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A NEW CHARACTERIZATION OF CHARACTERISTIC FUNCTIONS OF ABSOLUTELY CONTINUOUS DISTRIBUTIONS

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It is well known that if g belongs to L_2, then

$$\frac{\int g(x)\bar{g}(x+y)dx}{\int |g(x)|^2 dx}$$

is the characteristic function of an absolutely continuous distribution function. Conversely, every such characteristic function has the representation given above. Here we shown that if $R(s,t)$ is a covariance function such that $R(s,s)$ belongs to L_1, then

$$\frac{\int R(s,s+t)ds}{\int R(s,s)ds}$$

is the characteristic function of an absolutely continuous distribution. Conversely, every such characteristic function has the latter representation (put $R(s,t) = g(s)\bar{g}(t)$). The use of this new result is that certain functions are directly seen to be of the second form but not the first; hence, they can be identified as characteristic functions of absolutely continuous distributions.

1. The main theorem. Let $R(s,t)$, $-\infty < s,t < \infty$, be a complex-valued Borel function of two variables. It is a *covariance function* if for any positive integer n, and any set of pairs (s_i, u_i), $i = 1, \cdots, n$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} R(s_i, s_j)u_i u_j \geqq 0.$$

By Kolmogorov's existence theorem and the well known moment properties of Gaussian processes, for every covariance function there exists a probability space and a complex Gaussian process $X(t)$, $-\infty < t < \infty$, on the space such that

$$EX(t) = 0 \text{ for all } t, \qquad EX(s)\bar{X}(t) = R(s,t) \text{ for all } s,t.$$

We say that X is *associated* with R.

The function $R(s,s)$ is nonnegative because it is equal to $E|X(s)|^2$. If

$$\int_{-\infty}^{\infty} R(s,s)ds < \infty, \tag{1.1}$$

then there exists an associated process X which is measurable and satisfies

$$E\int_{-\infty}^{\infty} |X(s)|^2 ds = \int_{-\infty}^{\infty} E|X(s)|^2 ds < \infty. \tag{1.2}$$

It also follows that $X(t)$ belongs to L_2 almost surely, and so there is a measurable version of the Fourier transform process

$$\hat{X}(u) = \int_{-\infty}^{\infty} e^{ius} X(s)ds, \quad -\infty < u < \infty. \tag{1.3}$$

By Parseval's Theorem we also have

$$\int_{-\infty}^{\infty} E|X(s)|^2 ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} E|\hat{X}(u)|^2 du. \tag{1.4}$$

THEOREM 1. *Let R be a covariance function satisfying* (1.1). *Then the function*

$$r(t) = \frac{\int_{-\infty}^{\infty} R(s,s+t)ds}{\int_{-\infty}^{\infty} R(s,s)ds} \tag{1.5}$$

is a characteristic function. The corresponding distribution function is absolutely continuous with the derivative

$$g(u) = \frac{E|\hat{X}(-u)|^2}{\int_{-\infty}^{\infty} E|\hat{X}(y)|^2 dy}, \tag{1.6}$$

where X is the associated process satisfying (1.2). *Conversely if $r(t)$ is the characteristic function of an absolutely continuous distribution, then there exists R satisfying* (1.1) *such that r is representable as* (1.5).

Proof. First we prove the direct assertion. It follows from the definition of X that

$$\int_{-\infty}^{\infty} R(s, s+t)ds = \int_{-\infty}^{\infty} EX(s)\bar{X}(s+t)ds.$$

By virtue of condition (1.1) and the Cauchy-Schwarz inequality, we can take the expectation outside of the integral, and then apply the Parseval theorem:

$$E\int_{-\infty}^{\infty} X(s)\bar{X}(s+t)ds = E\left\{(1/2\pi)\int_{-\infty}^{\infty} e^{-iut}|\hat{X}(u)|^2 du\right\}.$$

It follows that the ratio in (1.5) is equal to

$$\frac{\int_{-\infty}^{\infty} e^{iut} E|\hat{X}(-u)|^2 du}{\int_{-\infty}^{\infty} E|\hat{X}(y)|^2 dy}$$

This is exactly the Fourier transform of the function $g(u)$ in (1.6).

The converse is simple: it is given in the above abstract.

2. Factorable covariances. R is said to be factorable if there exists a monotone function $A(y)$ and a "kernel" function $\phi(t, y)$ such that

$$R(s,t) = \int_{-\infty}^{\infty} \phi(s,y)\bar{\phi}(t,y)dA(y).$$

The condition (1.1) becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(s,y)|^2 dA(y)ds < \infty.$$

It follows (by Fubini's theorem) that $\phi(\cdot, y)$ belongs to L_2 for almost all y (with repect to dA); thus

$$\hat{\phi}(u,y) = \int_{-\infty}^{\infty} e^{iuz}\phi(z,y)dz$$

exists for all such y. By virtue of the isometry $X(t) \to \phi(t, \cdot)$ the density (1.6) takes the form

$$g(u) = \frac{\int_{-\infty}^{\infty} |\hat{\phi}(-u,y)|^2 dA(y)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{\phi}(u,y)|^2 du dA(y)}. \tag{2.1}$$

As a characteristic function $r(t)$ is also the covariance function of a stationary Gaussian process. When the spectral distribution is absolutely continuous, the process has a well known representation as a moving average of "white noise" on the line (see [2], p. 533). We will show that when R is factorable the stationary process with covariance of the form (1.5) also has a representation as a moving average of "noise" in the plane. The latter representation is more informative and easiler to derive in certain special cases (see §4 below).

