(B(G)) \cup \sigma(B(G))p \supset p\sigma(B(G)) \cap \sigma(B(G))p = p\sigma(B(G))p \supset G_p$.

Proof. The proof is rather easy and left to the reader.

Now consider the following conditions on $s \in \sigma(B(G))_+$.

Condition (A): There does not exist a net $\{s_\alpha\} \subset \sigma(B(G))_+$ satisfying $s_\alpha \lneqq s$ and $\lim_\alpha s_\alpha = s$.

Condition (B): There does not exist a net $\{s_\alpha\} \subset \sigma(B(G))_+$ satisfying $s_\alpha^2 \lneqq s^2$ (which implies $s_\alpha \lneqq s$) and $\lim_\alpha s_\alpha = s$.

Note that in both conditions weak and strong limits are equivalent. Also Condition (A) implies Condition (B). Both conditions imply that $s \in \sigma(B(G))_p$, since if $s^2 \lneqq s$ then $s^\alpha \lneqq s$ (respectively, $s^{2\alpha} \lneqq s^2$) and $\lim_{\alpha \downarrow 1} s^\alpha = s$. We should also observe that if s is central, then s satisfies condition (A) if and only if it satisfies condition (B), since if s_α and s commute, $0 \leqq s_\alpha \leqq s$ is equivalent to $0 \leqq s_\alpha^2 \leqq s^2$. What is much more important for us, however, is that if s satisfies

$s^2 = s \geqq 0$, then we have that $0 \leqq s_\alpha \leqq s^2 = s$ holds if and only if $0 \leqq s_\alpha^2 \leqq s^2 = s$. Note that $0 \leqq s_\alpha \leqq s^2 = s$ implies that $(e-s)s_\alpha = s_\alpha(e-s) = 0$, hence $s_\alpha s = s s_\alpha = s_\alpha$, and we are done, since for positive operators s_α, s, $s_\alpha^2 \leqq s^2$ always gives $s_\alpha \leqq s$. Thus we have the following generalization of the notion of critical point introduced in [14]:

DEFINITION. If $p \in \sigma(B(G))_+$ satisfies condition (A), or equivalently condition (B), then p is called a critical element of $\sigma(B(G))_+$.

Observe that p is critical if and only if p is weakly isolated in

$$\begin{aligned}(p\sigma(B(G)))_+ &= \{t \in \sigma(B(G))_+ : t^2 \leqq p\} = \{t \in \sigma(B(G))_+ : t \leqq p\} \\ &= p\sigma(B(G))_+ p = (p\sigma(B(G))p)_+.\end{aligned}$$

We now have the following characterization of critical elements:

PROPOSITION 7. (i) *$p \in \sigma(B(G))_+$ is critical;*

(ii) *$G_{p,\gamma}$ is left-strongly* (weakly) *open in $\sigma(B(G))p$;*

(iii) *$G_{p,\delta}$ is right-strongly* (weakly) *open in $p\sigma(B(G))$;*

(iv) *G_p is strongly* (weakly) *open in $p\sigma(B(G))p$;*

(v) *p is a minimal element of a strongly open and closed subset of $\sigma(B(G))_+$.*

Proof. Consider the map $\theta: s \in \sigma(B(G)) \mapsto s^*s \in \sigma(B(G))_+$ is continuous from the left-strong to the weak topology. Now in $\{t \in \sigma(B(G))_+ : t^2 \leqq p\}$, if p is critical $\{p\}$ is weakly open; and $G_{p,\gamma} = \theta^{-1}(p)$ is thus left-strongly open in

$$\sigma(B(G))p = \theta^{-1}(\{t \in \sigma(B(G))_+ : t^2 \leqq p\}).$$

That $G_{p,\gamma}$ is weakly open in $\sigma(B(G))p$ follows from Proposition 4 (i), thus (i) implies (ii). Clearly, (i) also imples (iii). Conversely, since $\{p\} = \sigma(B(G))_+ \cap G_{p,\gamma}$, and $(\sigma(B(G))p)_+ = \sigma(B(G))_+ \cap \sigma(B(G))p$, we have (ii) implies (i). Clearly (iii) implies (i) also. It is now easy to see that (i) is equivalent to (iv). Now suppose p is critical, then p is a minimal element of the strongly (weakly) open-closed set $\{t \in \sigma(B(G))_+ : p \leqq t^2\} = \{t \in \sigma(B(G))_+ : pt = p\}$ (which is the inverse image of weakly isolated point $\{p\}$ under weakly continuous map $t \in \sigma(B(G))_+ \mapsto ptp \in p\sigma(B(G))_+ p$). Conversely, if p is a minimal element of some strongly open and closed set $S \subset \sigma(B(G))_+$, then $\{p\} = S \cap \{t \in \sigma(B(G))_+ : t \leqq p\}$ is strongly (weakly) isolated in $p\sigma(B(G))_+ p = \{t \in \sigma(B(G))_+ : t \leqq p\}$, and p is thus critical. Hence (i) is equivalent to (v), and we are done. Since $G_{p,\delta}$, $G_{p,\gamma}$, and G_p are weakly open in weakly compact $p\sigma(B(G))$, $\sigma(B(G))p$, $p\sigma(B(G))p$ respectively, we have:

COROLLARY. *If p is critical $G_{p,\delta}$, $G_{p,\gamma}$ are locally compact spaces, and G_p is a locally compact topological group.*

We now investigate a special critical point, viz., z_F, which is critical by Theorem 2. Note that if z is a central critical element then a continuous homomorphism $\theta_z: g \in G \mapsto gz \in G_z$ results. In general, at the very least one has that ${}^t\theta_z: b \in B(G_z) \mapsto b \circ \theta_z \in B(G)$ is a normdecreasing homomorphism between the corresponding Fourier-Stieltjes algebras. In the case of z_F we have:

PROPOSITION 8. *G_{z_F} is the almost periodic compactification of G, and ${}^t\theta_{z_F}$ is an isometry of $B(G_F)$ onto $B(G) \cap AP(G) = z_F . B(G)$, where $AP(G)$ denotes the almost periodic functions on G.*

Proof. Let $\bar{G}$ denote the almost periodic compactification of G, then $i: G \to \bar{G}$ the canonical inclusion is such that ${}^ti: B(\bar{G}) \to AP(G) \cap B(G)$ (isometrically), where $AP(G) \cap B(G)$ is a bi-translation invariant, closed subalgebra of $B(G)$, i.e., $AP(G) \cap B(G) = z_0 . B(G)$ where z_0 is a central projection in $W^*(G)$, cf., **[5]** 2.27 and **[17]**. Now $z_F . B(G) \subseteq z_0 . B(G)$ since any element in $z_F . B(G)$ is almost periodic, **[3]**, 16.2.1. Now $z_F \in \sigma(B(G))$ implies $z_F \in \sigma(B(\bar{G}))$, where we identify $B(\bar{G})$ and $z_0 . B(G)$. But $z_0 \in \sigma(B(\bar{G}))$, namely, the identity. But $\bar{G}$ is compact, hence by the second corollary of Theorem 2, $z_F = z_0$. Thus $B(\bar{G}) \cong z_F . B(G)$. Now the dual group (in the sense of **[18]**) of $B(\bar{G})$ is uniquely determined, and is $\bar{G}$; while the dual group of $z_F . B(G)$ is the compact group $\sigma(B(G))z_F = G_{z_F}$. Thus $\bar{G}$ is topologically isomorphic with G_{z_F}.

A natural discussion now arises. Given a central critical element z, then the closure of $\theta_z(G)$, call it $G_{\theta z}$, in G_z is a locally compact group, and (with a slight abuse of notation) ${}^t\theta_z: B(G_{\theta_z}) \to B(G)$ is an isometric isomorphism onto a closed, bi-translation invariant subalgebra of $B(G)$. Also, of course, the inclusion $i: G_{\theta_z} \to G_z$ induces a norm-continuous homomorphism ${}^ti: B(G_z) \to B(G_{\theta_z})$ with the additional property that ${}^ti(A(G_z)) = A(G_{\theta_z})$, cf., **[8]**. One question then is $G_{\theta_z} = G_z$? By Proposition 8 the answer is yes if $z = z_F$. In general, it is not hard to see that the complete analysis of a central critical point z in $\sigma(B(G))_+$ depends ultimately on the resolution of the following question: Does the algebra of functions $z . B(G)$ contain an element of $A(G_z)$? The affirmative answer to this question in case G is abelian was furnished by Taylor, cf. **[14]**, **[15]** and references therein, with much machinery and considerable work. A closely related question is: what types of commutative Banach algebras are dual to a locally compact group G (in the sense of **[18]**)? Must such a dual algebra contain a copy of $A(G)$? A tool which we hope will help resolve these questions is considered in the next section.

