BIRNBAUM-ORLICZ SPACES OF FUNCTIONS ON GROUPS

IRACEMA M. BUND
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It is natural to ask how far the theory of closed invariant subspaces for $L_2(G)$ can be extended to Birnbaum-Orlicz spaces $L_A(G)$. If $G$ is a compact group and $A$ satisfies the $\Delta_2$-condition for $u \geq u_0 \geq 0$, the class of all closed invariant subspaces of $L_A(G)$ is exactly the family $\{(L_A)_P: P \subset \Sigma\}$ where $\Sigma$ is the dual object of $G$. Distinct subsets of $\Sigma$ engender distinct subspaces.

The generalization of the classical $L_p$-spaces foreshadowed by Z. W. Birnbaum in 1930 [1] was the subject of a long article by Z. W. Birnbaum and W. Orlicz [2]. In the next four decades their theory has been extended by many writers, among them G. Weiss [9] and W. Luxemburg who invented convenient new definitions. More recently M. Jodeit and A. Torchinsky [7] introduced a generalization of the concept of Young's function which we adopt here.

The essential introductory definitions and theorems are stated in §1; proofs may be found in [3], [8] and [9]. In §2 we show that if $G$ is a locally compact group, the Birnbaum-Orlicz space $L_A(G)$ is a left Banach $L_1$-module and a right Banach $(L_1 \cap L_\uparrow)$-module. Finally in §3 we establish the result stated in the synopsis. Our notation is as in [4], [5] and [6].

1. Preliminaries. (1.1) A function $A$ on $[0, \infty]$ will be called a generalized Young's function if it is left continuous on $[0, \infty]$, $A(u)/u$ is nondecreasing for $u > 0$, and $A(0) = 0$. It easily follows that

\[(i)\quad A(\alpha u) \leq \alpha A(u) \quad \text{for} \quad 0 \leq \alpha \leq 1 \quad \text{and} \quad 0 \leq u < \infty.\]

The zero function and the function $A(u) = \infty \cdot \xi_{[0,\infty]}(u)$ are trivial generalized Young’s functions. Throughout the remaining of this work the letter $A$ will denote a nontrivial generalized Young’s function. We also fix $a = \sup\{u: A(u) = 0\}$.

A Young’s function $A_0$ is associated to $A$ by the equality $A_0(u) = \int_0^u A(t)/t \, dt$.

(1.2) Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space. The set $L_A(X, \mathcal{M}, \mu)$ of all complex-valued, $\mathcal{M}$-measurable functions defined $\mu$-a.e. on $X$, such that $\int_X A(\alpha |f|) \, d\mu < \infty$ for some positive number $\alpha$ is
called a Birnbaum-Orlicz space. Where no confusion seems possible, we will write $L_A(X)$ for $L_A(X,\mathcal{M},\mu)$.

The equality

(i) $p_A(f) = \inf\{k \in [0,\infty]: \int_X A(|f|/k)\,d\mu \leq 1\}$

defines a nonnegative finite-valued function on $L_A(X)$ which is a norm in case $A$ is convex. This suggests that we define a norm on $L_A(X)$ by the equality $\|f\|_A = p_A(f)$. With this norm, $L_A(X)$ is a Banach space.

If $f \in L_A(G)$ the following hold:

(ii) $\|f\|_A \leq p_A(f) \leq 2\|f\|_A$;

(iii) $\int_X A(|f|/p_A(f))\,d\mu \leq 1$, provided that $p_A(f) > 0$.

Denoting the Young's complement of $A$ by $\tilde{A}$, for $f$ in $L_A(X)$ and $g$ in $L_A(X)$ we obtain

(iv) $\int_X |fg|\,d\mu \leq 2p_A(f)p_A(g)$.

If $\mu(X)$ is finite, $L_A(X)$ is contained in $L_1(X)$ and for $f \in L_A(X)$ we have

(v) $\|f\|_1 \leq [4/(\tilde{A})^{-1}(1/\mu(X))]\|f\|_A$,

where $(\tilde{A})^{-1}$ denotes the right inverse of $\tilde{A}$.

(1.3) **Theorem.** Let $f$ be a complex-valued measurable function vanishing outside of a $\sigma$-finite set. Suppose that

$$N_A(f) = \sup \left\{ \int_X |fg|\,d\mu : g \in L_A(X), p_A(g) \leq 1 \right\} < \infty.$$ Then $f \in L_A(X)$ and we have $\|f\|_A \leq N_A(f)$.

