

Pacific Journal of Mathematics

**A NEW CHARACTERIZATION OF CHARACTERISTIC
FUNCTIONS OF ABSOLUTELY CONTINUOUS
DISTRIBUTIONS**

SIMEON M. BERMAN

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It is well known that if g belongs to L_2 , then

$$\frac{\int g(x)\bar{g}(x+y)dx}{\int |g(x)|^2 dx}$$

is the characteristic function of an absolutely continuous distribution function. Conversely, every such characteristic function has the representation given above. Here we show that if $R(s, t)$ is a covariance function such that $R(s, s)$ belongs to L_1 , then

$$\frac{\int R(s, s+t)ds}{\int R(s, s)ds}$$

is the characteristic function of an absolutely continuous distribution. Conversely, every such characteristic function has the latter representation (put $R(s, t) = g(s)\bar{g}(t)$). The use of this new result is that certain functions are directly seen to be of the second form but not the first; hence, they can be identified as characteristic functions of absolutely continuous distributions.

1. The main theorem. Let $R(s, t)$, $-\infty < s, t < \infty$, be a complex-valued Borel function of two variables. It is a *covariance function* if for any positive integer n , and any set of pairs (s_i, u_i) , $i = 1, \dots, n$,

$$\sum_{i=1}^n \sum_{j=1}^n R(s_i, s_j) u_i u_j \geq 0.$$

By Kolmogorov's existence theorem and the well known moment properties of Gaussian processes, for every covariance function there exists a probability space and a complex Gaussian process $X(t)$, $-\infty < t < \infty$, on the space such that

$$EX(t) = 0 \text{ for all } t, \quad EX(s)\bar{X}(t) = R(s, t) \text{ for all } s, t.$$

We say that X is *associated* with R .

The function $R(s, s)$ is nonnegative because it is equal to $E|X(s)|^2$. If

$$(1.1) \quad \int_{-\infty}^{\infty} R(s, s) ds < \infty,$$

then there exists an associated process X which is measurable and satisfies

$$(1.2) \quad E \int_{-\infty}^{\infty} |X(s)|^2 ds = \int_{-\infty}^{\infty} E|X(s)|^2 ds < \infty.$$

It also follows that $X(t)$ belongs to L_2 almost surely, and so there is a measurable version of the Fourier transform process

$$(1.3) \quad \hat{X}(u) = \int_{-\infty}^{\infty} e^{ius} X(s) ds, \quad -\infty < u < \infty.$$

By Parseval's Theorem we also have

$$(1.4) \quad \int_{-\infty}^{\infty} E|X(s)|^2 ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} E|\hat{X}(u)|^2 du.$$

THEOREM 1. *Let R be a covariance function satisfying (1.1). Then the function*

$$(1.5) \quad r(t) = \frac{\int_{-\infty}^{\infty} R(s, s+t) ds}{\int_{-\infty}^{\infty} R(s, s) ds}$$

is a characteristic function. The corresponding distribution function is absolutely continuous with the derivative

$$(1.6) \quad g(u) = \frac{E|\hat{X}(-u)|^2}{\int_{-\infty}^{\infty} E|\hat{X}(y)|^2 dy},$$

where X is the associated process satisfying (1.2). Conversely if $r(t)$ is the characteristic function of an absolutely continuous distribution, then there exists R satisfying (1.1) such that r is representable as (1.5).

Proof. First we prove the direct assertion. It follows from the definition of X that

$$\int_{-\infty}^{\infty} R(s, s + t) ds = \int_{-\infty}^{\infty} EX(s)\bar{X}(s + t) ds.$$

By virtue of condition (1.1) and the Cauchy-Schwarz inequality, we can take the expectation outside of the integral, and then apply the Parseval theorem:

$$E \int_{-\infty}^{\infty} X(s)\bar{X}(s + t) ds = E \left\{ (1/2\pi) \int_{-\infty}^{\infty} e^{-iut} |\hat{X}(u)|^2 du \right\}.$$

It follows that the ratio in (1.5) is equal to

$$\frac{\int_{-\infty}^{\infty} e^{iut} E |\hat{X}(-u)|^2 du}{\int_{-\infty}^{\infty} E |\hat{X}(y)|^2 dy}$$

This is exactly the Fourier transform of the function $g(u)$ in (1.6).