Generalized inverse Fourier-transform. In **[7]**, **[8]** C. S. Herz demonstrates that $A(G)$ is the quotient of $L^2(G) \hat{\otimes} L^2(G)$ (the projective tensor product of $L^2(G)$ with itself) by the kernel of the continuous surjection $P: L^2(G) \hat{\otimes} L^2(G) \rightarrow A(G)$ determined by $P(\xi \otimes \eta) = \eta * \check{\xi}$, where $\eta * \check{\xi}(g) = \int \xi(g^{-1}x)\eta(x)dx$ for $\xi, \eta \in L^2(G)$. With this norm $A(G)$ is a Banach algebra. Now we note that $L^2(G)$ is a Hilbert G-module, i.e., there is a continuous unitary representation of G on $L^2(G)$, viz., the left regular representation λ, and that $A(G)$ is just the collection of coefficients of λ. A natural question is: can this result be generalized to an arbitrary Hilbert G-module H_π, i.e., to the case where we have a continuous, unitary representation π of G in H_π? We give an affirmative answer to this question, and in so doing introduce the notion of the *generalized inverse Fourier transform localized at* π, as well as the notion of the *standardization of* π. These concepts have been motivated by our desire to better understand closed, bi-translation invariant subspaces, subalgebras, and ideals in $B(G)$. We present this section with the hope that it will be a useful tool which will bring to bear on any unitary group representation almost the entire calculus previously only used in association with the left-regular representation. Technically we have been motivated by **[7]**, **[8]**, **[10]**, **[11]** as will become apparent, but the Tomita-Takesaki theory makes the dominant contribution.

We first note that $L^2(G) \hat{\otimes} L^2(G)$ may be identified with the nuclear (or trace class) operators, $\mathcal{T}(L^2(G))$, on $L^2(G)$ via the map $\tau: L^2(G) \hat{\otimes} \overline{L^2(G)} \rightarrow \mathcal{T}(L^2(G))$ determined by $\tau(\xi \otimes \eta) = \langle \ , \eta \rangle \xi$, where although $L^2(G)$ and its dual $\overline{L^2(G)}$ are "the same" we prefer to retain the distinction. Note that $\langle \ , \eta \rangle$ indicates we view η as in $\overline{L^2(G)}$, $(\ | \eta)$ indicates we view $\eta \in L^2(G)$.

REMARK. From an intuitive point of view we regard $\mathcal{T}(L^2(G))$ as a semi-abelianized, discretized version of another noncommutative L^1-measure algebra associated with a weight. The precise meaning of this statement will be made clear when we discuss the standardization of π. Suffice it to say that the map P of C. S. Herz behaves very much like an inverse Fourier-transform of an L^1-space onto $A(G)$.

A version of our next theorem, we have been informed by mail, was obtained independently by a student of P. Eymard, G. Arsac, in his Ph.D. thesis. The research of this paper was carried out independently by the present author without knowledge of the work of Arsac. Our point of view and motivation are different, and our "concrete" transform and standardization concepts, as far as we know, have not been discussed by Arsac. Whereas our proof of Theorem 3 is based on an inverse transform of nuclear (i.e., trace class) operators, Arsac's proof is based on the more abstract projective tensor product representation of this object as a Banach space. Our proofs differ in that we look at a "concrete" transform of nuclear operators; also, we have a C^*-algebra of operators to deal with; and thus we obtain more detailed results. Our approach emphasizes the action of G and closely resembles the classical Fourier-transform theory.

DEFINITION. Given a continuous, unitary, representation π of G on H_π, we denote the nuclear operators on H_π by $\mathcal{T}(H_\pi)$. We define the inverse Fourier-transform of $t \in \mathcal{T}(H_\pi)$ to be that complex-valued function on G defined by $\check{t}_\pi\colon g \in G \mapsto \operatorname{Tr}(\pi(g)t)$, where Tr is the normalized trace on $\mathcal{L}(H_\pi)$. We refer to this map as the *inverse Fourier transform (localized) at* π.

REMARK. This transform is obtained by considering $t \in \mathcal{T}(H_\pi)$ as an element in the predual of $\mathcal{L}(H_\pi)$ and then restricting to the von Neumann algebra $\{\pi(g)\colon g \in G\}'' \subset \mathcal{L}(H_\pi)$. In this way we shall see that $\check{t}_\pi \in z[\pi].B(G)$. If we define the transform by $g \in G \mapsto \operatorname{Tr}(\pi(g)^*t)$, then $\check{t}_\pi \in z[\bar{\pi}].B(G)$, where $\bar{\pi}$ is the representation "conjugate" to π.

THEOREM 3. (i) *The function $\check{t}_\pi(g) = \operatorname{Tr}(\pi(g)t)$ on G is in $z[\pi].B(G)$, where $z[\pi]$ is the support of π in $W^*(G)$, i.e., $z[\pi]B(G)$ is the closed, bi-translation invariant subspace of $B(G)$ determined by the coefficients $\{(\pi(\cdot)\xi \mid \eta)\colon \xi, \eta \in H_\pi\}$ of π.*

(ii) *If t is a positive operator in $\mathcal{T}(H_\pi)$, then $\check{t}_\pi \in z[\pi].P(G)$, and $\|\check{t}_\pi\|_{B(G)} = \check{t}_\pi(e) = \|t\|_{\mathcal{T}(H_\pi)} = \operatorname{Tr}(t)$. If $t = v\,|t| \in \mathcal{T}(H_\pi)$* (left polar decomposition in $\mathcal{L}(H_\pi)$), *then $\check{t}_\pi = v\,.\,|t|\check{}_\pi$* (left polar decomposition with respect to $\mathcal{L}(H_\pi)$), *and $\|\check{t}_\pi\|_{B(G)} \leqq \|t\|_{\mathcal{T}(H_\pi)}$.*

(iii) *For each $b \in z[\pi].B(G)$, there is a $t \in \mathcal{T}(H_\pi)$ such that $b = \check{t}_\pi$.*

(iv) *The map $t \in \mathcal{T}(H_\pi) \mapsto \check{t}_\pi \in B(G)$ is one-to-one if and only if π is irreducible.*

Proof. Given $t \in \mathcal{T}(H_\pi)$, let $t = v\,|t|$ be its polar decomposition, with $|t| = \sum_{i=1}^\infty \lambda_i(\cdot \mid \xi_i)\xi_i$ where $\xi_i \in H_\pi$, $\|\xi_i\| = 1$, and $\lambda_i \geqq 0$ for all i, $\sum_{i=1}^\infty \lambda_i = \operatorname{Tr}(|t|)$. Thus $\check{t}_\pi(g) = \operatorname{Tr}(\pi(g)t) = \operatorname{Tr}(\sum_{j=1}^\infty \lambda_i(\cdot \mid \xi_i)\pi(g)v\xi_i))$

$= \Sigma_{i=1}^{\infty} \lambda_i \operatorname{Tr}((\cdot \mid \xi_i)\pi(g)v\xi_i)$, where the last equality follows from the Hölder inequality

$$\left| \operatorname{Tr}\left(\pi(g)v \sum_{i=n}^{\infty} \lambda_i(\cdot \mid \xi_i)\xi_i \right) \right| \leqq \| \pi(g)v \|_{\mathscr{L}(H_\pi)} \left\| \sum_{i=n}^{\infty} \lambda_i(\cdot \mid \xi_i)\xi_i \right\|_{\mathscr{T}(H_\pi)}.$$