(1.4) **Theorem.** Let $X$ be a locally compact Hausdorff space. Let $\mu$ be a measure obtained from a nonnegative linear functional on $C_0(X)$, and let $\mathcal{M}$ be the $\sigma$-algebra of all $\mu$-measurable subsets of $X$. Then each function $f$ in $L_A(X)$ can be written as $f = f_1 + f_2$, where $f_1 = f \chi_F$ for some $\sigma$-compact set $F$, and $|f_2| \leq a p_A(f)$ $\mu$-a.e. on $X$. In particular, if $a = 0$, then $f$ vanishes $\mu$-a.e. outside of a $\sigma$-compact set.

2. **Birnbaum-Orlicz spaces of functions on groups.** From here on we consider spaces $L_A(G,\mathcal{M},\lambda)$, where $G$ is a locally compact group, $\lambda$ is a left Haar measure on $G$, and $\mathcal{M}$ is the $\sigma$-algebra of $\lambda$-measurable subsets of $G$. We will often write $\int f\,d\lambda$ as $\int_G f(x)\,dx$.

Our first theorem follows easily from (20.2) in [4], and the fact that $L_1(G,\mathcal{M},\max\{1,1/\Delta\}\lambda)$ is complete.
(2.1) Theorem. A complex-valued measurable function \( f \) belongs to \( \mathcal{L}_1(G) \cap \mathcal{L}_1^*(G) \) if and only if \( \max\{1,1/\Delta\}f \in \mathcal{L}_1(G) \). The equalities

(i) \( \|f\| = \|f\| + \|(1/\Delta)f\| \),

and

(ii) \( \|f\| = \max\{1,1/\Delta\}f \)

define equivalent norms on the linear space \( \mathcal{L}_1(G) \cap \mathcal{L}_1^*(G) \). Precisely, we have

(iii) \( \|f\| \leq \|f\| \leq 2\|f\| \) for all \( f \in \mathcal{L}_1(G) \cap \mathcal{L}_1^*(G) \).

With either of these two norms, \( \mathcal{L}_1(G) \cap \mathcal{L}_1^*(G) \) is a Banach space.

(2.2) Theorem. Let \( f \) be a function in \( \mathcal{L}_A(G) \) and let \( s \) be an arbitrary element of \( G \). Then the functions \( f_s \) and \( f_s \) belong to \( \mathcal{L}_A(G) \) and we have:

(i) \( p_A(f) = p_A(f_s) \);

(ii) \( p_A(f_s) \leq \max\{1,\Delta(s^{-1})\}p_A(f) \).

Proof. It is clear that \( f_s \) and \( f_s \) are \( \lambda \)-measurable. Relations (i) and (ii) trivially become equalities if \( p_A(f) = 0 \). Suppose that \( p_A(f) > 0 \).

Theorem (20.1.i) in [4], and (1.2.iii) yield the inequality \( p_A(f_s) \leq p_A(f) \), from which (i) easily follows. Using (20.1.ii) in [4], and once again (1.2.iii) we write

\[
\int_G A(\|f_s\|/p_A(f_s)) d\lambda \leq \Delta(s^{-1}),
\]

which establishes (ii) in case \( \Delta(s^{-1}) \leq 1 \). For \( \Delta(s^{-1}) > 1 \), use (1) and (1.1.i).

The following result is part of (20.7) in the Russian edition of Hewitt and Ross “Abstract Harmonic Analysis”, to be published.

(2.3) Lemma. Let \( f \) be a \( \lambda \)-measurable function on \( G \). The following functions are \( \lambda \times \lambda \)-measurable on \( G \times G \):

\[
(x,y) \mapsto f(xy^{-1}), \quad (x,y) \mapsto f(y^{-1}x), \quad (x,y) \mapsto f(x),
\]

\[
(x,y) \mapsto f(x^{-1}), \quad (x,y) \mapsto f(y), \quad (x,y) \mapsto f(y^{-1}).
\]

(2.4) Theorem. Let \( f \) be a function in \( \mathcal{L}_A(G) \) vanishing outside of a \( \sigma \)-compact set \( F \) and let \( g \) be a function in \( \mathcal{L}_1(G) \). The integral

(i) \( g * f(x) = \int_G f(y^{-1}x)g(y)dy \)

exists and is finite for almost all \( x \) in \( G \). The function \( g * f \) is in \( \mathcal{L}_A(G) \) and we have
(ii) \( \| g * f \|_A \leq 4 \| f \|_A \| g \| \).