The converse is simple: it is given in the above abstract.

2. Factorable covariances. R is said to be factorable if there exists a monotone function $A(y)$ and a “kernel” function $\phi(t, y)$ such that

$$R(s, t) = \int_{-\infty}^{\infty} \phi(s, y)\bar{\phi}(t, y) dA(y).$$

The condition (1.1) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi(s, y)|^2 dA(y) ds < \infty.$$

It follows (by Fubini’s theorem) that $\phi(\cdot, y)$ belongs to L_2 for almost all y (with respect to dA); thus

$$\hat{\phi}(u, y) = \int_{-\infty}^{\infty} e^{iuz} \phi(z, y) dz$$

exists for all such y . By virtue of the isometry $X(t) \rightarrow \phi(t, \cdot)$ the density (1.6) takes the form

$$(2.1) \quad g(u) = \frac{\int_{-\infty}^{\infty} |\hat{\phi}(-u, y)|^2 dA(y)}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{\phi}(u, y)|^2 dudA(y)}.$$

As a characteristic function $r(t)$ is also the covariance function of a stationary Gaussian process. When the spectral distribution is absolutely continuous, the process has a well known representation as a moving average of "white noise" on the line (see [2], p. 533). We will show that when R is factorable the stationary process with covariance of the form (1.5) also has a representation as a moving average of "noise" in the plane. The latter representation is more informative and easier to derive in certain special cases (see §4 below).

Let W be a real Gaussian random set function defined over the plane Borel sets, that is, $W(C)$ has a normal distribution for every plane Borel set C . Let W have the following moment structure:

$$EW(C) = 0 \text{ for all } C$$

$EW(C)W(C') = 0$ if C and C' are disjoint (independent increments)

$EW^2(C) = \int_B dx \cdot \int_{B'} dA(y)$ if $C = B \times B'$ is a product of two linear sets.

Consider the stochastic integral with respect to W , divided by a positive constant:

$$(2.2) \quad Y(t) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x+t, y) W(dx \times dy)}{\left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi(x, y)|^2 dx dA(y) \right\}^{\frac{1}{2}}}.$$

By a direct calculation and by means of the fundamental properties of the stochastic integral we find that the process $Y(t)$ is stationary (and Gaussian) with covariance function (1.5).

3. Change of time parameter in the covariance. Let $R(s, t)$ be a covariance, and $f(x)$ a real Borel function. Then the composite function $R(f(x), f(y))$ is also a covariance function (in x and y). According to Theorem 1, if

$$(3.1) \quad \int_{-\infty}^{\infty} R(f(x), f(x)) dx < \infty,$$

then

$$(3.2) \quad r(t) = \frac{\int_{-\infty}^{\infty} R(f(x), f(x+t)) dx}{\int_{-\infty}^{\infty} R(f(x), f(x)) dx}$$

is the characteristic function of an absolutely continuous distribution. By means of this result we can identify some general and interesting functions as such characteristic functions.

EXAMPLE. Let $\sigma^2(t)$, $-\infty < t < \infty$, be the incremental second moment function of a Gaussian process with mean 0 and stationary increments ($\sigma^2(t) = E|U(s+t) - U(s)|^2$ where U has stationary increments). Then the covariance function of the process is

$$R(s, t) = \frac{1}{2}[\sigma^2(s) + \sigma^2(t) - \sigma^2(t - s)].$$

Let $f(x)$ be a Borel function such that

$$\int_{-\infty}^{\infty} \sigma^2(f(x))dx < \infty;$$

then (3.1) is fulfilled, and so

$$(3.3) \quad r(t) = 1 - \frac{\int_{-\infty}^{\infty} \sigma^2(f(x+t) - f(x))dx}{2 \int_{-\infty}^{\infty} \sigma^2(f(x))dx}$$

is the characteristic function of an absolutely continuous distribution.