Thus $t_\pi(g) = \Sigma_{i=1}^{\infty} \lambda_i(\pi(g)v\xi_i \mid \xi_i) = \Sigma_{i=1}^{\infty} \lambda_i(v \,.\, \omega_{\xi_i})(g)$ is in $B(G)$; since (restricted to G) $\Sigma_{i=1}^{\infty} \lambda_i \omega_{\xi_i} \in P(G)$ norm — converges in $B(G)$ to an element in $P(G)$; and since $v \,.\, (\Sigma_{i=1}^{\infty} \lambda_i \omega_{\xi_i}) = \Sigma_{i=1}^{\infty} \lambda_i v \,.\, \omega_{\xi_i}$, when restricted to $\{\pi(g) \colon g \in G\}''$ is in $z[\pi] \,.\, B(G)$. Note that $\|\Sigma_{i=1}^{\infty} \lambda_i v \,.\, \omega_{\xi_i}\|_{B(G)} \leqq \Sigma_{i=1}^{\infty} \lambda_i = \operatorname{Tr}(|t|)$. Thus (i) and (ii) have been demonstrated. As for (iii) and (iv) they are almost obvious from the remark immediately above, since $z[\pi] \,.\, B(G)$ is the predual of the von Neumann algebra $\{\pi(g) \colon g \in G\}''$, $\mathscr{T}(H_\pi)$ is the predual of $\mathscr{L}(H_\pi)$, cf., **[4]** Chap. 1, §3 Théorème 1, and $\{\pi(g) \colon g \in G\}'' = \mathscr{L}(H_\pi)$ if and only if π is irreducible.

REMARK. Note that the partial isometry v in part (ii) of Theorem 3 is in general not in $\{\pi(g) \colon g \in G\}''$ but only in $\mathscr{L}(H_\pi)$. Thus $\|t_\pi\|_{B(G)}$ can be a zero even if t is not zero, and $z[\pi] \,.\, B(G)$ is the Banach space coimage of $\wedge$.

REMARK. Theorem 3 can immediately and obviously be applied to any group representation π such that, for example, $\pi(L^1(G)) \cap \mathscr{T}(H_\pi)$ is large; and there are many groups whose irreducible representations, for example, have this property. Thus one might say that $B(G)$ is "sufficient" for the Fourier analysis of such groups. We contend, however, in a forthcoming paper that $B(G)$ is "sufficient" for the Fourier analysis of any locally compact group, cf., the final remark of this paper.

We now introduce the concept of the standardization of a continuous, unitary group representation π. This procedure amounts basically to translation of the Tomita-Takesaki theory into the special context of group theory. This standardization process gains added significance when one realizes that with the machinery of this theory any continuous unitary, representation π of group G becomes a "modified left-regular" representation accompanied by the calculus thereof. As an application we will apply Theorem 3 in this setting.

Given any π, as above, let $M(\pi)$ (or M_π, whichever notation is more convenient) be the von Neumann algebra $\{\pi(g) \colon g \in G\}'' \subset \mathscr{L}(H_\pi)$. On $M(\pi)$ there exists a normal, faithful, semi-finite weight denoted by $\varphi(\pi)$, or φ_π; we can thus put the pair $\{M(\pi), \varphi(\pi)\}$ into standard form, cf., **[6]**, **[11]**, **[12]**, **[16]**. Very briefly, we take left-ideal $n_{\varphi(\pi)} = \{x \in M(\pi) \colon \varphi_\pi(x^*x) < +\infty\}$; $H_{\varphi(\pi)}$, the completion of $n_{\varphi(\pi)}$ with respect to the nondegenerate inner product induced by $\varphi(\pi)$, and $\eta: x \in n_\varphi \mapsto \eta(x) \in H_\pi$ the usual inclusion. We then denote by $\lambda(\varphi_\pi)$ the faithful *-representation of $M(\pi)$ on $H_{\varphi(\pi)}$ determined by $\lambda(\varphi_\pi)(x)\eta(y) = \lambda(\varphi_\pi)\eta(xy)$ for all $x \in M_\pi$, $y \in n_{\varphi(\pi)}$. But $\lambda(\varphi_\pi)\circ\pi$ is thus also a continuous, unitary representation of G; and it is *quasi-equivalent to* π. Thus in particular, $z[\lambda(\varphi_\pi)\circ\pi] = z[\pi]$, and both representations determine the same subspace of $B(G)$.

DEFINITION. Given any quasi-equivalence class $\{\pi\}$ of continuous, unitary representations of locally compact group G, then for $\pi \in \{\pi\}$ construct $\pi_s = \lambda(\varphi_\pi)\circ\pi \in \{\pi\}$, and call π_s the *standardization of* π.

REMARK. With abuse of notation we will often drop the subscript s and use π to denote both π and π_s, also H_π will henceforth refer only to $H_{\pi_s} = H_{\varphi(\pi)}$, etc.

We thus have the following corollary of Theorem 3:

COROLLARY. *Let π be the representation of G in standard form. Then the inverse Fourier-transform localized at π has, in addition to properties* (i), (ii), (iii), *of Theorem* 3,

(v) *If $b \in z[\pi].B(G)$, there exists an operator of rank one, $t = (\cdot\,|\,\eta)\xi \in \mathcal{T}(H_\pi)$, such that $b = t_\pi$. Furthermore, $\xi, \eta \in H_\pi$ can be so selected that* $\|b\|_{B(G)} = \|\xi\|_{H_\pi}\|\eta\|_{H_\pi}$.

REMARK. This corollary is obvious if one is familiar with the Tomita-Takesaki theory. A quick proof is as follows: Observe that if π is standard, i.e., M_π on H_π, with unitary involution J_π, and self-dual cone $P_\pi \subset H_\pi$, then any sigma-finite projection in M_π has a cyclic vector ξ (which can be chosen from P_π). But now we are done, cf., **[4]** Chap. II, §1 cor. of Thm. 4 and the discussion of standard forms following this corollary. Each positive, weakly continuous functional on M_π, i.e., in $z[\pi].P(G)$ is of the form ω_ξ, with $\xi \in P_\pi$. (In fact the map $\xi \in P_\pi \subset H_\pi \mapsto \omega_\xi \in (M_\pi)_+ = z[\pi].P(G)$ is a norm, homeomorphism, cf. **[1]**, **[2]**, **[6]**.) Thus given $b \in z[\pi].B(G)$, let $v.p = b$ be the (left) polar decomposition of b with respect to M_π, i.e., $v \in M_\pi$, $p \in z[\pi].P(G)$. Then $\|p\|_{B(G)} = \omega_\xi(e) = \|\xi\|^2_{H_\pi}$, and $p = t_1$, where $t_1 = (\ \ |\xi)\xi$, and $b = t_2$, where $t_2 = (\ \ |\xi)v\xi$, where $\|b\|_{B(G)} = \|\xi\|\,\|v\xi\|$. Thus every element in $z[\pi].B(G)$ is a transform of a rank-one operator of "minimal cross-norm". (We have in fact shown more, since we can select $\xi \in P_\pi$.)

REMARK. As a corollary of the above discussion we get a more detailed version of **[5]**, Thm. p. 218. Thus we may think of $b \in$

$z[\pi] \cdot B(G)$ as a generalized convolution (with a "twist") of two elements from H_π, cf., $L^2(G) * L^2(G)^{\sim} = A(G)$.

REMARK. We mentioned earlier that $\mathcal{T}(H_\pi)$ was a semi-abelianized, discretized version of another noncommutative L^1-measure "algebra". The measure "algebra" we have in mind is the L^1-space of weight φ_π. We have a definition and embryonic theory for this space analogous to the work done in **[9]** and **[10]** for the unimodular (trace) case. This L^1-space is the "proper" domain for the inverse Fourier transform; however, to go into details here would take us beyond the scope of this paper. We intend to go into this subject in depth in an upcoming paper.