If \( g \in \mathcal{L}_1(G) \cap \mathcal{C}_1(G) \), the integral

\[
(iii) \quad f * g(x) = \int_G \Delta(y^{-1}) f(xy^{-1}) g(y) \, dy
\]

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \). The function \( f * g \) is in \( \mathcal{L}_A(G) \) and we have

\[
(iv) \quad \| f * g \|_A \leq 4 \| f \|_A \| g \|,
\]

where \( \| \cdot \| \) is as in (2.1.i).

**Proof.** We may suppose that \( g \) vanishes outside of a \( \sigma \)-compact set \( E \). Thus the function \( (x, y) \rightarrow f(y^{-1}x) g(y) \) vanishes outside of the \( \sigma \)-compact set \( (EF) \times E \).

Let \( v \) be an arbitrary function in \( \mathcal{L}_A(G) \). From (2.3) we know that the mapping \( (x, y) \rightarrow v(x)f(y^{-1}x) g(y) \) is \( \lambda \times \lambda \)-measurable. Plainly this function vanishes outside of \( (EF) \times E \).

Recalling (1.2.iv) and (2.2.i), we obtain

\[
\int_G \int_G |v(x)f(y^{-1}x) g(y)| \, dx \, dy
\]

(1)

\[
\leq 2p_A(f) p_A(v) \| g \|.
\]

Thus we may apply (13.10) of [4] to conclude that

\[
\int_G \int_G |v(x)f(y^{-1}x) g(y)| \, dy \, dx
\]

(2)

\[
= \int_G \int_G |v(x)f(y^{-1}x) g(y)| \, dx \, dy.
\]

From (13.10) and (13.8) in [4], we see that the integral

\[
\int_G v(x)f(y^{-1}x) g(y) \, dy
\]

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \), and that

\[
x \rightarrow v(x) \int_G f(y^{-1}x) g(y) \, dy.
\]

(3)

is a function in \( \mathcal{L}_1(G) \); in particular it is a \( \lambda \)-measurable function.

We define \( g * f(x) \) by the equality (i), provided the integral exists, and put \( g * f(x) = 0 \), otherwise. It is easy to see that \( g * f(x) \) is finite \( \lambda \)-a.e. on \( G \).

In (3) we may take \( v \) to be any function in \( \mathcal{C}_\infty(G) \). Recalling (11.42) in [4], we see that \( g * f \) is \( \lambda \)-measurable.
Consider $v$ in $\mathcal{L}_A(G)$ with $p_A(v) \leq 1$. Taking account of (1) and (2), we obtain

$$\int_G |v(x)(g*f)(x)| \, dx \leq \int_G \int_G |v(x)f(y^{-1}x)g(y)| \, dy \, dx$$

$$= \int_G \int_G [v(x)f(y^{-1}x)g(y)] \, dx \, dy \leq 2p_A(f)\|g\|.$$ 

This implies that

(4) 

$$N_A(g*f) \leq 2p_A(f)\|g\|.$$ 

Now we observe that $g*f(x) = 0$ for $x$ outside of the $\sigma$-compact set $EF$. Thus from (4) and (1.3), we conclude that $g*f \in \mathcal{L}_A(G)$ and that $\|g*f\|_A \leq 2p_A(f)\|g\|$. Applying (1.2.ii) to this last inequality, we obtain (ii).

Next suppose that $g \in \mathcal{L}_1(G) \cap \mathcal{L}^\dagger_1(G)$. Consider the function

(5) 

$$(x,y) \to v(x)f(xy^{-1})g(y)\Delta(y^{-1}),$$

where $v$ is an arbitrary function in $\mathcal{L}_A(G)$. As in the previous case, we see that the function (5) is $\lambda \times \lambda$-measurable and vanishes outside of the $\sigma$-compact set $(FE) \times E$. From (1.2.iv) and (2.2.H) we obtain

$$\int_G |v(x)f(xy^{-1})| \, dx \leq 2\max\{1,\Delta(y)\}p_A(f)p_A(v).$$

Thus we have

$$\int_G \int_G |v(x)f(xy^{-1})g(y)\Delta(y^{-1})| \, dx \, dy$$

$$\leq 2p_A(f)p_A(v)\int_G \max\{1,\Delta(y^{-1})\}|g(y)| \, dy$$

$$= 2p_A(f)p_A(v)\max\{1,1/\Delta\}g\|,$$

$$\leq 2p_A(f)p_A(v)\|g\|,$$

the last inequality being a consequence of (2.1.ii) and (2.1.iii).