Let W be a real Gaussian random set function defined over the plane Borel sets, that is, $W(C)$ has a normal distribution for every plane Borel set C. Let W have the following moment structure:

$EW(C) = 0$ for all C

$EW(C)W(C') = 0$ if C and C' are disjoint (independent increments)

$EW^2(C) = \int_B dx \cdot \int_{B'} dA(y)$ if $C = B \times B'$ is a product of two linear sets.

Consider the stochastic integral with respect to W, divided by a positive constant:

$$Y(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x+t, y)W(dx \times dy)}{\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(x, y)|^2 dxdA(y)\right\}^{\frac{1}{2}}}. \tag{2.2}$$

By a direct calculation and by means of the fundamental properties of the stochastic integral we find that the process $Y(t)$ is stationary (and Gaussian) with covariance function (1.5).

3. Change of time parameter in the covariance. Let $R(s,t)$ be a covariance, and $f(x)$ a real Borel function. Then the composite function $R(f(x),f(y))$ is also a covariance function (in x and y). According to Theorem 1, if

$$\int_{-\infty}^{\infty} R(f(x),f(x))dx < \infty, \tag{3.1}$$

then

$$r(t) = \frac{\int_{-\infty}^{\infty} R(f(x),f(x+t))dx}{\int_{-\infty}^{\infty} R(f(x),f(x))dx} \tag{3.2}$$

is the characteristic function of an absolutely continuous distribution. By means of this result we can identify some general and interesting functions as such characteristic functions.

EXAMPLE. Let $\sigma^2(t)$, $-\infty < t < \infty$, be the incremental second moment function of a Gaussian process with mean 0 and stationary increments ($\sigma^2(t) = E\,|U(s+t) - U(s)|^2$ where U has stationary increments). Then the covariance function of the process is

$$R(s,t) = \tfrac{1}{2}[\sigma^2(s) + \sigma^2(t) - \sigma^2(t-s)].$$

Let $f(x)$ be a Borel function such that

$$\int_{-\infty}^{\infty} \sigma^2(f(x))dx < \infty;$$

then (3.1) is fulfilled, and so

$$r(t) = 1 - \frac{\int_{-\infty}^{\infty} \sigma^2(f(x+t) - f(x))dx}{2\int_{-\infty}^{\infty} \sigma^2(f(x))dx} \tag{3.3}$$

is the characteristic function of an absolutely continuous distribution.

Let $f(x)$ be a Borel function such that

$$\int_{-\infty}^{\infty} |f(x)|^\alpha dx = 1$$

for some α, $0 < \alpha \leqq 2$; then

$$r(t) = 1 - \tfrac{1}{2}\int_{-\infty}^{\infty} |f(x+t) - f(x)|^\alpha dx \tag{3.4}$$

is a characteristic function with an absolutely continuous distribution; indeed, it is a special case of (3.3) with $\sigma^2(t) = t^\alpha$. (The fact that (3.4) is a characteristic function was first proved by Lawrence Shepp in a private communication.)

This can be used to prove a general result about the space L_α. According to the classical representation of the characteristic function of an absolutely continuous distribution as a convolution, there exists a function $\tilde{f}$ such that

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^2 dx = 1$$

and $r(t)$ is representable as

$$1-\tfrac{1}{2}\int_{-\infty}^{\infty}|\tilde{f}(x+t)-\tilde{f}(x)|^2\,dx.$$

From (3.4) we then conclude that

$$\int_{-\infty}^{\infty}|f(x+t)-f(x)|^\alpha\,dx=\int_{-\infty}^{\infty}|\tilde{f}(x+t)-\tilde{f}(x)|^2\,dx, \text{ for all } t.$$

As far as I can determine the existence of such an $\tilde{f}$ in L_2 for each f in L_α is a result unknown up to now.

4. A new proof of Polya's theorem and related results. Polya described a class of characteristic functions, now called "Polya characteristic functions". [4] He showed that if $r(t)$ is a convex function such that $r(t)\geqq 0$, $r(t)=r(-t)$, $r(0)=1$, and $\lim_{t\to\infty}r(t)=0$, then $r(t)$ is the characteristic function of an absolutely continuous distribution. We will show that such a function is representable as in Theorem 1, and so provide a new proof of Polya's theorem; furthermore, we will get an explicit form of the density from the results of §2, and a stochastic integral representation.

As a convex function, $r(t)$ has an integral representation

$$r(t)=\int_{|t|}^{\infty}f(x)dx, \tag{4.1}$$

where $f(x)$, $x\geqq 0$, is nonnegative and nonincreasing, and

$$\int_0^{\infty}f(x)dx=1.$$

Indeed, take f as the negative of the right hand derivative of r (see [3], p. 5). Extend f to all x by assigning it the value 0 on the negative axis. Then $r(t)$ is representable as

$$r(t)=\int_{-\infty}^{\infty}\min(f(x),f(x+t))dx.$$

This is of the form (1.5) with $R(s,t)=\min(s,t)$. Thus r is a characteristic function of the indicated type.

Let $I(t)$ be the indicator function of the positive t-axis. The covariance $\min(s,t)$ is factorable for positive s and t:

$$\min(s,t) = \int_0^\infty I(s-y)I(t-y)dy.$$

Put $\phi(x,y) = I(f(x)-y)$ and $A(y) = y$ for $y > 0$, and $A(y) = 0$ for $y \leqq 0$. It follows from (2.1) that the density function of the Polya characteristic function is

$$(1/2\pi)\int_0^\infty \left| \int_{-\infty}^\infty e^{iux} I(f(x)-y)dx \right|^2 dy.$$

The random set function W in (2.2) is the 2-dimensional Brownian motion with independent increments, and the stochastic integral becomes

$$Y(t) = \iint_{\{y>0\}} I(f(x+t)-y)W(dx \times dy).$$

Such a representation for the Polya covariance process was recently given by Cabana and Wschebor [1].

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