REFERENCES

1. H. Araki, *A Radon-Nikodym theorem for W^*-algebras with chain rule*, Queen's—Math. preprint, 1972.
2. A. Connes, C. R. Acad. Sc. Paris, **274** (1972), 1923–1926.
3. J. Dixmier, *Les C^*-algèbres et leurs Representations*, Gauthier-Villars, Paris, 1964.
4. ———, *Les Algèbres d'Operateurs dans l'Espace Hilbertien*, Gauthier-Villars, deuxième édition, Gauthier-Villars, Paris, 1969.
5. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France, **92** (1964), 181–236.
6. U. Haagerup, *The standard form of von Neumann algebras*, Kobenhavns Universitet Math. preprint No. 15, 1973.
7. C. S. Herz, *Remarques sur la note précédente de M. Varopolus*, C. R. Acad. Sc. Paris, **260** (1965), 6001–6004.
8. ———, *La rapport entre l'algèbre A d'un groupe et d'un sous-groupe*, C. R. Acad. Sc. Paris, **271** (1970), 244–246.
9. I. E. Segal, *A non commutative extension of abstract integration*, Ann. of Math., **57** (1953), 401–457.
10. W. F. Stinespring, *Integration theorems for gages and duality for unimodular groups*, Trans. Amer. Math. soc., **90** (1959), 15–56.
11. M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and Its Applications*, Springer-Verlag, Berlin, 1970.
12. ———, *Lectures on Operator Algebras*, U.C.L.A. Lecture Notes, 1970.
13. N. Tatsuuma, *Prime ideals in the dual objects of locally compact groups*, Proc. Japan Acad., **47**, No. 3 (1971), 249–251.
14. J. L. Taylor, *Inverses, logarithms, and idempotents, in $M(G)$*, Rocky Mountain J. Math., **2** (1972), 183–206.
15. ———, *Measure Algebras*, notes from N.S.F. regional conference on measure algebras, Univ. of Montana, 1972.
16. M. Tomita, *Standard forms of von Neumann algebras*, the Vth functional analysis symposium of the Mathematical Society of Japan, Sendai, (1967).
17. M. E. Walter, *W^*-algebras and non-abelian harmonic analysis*, J. Functional Analysis, **11** (1972), 17–38.
18. ———, *A duality between locally compact groups and certain Banach algebras*, J. Functional analysis, (to appear).

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SUBHARMONICITY AND HULLS

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For X a compact set in C^2, $h(X)$ denotes the polynomially convex hull of X. We are concerned with the existence of analytic varieties in $h(X)\setminus X$. X is called "invariant" if (z, w) in X implies $(e^{i\theta}z, e^{-i\theta}w)$ is in X, for all real θ. X is called an "invariant disk" if there is a continuous complex-valued function a defined on $0 \leqq r \leqq 1$ with $a(0) = a(1) = 0$, such that $X = \{(z, w) \mid |z| \leqq 1,\ w = a(|z|)/z\}$. Let X be an invariant set and put $f(z, w) = zw$. Let Ω be an open disk in $C\setminus f(X)$ and put $f^{-1}(\Omega) = \{(z, w) \text{ in } h(X) \mid zw \in \Omega\}$. In Theorem 2 we show that if $f^{-1}(\Omega)$ is not empty, then $f^{-1}(\Omega)$ contains an analytic variety. Let now X be an invariant disk, with certain hypotheses on the function a. Then we show in Theorem 3 that $f^{-1}(\Omega)$ is the union of a one-parameter family of analytic varieties. A key tool in the proofs is a general subharmonicity property of certain functions associated to a uniform algebra. This property is given in Theorem 1.

1. Let X be a compact Hausdorff space, let A be a uniform algebra on X and let M be the maximal ideal space of A.

Fix $f \in A$. For each $\zeta \in \mathbf{C}$ put $f^{-1}(\zeta) = \{p \in M \mid f(p) = \zeta\}$ and for each subset Ω of $\mathbf{C}$, put $f^{-1}(\Omega) = \{p \in M \mid f(p) \in \Omega\}$. Consider an open subset Ω of $\mathbf{C}\setminus f(X)$. Supposing $f^{-1}(\Omega)$ to be nonempty, what can be said about the structure of $f^{-1}(\Omega)$? Work of Bishop [2] and Basener [1] yields that if $f^{-1}(\zeta)$ is at most countable for each $\zeta \in \Omega$, then $f^{-1}(\Omega)$ contains analytic disks. On the other hand, Cole [4] has given an example where no analytic disk is contained in $f^{-1}(\Omega)$. In §2 we prove.

THEOREM 1. *Let Ω be an open subset of $\mathbf{C}\setminus f(X)$. Choose $g \in A$. Define $Z(\zeta) = \sup_{f^{-1}(\zeta)} |g|$, $\zeta \in \Omega$. Then $\log Z$ is subharmonic in Ω.*

This theorem is proved by a method of Oka in [5].

In §3 we apply Theorem 1 to the following situation: X is a compact set in $\mathbf{C}^2$, A is the uniform closure on X of polynomials in z and w. Here $M = h(X)$, the polynomially convex hull of X. We assume that X is invariant under the map T_θ:

$$(z, w) \to (e^{i\theta} z, e^{-i\theta} w) \quad \text{for} \quad 0 \leqq \theta < 2\pi.$$

Put $f = zw$. Let Ω be an open disk contained in $\mathbf{C} \setminus f(X)$ with $0 \notin \Omega$. Here $f^{-1}(\Omega) = \{(z, w) \in h(X) \mid zw \in \Omega\}$.

THEOREM 2. *If $f^{-1}(\Omega)$ is not empty, then $f^{-1}(\Omega)$ contains an analytic disk.*

In §4, we consider the case when X is a disk in $\mathbf{C}^2$, defined:

$$X = \left\{(z, w) \,\middle|\, |z| \leqq 1,\ w = \frac{a(|z|)}{z}\right\},$$

where a is a continuous complex valued function defined on $0 \leqq r \leqq 1$, with $a(r) = o(r)$.

X is evidently invariant under T_θ for all θ. In Theorem 3 we give an explicit description of $h(x)$ for a certain class of such disks X.

2. *Proof of Theorem* 1. (Cf. **[5]**, §2.) Fix $\zeta_0 \in \Omega$ and let $\zeta_n \to \zeta_0$. Assume $Z(\zeta_n) \to t$. We claim $Z(\zeta_0) \geqq t$. For choose p_n in $f^{-1}(\zeta_n)$ with $|g(p_n)| = Z(\zeta_n)$. Let p be an accumulation point of $\{p_n\}$. Then $|g(p)| \geqq t$, whence $Z(\zeta_0) \geqq t$, as claimed. Thus Z is upper-semicontinuous at ζ_0, and so Z is upper-semicontinuous in Ω.

Theorem 1.6.3 of **[6]** gives that an upper-semicontinuous function u in Ω is subharmonic provided for each closed disk $D \subset \Omega$ and each polynomial P we have

(1) $$u \leqq \operatorname{Re} P \quad \text{on } \partial D \quad \text{implies} \quad u \leqq \operatorname{Re} P \quad \text{on } D.$$

Fix a closed disk D contained in Ω and let $\mathring{D}$ be its interior. Choose a polynomial P such that $\log Z \leqq \operatorname{Re} P$ on ∂D. Then

$$Z(\zeta)\,|\exp(-P(\zeta))| \leqq 1 \quad \text{on } \partial D.$$

Hence for each ζ in ∂D, if x is in $f^{-1}(\zeta)$, then

(2) $$|g(x)| \cdot |\exp(-P(f))(x)| \leqq 1, \quad \text{or}$$

$$|g \cdot \exp(-P(f))| \leqq 1 \quad \text{at} \quad x.$$

Now $g \cdot \exp(-P(f))$ is in A. Put $N = f^{-1}(\mathring{D})$. The boundary of N is contained in $f^{-1}(\partial D)$. Hence by the Local Maximum Modulus Principle for uniform algebras, for each y in N we can find x in $f^{-1}(\partial D)$ with

$$|g \exp(-P(f))(y)| \leqq |g \cdot \exp(-P(f))(x)|,$$

whence by (2) we have

$$|g\cdot\exp(-P(f))(y)|\leqq 1. \tag{3}$$

Fix ζ_0 in $\mathring{D}$. Choose y in $f^{-1}(\zeta_0)$ with $|g(y)|=Z(\zeta_0)$. Applying (3) to this y, we get

$$Z(\zeta_0)|\exp(-P(\zeta_0))|\leqq 1. \tag{4}$$

Hence $\log Z(\zeta_0)\leqq \operatorname{Re} P(\zeta_0)$. So (1) is satisfied, and so $\log Z$ is subharmonic in Ω, as desired.