From this point on the proof is completely analogous to that presented above for $g*f$ and we omit it.
Theorem (2.4) serves as a lemma for the following general result.

(2.5) **Theorem.** Suppose that \( f \in \mathcal{L}_A(G) \) and \( g \in \mathcal{L}_I(G) \). Then the integral

(i) \( g * f(x) = \int_G f(y^{-1}x)g(y)dy \)

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \). The function \( g * f \) is in \( \mathcal{L}_A(G) \) and we have

(ii) \( \|g * f\|_A \leq k \|f\|_A \|g\| \),

where \( k = 4 \) if \( a = 0 \) or if \( G \) is \( \sigma \)-compact, and \( k = 6 \) otherwise.

If \( g \in \mathcal{L}_I(G) \cap \mathcal{L}_I^*(G) \), the integral

(iii) \( f * g(x) = \int_G \Delta(y^{-1})g(y)dy \)

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \). The function \( f * g \) is in \( \mathcal{L}_A(G) \) and we have

(iv) \( \|f * g\|_A \leq k \|f\|_A \|g\| \),

where \( k \) is as above and \( \| \cdot \| \) is as in (2.1.i).

**Proof.** If \( G \) is \( \sigma \)-compact, the assertion follows immediately from (2.4). If \( a = 0 \), it follows from (1.4) and (2.4). Thus we may suppose that \( a > 0 \) and that \( G \) fails to be \( \sigma \)-compact.

Using (1.4), we may write \( f = f_1 + f_2 \), where \( f_1 = f\chi_F \) for some \( \sigma \)-compact set \( F \), and \( |f_2| \leq ap_A(f) \). It follows that

(1) \( \int_G |f_2(y^{-1}x)g(y)|dy \leq ap_A(f)\|g\| \)

for all \( x \) in \( G \), and hence that \( g * f_2(x) \) exists and is finite for all \( x \) in \( G \). A short computation, in which we use (1), gives us

\[
  g * f_2(x) \|g * f_2\|_A \leq p_A(g * f_2) \leq a^{-1}\|g * f_2\|_A \leq 2\|f\|_A \|g\|.
\]

Applying (2.4.i) to \( f_1 \), we conclude that

\[
  \int_G f_1(y^{-1}x)g(y)dy + \int_G f_2(y^{-1}x)g(y)dy
\]

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \). Hence the same is true of \( g * f(x) \).

Inequality (ii) follows from (2) and (2.4.ii) applied to \( f_1 \). The remaining assertions are similarly established.

(2.6) **Theorem.** The space \( \mathcal{L}_I(G) \cap \mathcal{L}_I^*(G) \) is a Banach algebra.
Proof. For $f$ and $g$ in $£_1(G) \cap £_1^*(G)$ we obtain

$$((1/\Delta)g) * ((1/\Delta)f) = (1/\Delta)(g * f).$$

Thus (2.1) and (2.5.i) tell us that $g * f \in £_1(G) \cap £_1^*(G)$. We use (1) to prove that $£_1(G) \cap £_1^*(G)$, with the norm $\| \cdot \|$ defined in (2.1.i), is a normed algebra:

$$\| g * f \| \leq \| g \| \| f \| + \| (1/\Delta)g \| \| (1/\Delta)f \|, \leq \| g \| \| f \|.$$

(2.7) Theorem. The space $£_\lambda(A(G))$ is a left Banach $£_1$-module and a right Banach $£_1 \cap £_1^*$-module.

Proof. For $g$ in $£_1(G)$ and $f$ in $£_\lambda(A(G))$, (2.5.ii) tells us that there is a positive number $k$ such that $\| g * f \|_\lambda \leq k \| f \|_\lambda \| g \|_\lambda$.