Let $f(x)$ be a Borel function such that

$$\int_{-\infty}^{\infty} |f(x)|^\alpha dx = 1$$

for some α , $0 < \alpha \leq 2$; then

$$(3.4) \quad r(t) = 1 - \frac{1}{2} \int_{-\infty}^{\infty} |f(x+t) - f(x)|^\alpha dx$$

is a characteristic function with an absolutely continuous distribution; indeed, it is a special case of (3.3) with $\sigma^2(t) = t^\alpha$. (The fact that (3.4) is a characteristic function was first proved by Lawrence Shepp in a private communication.)

This can be used to prove a general result about the space L_α . According to the classical representation of the characteristic function of an absolutely continuous distribution as a convolution, there exists a function \tilde{f} such that

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^2 dx = 1$$

and $r(t)$ is representable as

$$1 - \frac{1}{2} \int_{-\infty}^{\infty} |\tilde{f}(x+t) - \tilde{f}(x)|^2 dx.$$

From (3.4) we then conclude that

$$\int_{-\infty}^{\infty} |f(x+t) - f(x)|^\alpha dx = \int_{-\infty}^{\infty} |\tilde{f}(x+t) - \tilde{f}(x)|^2 dx, \text{ for all } t.$$

As far as I can determine the existence of such an \tilde{f} in L_2 for each f in L_α is a result unknown up to now.

4. A new proof of Polya's theorem and related results. Polya described a class of characteristic functions, now called "Polya characteristic functions". [4] He showed that if $r(t)$ is a convex function such that $r(t) \geq 0$, $r(t) = r(-t)$, $r(0) = 1$, and $\lim_{t \rightarrow \infty} r(t) = 0$, then $r(t)$ is the characteristic function of an absolutely continuous distribution. We will show that such a function is representable as in Theorem 1, and so provide a new proof of Polya's theorem; furthermore, we will get an explicit form of the density from the results of §2, and a stochastic integral representation.

As a convex function, $r(t)$ has an integral representation

$$(4.1) \quad r(t) = \int_{|t|}^{\infty} f(x) dx,$$

where $f(x)$, $x \geq 0$, is nonnegative and nonincreasing, and

$$\int_0^{\infty} f(x) dx = 1.$$

Indeed, take f as the negative of the right hand derivative of r (see [3], p. 5). Extend f to all x by assigning it the value 0 on the negative axis. Then $r(t)$ is representable as

$$r(t) = \int_{-\infty}^{\infty} \min(f(x), f(x+t)) dx.$$

This is of the form (1.5) with $R(s, t) = \min(s, t)$. Thus r is a characteristic function of the indicated type.

Let $I(t)$ be the indicator function of the positive t -axis. The covariance $\min(s, t)$ is factorable for positive s and t :

$$\min(s, t) = \int_0^\infty I(s - y)I(t - y)dy.$$

Put $\phi(x, y) = I(f(x) - y)$ and $A(y) = y$ for $y > 0$, and $A(y) = 0$ for $y \leq 0$. It follows from (2.1) that the density function of the Polya characteristic function is

$$(1/2\pi) \int_0^\infty \left| \int_{-\infty}^\infty e^{iux} I(f(x) - y) dx \right|^2 dy.$$

The random set function W in (2.2) is the 2-dimensional Brownian motion with independent increments, and the stochastic integral becomes

$$Y(t) = \iint_{\{y>0\}} I(f(x + t) - y) W(dx \times dy).$$

Such a representation for the Polya covariance process was recently given by Cabana and Wschebor [1].

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