3. *Proof of Theorem* 2. Since X is invariant under the maps T_θ, $h(X)$ is invariant under each T_θ. Fix $\zeta\in\Omega$. There are two possibilities:

(a) $|z|$ is constant on $f^{-1}(\zeta)$.

(b) $\exists\ r_1, r_2$ with $0<r_1<r_2$ and $\exists$

$$(z_1, w_1), (z_2, w_2)\in f^{-1}(\zeta) \quad \text{with} \quad |z_1|=r_1,\ |z_2|=r_2.$$

Suppose (b) occurs. Then the circles: $z=r_1e^{i\theta}$, $w=\zeta/r_1e^{i\theta}$, $0\leqq\theta\leqq 2\pi$ and $z=r_2e^{i\theta}$, $w=\zeta/r_2e^{i\theta}$, $0\leqq\theta\leqq 2\pi$ both lie in $h(X)$. Hence the analytic annulus: $r_1<|z|<r_2$, $w=\zeta/z$ lies in $f^{-1}(\zeta)$. Thus if (b) occurs at any point ζ in Ω, $f^{-1}(\Omega)$ does contain an analytic disk. Hence to prove the Theorem, we may assume that (a) holds for each $\zeta\in\Omega$. Define, for $\zeta\in\Omega$, $Z(\zeta)=\sup_{f^{-1}(\zeta)}|z|$, $W(\zeta)=\sup_{f^{-1}(\zeta)}|w|$. Fix $(z_0, w_0)\in f^{-1}(\zeta)$. Since we have case (a), $Z(\zeta)=|z_0|$. Hence $W(\zeta)=|w_0|$ and so $Z(\zeta)W(\zeta)=|\zeta|$, whence

$$\log Z(\zeta)+\log W(\zeta)=\log|\zeta|.$$

Since $\log Z$ and $\log W$ are subharmonic in Ω while $\log|\zeta|$ is harmonic, $\log Z$, $\log W$ are in fact harmonic in Ω. Put $U=\log Z$ and let V be the harmonic conjugate of U in Ω. Put $\phi(\zeta)=e^{U+iV}(\zeta)$. Then ϕ is analytic in Ω and $|\phi|=Z$ in Ω.

Assertion. The variety $z=\phi(\zeta)$, $w=\zeta/\phi(\zeta)$, $\zeta\in\Omega$, is contained in $h(X)$.

Fix $\zeta\in\Omega$. Choose $(z_1, w_1)\in f^{-1}(\zeta)$. Then $Z(\zeta)=|z_1|$, so $|\phi(\zeta)|=|z_1|$, i.e., $\exists$ real α with $z_1=\phi(\zeta)e^{i\alpha}$. Then $w_1=\zeta/\phi(\zeta)e^{i\alpha}$. But $(e^{-i\alpha}z_1, e^{i\alpha}w_1)\in h(X)$. Hence $(\phi(\zeta), \zeta/\phi(\zeta)\in h(X)$. The Assertion is proved, and Theorem 2 follows.

Note. Questions related to the result just proved are studied by J. E. Björk in **[3]**.

4. Invariant disks in C^2. Let P be a polynomial with complex coefficients, $P(t) = \sum_{n=1}^{N} c_n t^n$, which is one-one on the unit interval with endpoints identified, i.e., we assume that $P(1) = P(0) = 0$ and $P(t_1) \neq P(t_2)$ if $0 \leqq t_1 < t_2 < 1$. Also assume $P'(t) \neq 0$ for $0 \leqq t \leqq 1$. Then the curve β given parametrically: $\zeta = P(t)$, $0 \leqq t \leqq 1$, is a simple closed analytic curve in the ζ-plane whose only singularity is a double-point at the origin. Denote by θ the angle between the two arcs of β meeting at 0. Assume $\theta < \pi$. Define $a(r) = P(r^2)$, i.e.,

$$a(r) = \sum_{n=1}^{N} c_n r^{2n}. \tag{5}$$

Let X be the disk in $\mathbf{C}^2$ defined

$$X = \left\{ \left(z, \frac{a(|z|)}{z} \right) \middle| \, |z| \leqq 1 \right\}. \tag{6}$$

The function $f = zw$ maps X on β. Denote by Ω the interior of β.

THEOREM 3. *$\exists$ function ϕ analytic in Ω such that $h(X)$ is the union of X and $\{(z,0) \mid |z| \leqq 1\}$ and*

$$\{(z, w) \mid zw \in \Omega \quad \textit{and} \quad |z| = |\phi(zw)|\}.$$

COROLLARY. *Every point of $h(X) \setminus X$ lies on some analytic disk contained in $h(X)$.*

NOTATION. $A(\Omega)$ denotes the class of functions F defined and continuous in $\bar{\Omega}$ and analytic in Ω.

$\mathfrak{A}$ denotes the algebra of functions on $|z| \leqq 1$ which are uniformly approximable by polynomials in z and $a(|z|)/z$.

LEMMA 1. *Let $G \in C[0, 1]$. If $G(|z|) \in \mathfrak{A}$, then $\exists\, F \in A(\Omega)$ such that $G(r) = F(a(r))$ for $0 \leqq r \leqq 1$.*

Proof. Let g be a polynomial in z and $a(|z|)/z$. Calculation gives that there is a polynomial $\tilde{g}$ in one variable with

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) d\theta = \tilde{g}(a(r)), \quad 0 \leqq r \leqq 1.$$

Choose a sequence $\{g_n\}$ of polynomials in z and $a(|z|)/z$ approaching

$G(|z|)$ uniformly on $|z|\leqq 1$. Then $\tilde{g}_n(a(r))\to G(r)$ uniformly on $0\leqq r\leqq 1$. Hence $\exists\, F\in A(\Omega)$ with $\tilde{g}_n\to F$ uniformly on β, so $G(r)=F(a(r))$.

Lemma 2. *If $f=zw$, then $f^{-1}(\Omega)$ is not empty.*

Proof. Fix $\zeta_0\in\Omega$. If $f^{-1}(\Omega)$ is empty, then $f-\zeta_0\neq 0$ on $h(X)$ and so $(zw-\zeta_0)^{-1}$ lies in the closure of the polynomials in z and w on X. Then $(a(|z|)-\zeta_0)^{-1}\in\mathfrak{A}$. By Lemma 1, $\exists\, F\in A(\Omega)$ with $F(a(r))=(a(r)-\zeta_0)^{-1}$. Then $(\zeta-\zeta_0)^{-1}\in A(\Omega)$, which is false. So $f^{-1}(\Omega)$ is not empty.

Lemma 3. *Fix $\zeta\in\beta\setminus\{0\}$. Let (z_0,w_0) be a point in $h(X)$ with $z_0w_0=\zeta$. Then $(z_0,w_0)\in X$.*

Proof. Assume $(z_0,w_0)\notin X$. Let r be the point in $(0,1)$ with $a(r)=\zeta$. Put, for each r, $\gamma_r=\{(re^{i\theta},(a(r)/re^{i\theta}))\,|\,0\leqq\theta<2\pi\}$. Then γ_r is a polynomially convex circle contained in X. Hence $\exists$ polynomial P with $|P(z_0,w_0)|>2$, $|P|<1$ on γ_r. Choose a neighborhood N of γ_r on X where $|P|<1$. The image of $X\setminus N$ under the map $(z,w)\to zw$ is a closed subarc β_1 of β which excludes ζ. Choose $F\in A(\Omega)$ with $F(\zeta)=1$, $|F|<1$ on $\beta\setminus\{\zeta\}$. Then $\exists\,\delta>0$ such that $|F|<1-\delta$ on β_1. Hence $|F(zw)|<1-\delta$ on $X\setminus N$. Also $|F(zw)|\leqq 1$ on X. Fix n and put

$$Q=F(zw)^n\cdot P(z,w).$$

$|Q(z_0,w_0)|>2$. On N, $|Q|\leqq|P|<1$. On $X\setminus N$, $|Q|<(1-\delta)^n\cdot\max_X|P|$, and so $|Q|<1$ on $X\setminus N$ for large n. Then $|Q|<1$ on X. Since F is a uniform limit on β of polynomials in ζ, Q is a uniform limit on $X\cup\{(z_0,w_0)\}$ of polynomials in z and w. This contradicts that $(z_0,w_0)\in h(X)$. Thus $(z_0,w_0)\in X$. We are done.