Next we show that, for $f$ as above, and $g_1$ and $g_2$ in $£_1(G)$, we have $g_1 * (g_2 * f) = (g_1 * g_2) * f$. Using (20.1) of [4], we obtain the equality

$$\int_G f(v^{-1}y^{-1}x) g_2(v) dv = \int_G f(v^{-1}x) g_2(y^{-1}v) dv,$$

which implies that

$$g_1 * (g_2 * f)(x) = \int_G \int_G f(v^{-1}x) g_2(y^{-1}v) g_1(y) dy dv.$$

By (2.5.i), $g_1 * (g_2 * f)$ is in $£_\lambda(A(G))$, and hence the integral in (1) exists and is finite $\lambda$-almost everywhere in $G$. From (1.4) we know that $g_1$ and $g_2$ vanish outside of $\sigma$-compact sets $E_1$ and $E_2$, respectively. Thus the function $(v,y) \to f(v^{-1}x) g_2(y^{-1}v) g_1(y)$ vanishes outside of the $\sigma$-compact set $(E_1 E_2) \times E_1$. By (2.3) this function is $\lambda \times \lambda$-measurable.

We apply (13.10) in [4] to conclude that for $\lambda$-almost all $x$ in $G$ we have

$$g_1 * (g_2 * f)(x) = \int_G \int_G f(v^{-1}x) g_2(y^{-1}v) g_1(y) dy dv$$

$$= \int_G f(v^{-1}x) (g_1 * g_2)(v) dv = (g_1 * g_2) * f(x).$$

It is now clear that $£_\lambda(A(G))$ is a left Banach $£_1$-module. The proof that $£_\lambda(A(G))$ is a right Banach $£_1 \cap £_1^*$-module is similar and we omit it.

3. Closed ideals in $£_\lambda(A(G))$ for $G$ a compact group. Throughout this section we suppose that $G$ is compact and that $\lambda(G) = 1$. 
(3.1) **Theorem.** If \( f \) and \( g \) are in \( \mathcal{L}_A(G) \) the equality \( g * f(x) = \int_G f(y^{-1}x)g(y)dy \) defines a function in \( \mathcal{L}_A(G) \). We have

\[
\|g * f\|_A \leq (16/(\bar{A})^{-1}(1))\|f\|_A \|g\|_A.
\]

**Proof.** Follows from (2.5.i), (1.2.v) and (2.5.ii).

(3.2) **Theorem.** The Birnbaum-Orlicz space \( \mathcal{L}_A(G) \) is a Banach algebra under a norm which is a positive constant times \( \| \cdot \|_A \).

**Proof.** Define \( n_A(f) = (16/(\bar{A})^{-1}(1))\|f\|_A \) and use (3.1).

(3.3) **Theorem.** Suppose that \( A \) satisfies the \( \Delta_2 \)-condition for \( u \equiv u_0 \geq 0 \). Then the space \( \mathcal{E}(G) \) of trigonometric polynomials on \( G \) is \( \| \cdot \|_A \)-dense in \( L_A(G) \).

**Proof.** Our hypothesis imply that \( \mathcal{E}(G) \) is \( \| \cdot \|_A \)-dense in \( \mathcal{L}_A(G) \): see [3] or [8]. Theorem (27.39.ii) of [5] tells us that \( \mathcal{E}(G) \) is uniformly dense in \( \mathcal{E}(G) \), and it is easy to see that \( \mathcal{E}(G) \) is also \( \| \cdot \|_A \)-dense in \( \mathcal{E}(G) \).

(3.4) **Theorem.** Let \( A \) be as in (3.3). Suppose that \( S \) is a closed linear subspace of \( \mathcal{L}_A(G) \). Then \( S \) is a left [right] ideal in \( \mathcal{L}_A(g) \) if and only if \( S \) is closed under the formation of left [right] translates.

**Proof.** Since \( G \) is unimodular, it follows from (2.1) and (2.7) that \( \mathcal{L}_A(G) \) is a Banach \( \mathcal{L}_1 \)-module with respect to convolution. From (3.2) we know that \( \mathcal{L}_A(G) \) is a subalgebra of \( \mathcal{L}_1(G) \) which is a Banach algebra with the norm \( n_A \). Taking (3.3) into account, we see that \( L_A(G) \) has the properties stated in (38.6.a) in [5]. Thus the theorem follows immediately from (38.22.b) of [5].

(3.5) **Theorem.** Let \( A \) be as in (3.3). Then the class of all closed two-sided ideals in \( \mathcal{L}_A(G) \) is exactly the family \( \{ \mathcal{L}_P : P \subset \Sigma \} \). Distinct subsets of \( \Sigma \) engender distinct ideals.

**Proofs.** This is a direct application of (38.7) in [5].

**References**


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