Note. Since f maps X on β and $\mathbf{C}\setminus f(X)$ is the union of the interior and exterior of β, we conclude from the last Lemma that $h(X)$ is the union of X and $f^{-1}(\{0\})$ and $f^{-1}(\Omega)$.

We need some notation now. For each $\zeta\in\beta\setminus\{0\}$, denote by $r(\zeta)$ the unique r in $(0,1)$ with $a(r)=\zeta$.

Since a is a polynomial in r vanishing at 0, there is a constant $d>0$ such that

$$r(\zeta)>d\,|\zeta|,\quad \text{all}\quad \zeta\in\beta. \tag{7}$$

For $\zeta_0 \in \Omega$, denote by μ_{ζ_0} harmonic measure at ζ_0 relative to Ω. Since β consists of analytic arcs, with one jump-discontinuity for the tangent at $\zeta = 0$, $\mu_{\zeta_0} = K_{\zeta_0}\, ds$, where K_{ζ_0} is a bounded functions on β and ds is arc-length. Define

$$U(\zeta_0) = \int_\beta \log r(\zeta) d\mu_{\zeta_0}(\zeta).$$

Since (7) holds, this integral converges absolutely. U is a harmonic function in Ω, bounded above, and continuous at each boundary point $\zeta \in \beta \setminus \{0\}$ with boundary value $\log r(\zeta)$ at ζ.

For $\zeta \in \Omega$, define

$$Z(\zeta) = \sup_{f^{-1}(\zeta)} |z|, \quad W(\zeta) = \sup_{f^{-1}(\zeta)} |w|.$$

LEMMA 4. *For all* $\zeta \in \Omega$, $\log Z(\zeta) \leqq U(\zeta)$ *and* $\log W(\zeta) \leqq \log|\zeta| - U(\zeta)$.

Proof. Fix $\zeta \in \beta \setminus \{0\}$, choose $\zeta_n \in \Omega$ with $\zeta_n \to \zeta$ and suppose $Z(\zeta_n) \to \lambda$. Choose $p_n \in f^{-1}(\zeta_n)$ with $Z(\zeta_n) = |z(p_n)|$. Without loss of generality, $p_n \to p$ for some point $p \in h(X)$. Then $f(p) = \zeta$. By Lemma 3, $p \in X$, i.e., $p = (\mathrm{re}^{i\theta}, (a(r)/\mathrm{re}^{i\theta}))$ for some r, θ. Also $a(r) = \zeta$ and so $r = r(\zeta)$, whence $|z(p_n)| \to r(\zeta)$ and so $\lambda = r(\zeta)$. Thus $Z(\zeta') \to r(\zeta)$ as $\zeta' \to \zeta$ from within Ω, and so $\log Z$ assumes the same boundary values as U, continuously on $\beta \setminus \{0\}$.

For each positive integer k, let $\Omega_k = \{\zeta \in \Omega \,\big|\, |\zeta| > 1/k\}$. $\partial \Omega_k$ is the union of a closed subarc β_k of $\beta \setminus \{0\}$ and an arc α_k on the circle $|\zeta| = 1/k$.

Fix $\zeta_0 \in \Omega$. For large k, $\zeta_0 \in \Omega_k$. Denote by $\mu_{\zeta_0}^{(k)}$ the harmonic measure at ζ_0 relative to Ω_k. An elementary estimate gives that there is a constant C_{ζ_0} independent of k such that

$$\mu_{\zeta_0}^{(k)}(\alpha_k) \leqq C_{\zeta_0} \cdot \frac{1}{\sqrt{k}} \text{ for all } k. \tag{8}$$

Let S be any function subharmonic in Ω and assuming continuous boundary values, again denoted S, on $\beta \setminus \{0\}$. Assume $\exists$ constant M with $S \leqq M$ in Ω. Then for all k,

$$S(\zeta_0) \leqq \int_{\beta_k} S d\mu_{\zeta_0}^{(k)} + \int_{\alpha_k} M d\mu_{\zeta_0}^{(k)}, \quad \text{whence} \tag{9}$$

$$S(\zeta_0) \leqq \int_{\beta_k} S d\mu_{\zeta_0}^{(k)} + M \cdot C_{\zeta_0} \cdot \frac{1}{\sqrt{k}}.$$

Applying (9) with $S = \log Z$, we get

$$\log Z(\zeta_0) \leqq \int_{\beta_k} U d\mu_{\zeta_0}^{(k)} + MC_{\zeta_0} \cdot \frac{1}{\sqrt{k}}, \tag{10}$$

since as we saw earlier, $\log Z = U$ on $\beta \setminus \{0\}$.

By (7), if $\zeta' \in \alpha_k$,

$$U(\zeta') = \int_\beta \log r(\zeta) d\mu_{\zeta'}(\zeta) > C + \int_\beta \log |\zeta| \, d\mu_{\zeta'}(\zeta),$$

where C is a constant, so

$$U(\zeta') > C + \log |\zeta'| = C + \log \frac{1}{k}. \quad \text{Hence}$$

$$\begin{aligned} U(\zeta_0) &= \int_{\beta_k} U d\mu_{\zeta_0}^{(k)} + \int_{\alpha_k} U d\mu_{\zeta_0}^{(k)} \\ &\geqq \int_{\beta_k} U d\mu_{\zeta_0}^{(k)} + \left(C + \log \frac{1}{k}\right) \frac{C_{\zeta_0}}{\sqrt{k}}. \end{aligned}$$

Combining this with (10) and letting $k \to \infty$, we get that $\log Z(\zeta_0) \leqq U(\zeta_0)$, as desired. A parallel argument gives the assertion regarding W. We are done.

LEMMA 5. *With Z defined as above,* $\log Z(\zeta) = U(\zeta)$ *for all* $\zeta \in \Omega$, *and* $\log W(\zeta) = \log |\zeta| - U(\zeta)$.

Proof. Suppose either equality fails at some point ζ_0. By the last Lemma, this implies that

$$\log Z(\zeta_0) + \log W(\zeta_0) < \log |\zeta_0|.$$

Fix $p \in f^{-1}(\zeta_0)$. Then $|z(p)| \leqq Z(\zeta_0)$, $|w(p)| \leqq W(\zeta_0)$, so

$$\log |z(p)w(p)| < \log |\zeta_0|.$$

But $z(p)w(p) = \zeta_0$, so we have a contradiction, proving the Lemma.

Proof of Theorem 3. Let V denote the harmonic conjugate of U in Ω and put $\phi = e^{U+iV}$. Fix $(z_0, w_0) \in f^{-1}(\Omega)$ and put $\zeta_0 = z_0 \cdot w_0$. Unless $|z_0| = Z(\zeta_0)$ and $|w_0| = W(\zeta_0)$, we have

$$|\zeta_0| = |z_0| \, |w_0| < Z(\zeta_0) W(\zeta_0) = |\zeta_0|$$

by the last Lemma. So we must have $|z_0| = Z(\zeta_0) = |\phi(\zeta_0)|$.

Conversely fix $\zeta_0 \in \Omega$ and let (z_0, w_0) be a point in $\mathbf{C}^2$ such that $z_0 \cdot w_0 = \zeta_0$ and $|z_0| = |\phi(\zeta_0)|$. Choose $(z_1, w_1) \in f^{-1}(\zeta_0)$. By the preceding $|z_1| = |\phi(\zeta_0)|$, so $\exists$ real α with $z_0 = e^{i\alpha}z_1$, $w_0 = e^{-i\alpha}w_1$. Hence $(z_0, w_0) \in h(X)$, so $(z_0, w_0) \in f^{-1}(\Omega)$. Thus $f^{-1}(\Omega)$ consists precisely of those points (z, w) with $zw \in \Omega$ and $|z| = |\phi(zw)|$.

To finish the proof we need only identify $f^{-1}(0)$. The circle $\{(z,0) \mid |z| = 1\}$ lies in X, so the disk D: $\{(z,0) \mid |z| \leqq 1\}$ is contained in $f^{-1}(0)$. If $(z_0, w_0) \in f^{-1}(0)$ and does not lie in D, then $z_0 = 0$, $w_0 \neq 0$. The same argument as was used in proving Lemma 3 shows that then $(z_0, w_0) \notin h(X)$, contrary to assumption. So $f^{-1}(0) = D$, and the proof of Theorem 3 is finished.

REMARK. As we have just seen, $f^{-1}(\Omega)$ is the union of varieties V_α, $0 \leqq \alpha < 2\pi$, where V_α is defined:

$$z = e^{i\alpha}\phi(\zeta), \quad w = e^{-i\alpha}\frac{\zeta}{\phi(\zeta)}, \quad \zeta \in \Omega.$$

What does the boundary of such a variety V_α in $h(X)$ look like? It splits into two sets:

$$S = \{(z, w) \in \partial V_\alpha \mid zw \in \beta \setminus \{0\}\} \quad \text{and}$$
$$T = \{(z, w) \in \partial V_\alpha \mid zw = 0\}.$$

It is easy to see that S is an arc on X cutting each circle: $\{(z, w) \in X \mid |z| = r\}$, $0 < r < 1$, exactly once while T is a closed subset of the disk $D = \{(z,0) \mid |z| \leqq 1\}$.

It is remarkable that even though X is itself very regular, the rest of the hull of X is attached to X in a very complicated way.

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REFERENCES

1. R. Basener, *A condition for analytic structure*, Proc. Amer. Math. Soc., **36**, No. 1 (1972), 156–160.

2. E. Bishop, *Holomorphic completions, analytic continuation and the interpolation of semi-norms*, Ann. of Math., (2), **78** (1963), 468–500.

3. J. E. Björk, *Compact groups operating on Banach algebras*, (to appear).

4. B. Cole, *One-point parts and the peak point conjecture*, Dissertation, Yale University (1968).

5. K. Oka. *Domaines d'holomorphie*, J. Sci. Hiroshima Univ., **7** (1937), 115–130.

6. L. Hörmander, *An Introduction to complex analysis in several variables*, D. Van Nostrand Co. (1966).

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A MAP OF E^3 ONTO E^3 TAKING NO DISK ONTO A DISK

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An example is given of an u.s.c. decomposition in which no disk in E^3 maps onto a disk under the natural projection map P, and, furthermore, the decomposition space E^3/G is homeomorphic to E^3. Each nondegenerate element is a tame arc. The image $P(H)$ of the set of nondegenerate elements is 0-dimensional, although $\mathrm{Cl}\, P(H)$ is E^3. The basic construction used is called a knit Cantor set of nondegenerate elements.

Bing and Borsuk [3] have given an example of a 3-dimensional absolute retract R containing no disk. They define a particular u.s.c. decomposition of E^3 that yields R as the decomposition space. Hence, their example is a closed map of E^3 taking no disk onto a disk, but, of course, their image is not E^3.

In [8] the author defined a set $X \subset E^3$ to be the P-lift of a set Y contained in the decomposition space E^3/G if and only if X and Y are homeomorphic and the image of X under the natural projection is Y. A disk is said to said to be P-liftable if and only if it has a P-lift. Using this terminology, the example that is constructed in this note has no P-liftable disk in the image space.

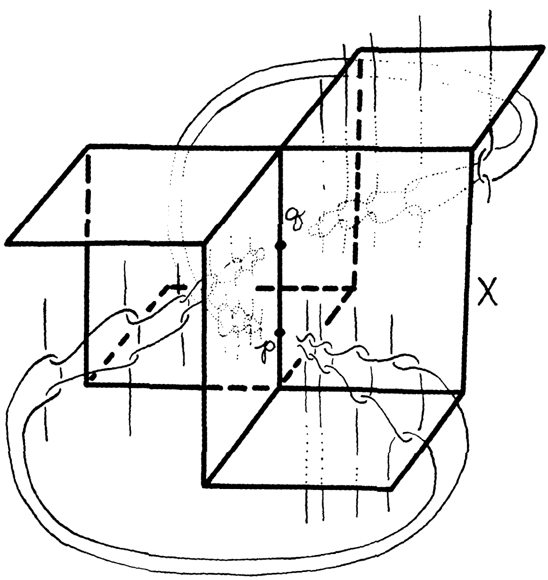
In [1] Armentrout asked whether there exists a pointlike decomposition G of E^3 such that there is a 2-sphere S in E^3/G that can not be approximated by a P-liftable sphere. This was first answered by the author in [9] by giving an example of a space E^3/G containing such a 2-sphere. In the decomposition space of this note no 2-sphere is P-liftable. Hence, this space is another answer to Armentrout's query.

The construction we describe in this note is based on a knit example in the author's papers [6], [7], and [9]. It is assumed that the reader is familiar with this example and the notations in [6]. We also need the following definitions.

DEFINITION. Let J_1' be the circle in the x—y plane with radius 1 and center at the origin, and J_2' be the circle in the y—z plane with radius 1 and center at $y = -1$, $z = 0$. Any two tame simple closed curves J_1 and J_2 in E^3 are said to *simply link* if and only if there is a homeomorphism of E^3 onto itself taking J_1 and J_2 onto J_1' and J_2', respectively. Two disjoint compact sets S_1 and S_2 are said to *simply link* if and only if there exist simple closed curves $J_1 \subset S_1$ and $J_2 \subset S_2$ such that J_1 and J_2 simply link.

DEFINITION. Let $\{M_i\}$ be a defining sequence for a decomposition G. The sequence $M_1^1, M_2^1, M_2^2, M_3^1, M_3^2, M_3^3, M_4^1, \cdots$, denoted by $\{M_i^j\}$, is called a *compound defining sequence* for G if and only if (1) $i = 1,2,3,\cdots$; (2) $1 \leqq j \leqq i$; (3) $\{M_i^j\}$ with the lexicographic order indicated above on the indices is a defining sequence; (4) M_i^{j-1} is a regular neighborhood of M_i^j; and (5) $M_i^i = M_1$. Given any decomposition of E^3 with a defining sequence, there exists a compound defining sequence.

In [6] and [9] the author gave an example of a knit decomposition G_0 of E^3, a 2-complex X in E^3, and an $\varepsilon > 0$ such that $P(X) \subset E^3/G_0$ is a disk D having the property that no disk D_ε which is ε-homeomorphic to D is P-liftable. This decomposition used "knit Cantor sets of nondegenerate elements". In the figure two countably infinite sets of arcs knit from the point p to the point q are indicated. Each arc pictured represents a Cantor set of arcs. These Cantor sets of arcs and the limiting arc g_p containing p and q are the nondegenerate elements of the decomposition G_0. The 2-complex X consists of eight squares that do not form a disk. Notice that each arc except g_p pierces X in a point. Since in E^3/G_0 the arc g_p has an image that is a point, the image of X is a disk D. The decomposition is a modification of two (2,1) toroidal decompositions. The entwining of the nondegenerate elements caused by the (2,1) toroidal decompositions is not indicated in the figure. It would be above and below the portions shown. The result



of the entwining is that Bd T_0 is not homotopic to a point in $E^3 - H^*$. It was shown that E^3/G_0 is homeomorphic to E^3. In the proof only disks that are ε-homeomorphic to D were considered. Using similar arguments, it can be shown that there is no disk δ in E^3 such that $P(\delta)$ is a disk in E^3/G_0 and Bd X is homotopic in $E^3 - M_1$ to Bd δ. (In **[6]** M_1 is the first manifold in a particular defining sequence for G_0.) Hence, for this decomposition G_0, there is a regular neighborhood T_0 of Bd X in E^3 which is an unknotted polyhedral solid torus such that no simple closed curve homotopic in T_0 to its core bounds a disk δ that has an image in E^3/G_0 that is a disk.

Simply linking T_0, there is an unknotted polyhedral solid torus S_0 that contains M_1, which in turn contains all the nondegenerate elements of G_0. Let T_C and S_C be any pair of simply linked unknotted polyhedral solid tori. There is a homeomorphism of E^3 onto itself that takes T_0 and S_0 onto T_C and S_C, respectively. Given $d > 0$, this homeomorphism can be chosen so that the diameter of each nondegenerate element is less than d. (This follows from the proof that E^3/G_0 is homeomorphic to E^3.) Hence, given simply linked unknotted polyhedral solid tori T_C and S_C and given $d > 0$, there is a decomposition G_C of E^3 with nondegenerate elements H_C such that (1) $H_C^* \subset S_C$; (2) for each $g \in H_C$, diam $g < d$; and (3) no disk δ with Bd δ homotopic in T_C to its core has an image in E^3/G_C which is a disk.

We now construct a family $\mathcal{T}$ of solid tori T_C. Associated with each T_C there are an S_C and H_C having the above properties. The family is dense in E^3 and so chosen that for any disk D in E^3 there are a solid torus $T_C \in \mathcal{T}$ and a tame simple closed curve $J \subset D \cap T_C$ such that J is homotopic in T_C to its core. Let G be the union of $H = \{g \in H_C : H_C$ is associated with some $T_C \in \mathcal{T}\}$ and points in $E^3 - H$. Then no disk projects onto a disk under the mapping $P: E^3 \to E^3/G$.

The family $\mathcal{T}$ is constructed in stages. To define the first stage, we start with the set of points $\mathcal{V}_1 = \{(p/2,\ q/2,\ r/2): p,\ q,$ and r are integers$\}$. Associated with $\mathcal{V}_1$ is the set $\mathcal{C}_1$ of all unknotted polygonal simple closed curves having vertices in $\mathcal{V}_1$ and diameters not greater than one.

For any tame unknotted simple closed curve J, let $L_J =$ lub$\{d$: there is a polygonal simple closed curve K in the unbounded component of $E^3 - N_d(J)$ such that K simply links $J\}$. (Here $N_d(J)$ denotes the d-neighborhood of J.) For each $C \in \mathcal{C}_1$, choose a polygonal simple closed curve K_C that simply links C and lies in the unbounded component of the complement of the $L_C/2$-neighborhood of C. These can certainly be chosen so that the diameter of each K_C is less than four and each K_C fails to intersect the union of the other such simple closed curves associated with elements of $\mathcal{C}_1$. For each K_C, choose a polyhedral solid torus S_C with core K_C and contained in the $L_C/4$-neighborhood of K_C. This implies that C simply links S_C. The set of these solid tori S_C can be chosen to be mutually disjoint.

The method for choosing the solid torus T_C that simply links S_C depends on the fact that the union of two solid tori with common boundaries and disjoint interiors is the 3-sphere obtained as the union of E^3 and the point at infinity. Since simply linked tori do not have common boundaries, for each C enlarge S_C slightly: choose a solid torus S'_C that satisfies the definition of S_C and contains S_C in its interior. Let N be the $L_C/4$-neighborhood of C. Let A_C be the complement of a polyhedral 3-ball containing $N \cup S_C$ and having diameter less than eight. Let B_C be a polyhedral 3-ball (a tubular neighborhood of a polygonal arc) in $E^3 - (A_C \cup N \cup S'_C)$ connecting A_C and S'_C in such a way that $\mathrm{Cl}(E^3 - (A_C \cup B_C \cup S'_C))$ is a polyhedral unknotted solid torus having C as a core. Denote this solid torus with core C by T_C. Observe that T_C contains the $L_C/4$-neighborhood of C and simply links S_C. Let $\mathcal{T}_1 = \{T_C : C \in \mathcal{C}_1\}$. This is the first stage of the construction of the family $\mathcal{T}$.

For each T_C and S_C, we choose a decomposition G_C, having the above properties with respect to T_C and S_C and having no nondegenerate element with diameter greater than one. It can be assumed that each nondegenerate element is polygonal.

From the definition of a compound defining sequence it follows that each component which is a solid torus is one of a finite nest of solid tori which are regular neighborhoods of the innermost one of the nest. We assume that all solid tori in a nest are tubular neighborhoods of the same polygonal simple closed curve.

To define the nth stage, let $\mathcal{V}_n = \{(p/(2^n), q/(2^n), r/(2^n)) : p, q,$ and r are integers$\}$. The family $\mathcal{C}_n$ is the set of all unknotted polygonal simple closed curves having vertices in $\mathcal{V}_n$ and diameters again not greater than one. Complete choices of T_C and S_C as in the first stage with the added requirement that each K_C miss all nondegenerate elements from previous stages.

We next determine the size requirement for nondegenerate elements at the nth stage. For any $C \in \mathcal{C}_n$, the associated solid torus S_C intersects the compound defining sequences of only a finite number of the sets H_C previously defined. Call them $H_k, 1 \leqq k \leqq k_C$, where k_C is the appropriate integer. For each H_k, let $\{(M_k)_i^j\}$ be the compound defining sequence. Because S_C misses each set H_k^*, there are only a finite number of $(M_k)_i^j$ whose boundaries intersect S_C. For each C, let $x_C = \min\{d : d$ is the distance between two sets $\mathrm{Bd}\,(M_k)_i^j \cap S_C$ for some values of $i, j,$ and $k\}$. This x_C is strictly positive. We require that each nondegenerate element in H_C have diameter less than x_C and less than $1/n$. There is a decomposition G_C satisfying this and the conditions above with respect to S_C and the corresponding T_C. Again assume that each image of a nondegenerate element is polygonal and that manifolds in compound defining sequences are unions of prisms. This completes the construction of the nth stage.

Let G be the union of $H = \{g \in H_C: C \in \mathscr{C}_n$ for some positive integer $n\}$ and points in $E^3 - H^*$. This decomposition G defines the map claimed in the title.

The proof is based on McAuley's countably shrinkable theorem **[4]**, as slightly revised by Reed **[5]**, To use the theorem it is necessary to shrink certain elements without permitting others to grow too much. Some of the shrinking is based on Bing's shrinking of the (2,1) toroidal decomposition **[2]**. Recall that, in the construction, elements at a later stage are not permitted to intersect boundaries of more than two manifold stages in previous compound defining sequences. This allows growth of later stage nondegenerate elements to be controlled during the shrinking of a particular stage. The proof is tedious, but straightforward.

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References

1. Steve Armentrout, *Monotone decompositions of* E^3, Annals of Math. Studies, **60** (1966), 1–25.

2. R. H. Bing, *A homeomorphism between the* 3*-sphere and the sum of two horned spheres*, Ann. of Math., (2) **56** (1952), 354–362.

3. R. H. Bing and K. Borsuk, *A* 3*-dimensional absolute retract which does not contain any disk*, Fund. Math., **54** (1964), 159–175.

4. Louis F. McAuley, *Upper semicontinuous decompositions of* E^3 *into* E^3 *and generalizations to metric spaces*, *Topology of* 3*-Manifolds and Related Topics*, Proc. Univ. of Georgia Inst., 1961, 21–26.

5. Myra Jean Reed, *Decomposition spaces and separation properties*, Doctoral Dissertation, SUNY/Binghamton, May, 1971.

6. Edythe P. Woodruff, *Concerning the condition that a disk in* E^3/G *be the image of a disk in* E^3, Proc. Conf. on Monotone Mappings and Open Mappings at SUNY/Binghamton (1970).

7. ———, *Concerning the condition that a disk in* E^3/G *be the image of a disk in* E^3, Doctoral Dissertation, SUNY/Binghamton, May, 1971.

8. ———, *Conditions in which disks are P-liftable*, Trans. Am. Math. Soc., **186** (1973), 403–418.

9. ———, *Examples of disks in* E^3/G *which cannot be approximated by P-liftable disks*, Fund. Math., **86** (1974), 117–136.

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