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BIRNBAUM-ORLICZ SPACES OF FUNCTIONS ON GROUPS

IRACEMA M. BUND

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It is natural to ask how far the theory of closed invariant subspaces for $\mathfrak{L}_p(G)$ can be extended to Birnbaum-Orlicz spaces $\mathfrak{L}_A(G)$. If G is a compact group and A satisfies the Δ_2 -condition for $u \cong u_0 \cong 0$, the class of all closed invariant subspaces of $\mathfrak{L}_A(G)$ is exactly the family $\{(\mathfrak{L}_A)_P : P \subset \Sigma\}$ where Σ is the dual object of G . Distinct subsets of Σ engender distinct subspaces.

The generalization of the classical \mathfrak{L}_p -spaces foreshadowed by Z. W. Birnbaum in 1930 [1] was the subject of a long article by Z. W. Birnbaum and W. Orlicz [2]. In the next four decades their theory has been extended by many writers, among them G. Weiss [9] and W. Luxemburg who invented convenient new definitions. More recently M. Jodeit and A. Torchinsky [7] introduced a generalization of the concept of Young's function which we adopt here.

The essential introductory definitions and theorems are stated in §1; proofs may be found in [3], [8] and [9]. In §2 we show that if G is a locally compact group, the Birnbaum-Orlicz space $\mathfrak{L}_A(G)$ is a left Banach \mathfrak{L}_1 -module and a right Banach $(\mathfrak{L}_1 \cap \mathfrak{L}_1^*)$ -module. Finally in §3 we establish the result stated in the synopsis. Our notation is as in [4], [5] and [6].

1. Preliminaries. (1.1) A function A on $[0, \infty[$ into $[0, \infty]$ will be called a generalized Young's function if it is left continuous on $]0, \infty[$, $A(u)/u$ is nondecreasing for $u > 0$, and $A(0) = 0$. It easily follows that

$$(i) \quad A(\alpha u) \leq \alpha A(u) \quad \text{for} \quad 0 \leq \alpha \leq 1 \quad \text{and} \quad 0 \leq u < \infty.$$

The zero function and the function $A(u) = \infty \cdot \chi_{]0, \infty[}(u)$ are trivial generalized Young's functions. Throughout the remaining of this work the letter A will denote a nontrivial generalized Young's function. We also fix $a = \sup\{u : A(u) = 0\}$.

A Young's function A_0 is associated to A by the equality $A_0(u) = \int_0^u A(t)/t \, dt$.

(1.2) Let (X, \mathcal{M}, μ) be an arbitrary measure space. The set $\mathfrak{L}_A(X, \mathcal{M}, \mu)$ of all complex-valued, \mathcal{M} -measurable functions defined μ -a.e. on X , such that $\int_X A(\alpha |f|) \, d\mu < \infty$ for some positive number α is

called a Birnbaum-Orlicz space. Where no confusion seems possible, we will write $\mathfrak{L}_A(X)$ for $\mathfrak{L}_A(X, \mathcal{M}, \mu)$.

The equality

$$(i) \quad p_A(f) = \inf\{k \in]0, \infty[: \int_X A(|f|/k) d\mu \leq 1\}$$

defines a nonnegative finite-valued function on $\mathfrak{L}_A(X)$ which is a norm in case A is convex. This suggests that we define a norm on $\mathfrak{L}_A(X)$ by the equality $\|f\|_A = p_{A_0}(f)$. With this norm, $\mathfrak{L}_A(X)$ is a Banach space.

If $f \in \mathfrak{L}_A(G)$ the following hold:

$$(ii) \quad \|f\|_A \leq p_A(f) \leq 2\|f\|_A ;$$

$$(iii) \quad \int_X A(|f|/p_A(f)) d\mu \leq 1, \text{ provided that } p_A(f) > 0.$$

Denoting the Young's complement of A by \bar{A} , for f in $\mathfrak{L}_A(X)$ and g in $\mathfrak{L}_{\bar{A}}(X)$ we obtain

$$(iv) \quad \int_X |fg| d\mu \leq 2p_A(f)p_{\bar{A}}(g).$$

If $\mu(X)$ is finite, $\mathfrak{L}_A(X)$ is contained in $\mathfrak{L}_1(X)$ and for $f \in \mathfrak{L}_A(X)$ we have

$$(v) \quad \|f\|_1 \leq [4/(\bar{A})^{-1}(1/\mu(X))] \|f\|_A,$$

where $(\bar{A})^{-1}$ denotes the right inverse of \bar{A} .

(1.3) THEOREM. *Let f be a complex-valued measurable function vanishing outside of a σ -finite set. Suppose that*

$$N_A(f) = \sup \left\{ \int_X |fg| d\mu : g \in \mathfrak{L}_{\bar{A}}(X), p_{\bar{A}}(g) \leq 1 \right\} < \infty.$$

Then $f \in \mathfrak{L}_A(X)$ and we have $\|f\|_A \leq N_A(f)$.

(1.4) THEOREM. *Let X be a locally compact Hausdorff space. Let μ be a measure obtained from a nonnegative linear functional on $\mathfrak{C}_0(X)$, and let \mathcal{M} be the σ -algebra of all μ -measurable subsets of X . Then each function f in $\mathfrak{L}_A(X)$ can be written as $f_1 + f_2$, where $f_1 = f \xi_F$ for some σ -compact set F , and $|f_2| \leq ap_A(f)$ μ -a.e. on X . In particular, if $a = 0$, then f vanishes μ -a.e. outside of a σ -compact set.*

2. Birnbaum-Orlicz spaces of functions on groups. From here on we consider spaces $\mathfrak{L}_A(G, \mathcal{M}, \lambda)$, where G is a locally compact group, λ is a left Haar measure on G , and \mathcal{M} is the σ -algebra of λ -measurable subsets of G . We will often write $\int_G f d\lambda$ as

$$\int_G f(x) dx.$$

Our first theorem follows easily from (20.2) in [4], and the fact that $\mathfrak{L}_1(G, \mathcal{M}, \max\{1, 1/\Delta\}\lambda)$ is complete.

(2.1) THEOREM. *A complex-valued measurable function f belongs to $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ if and only if $\max\{1, 1/\Delta\}f \in \mathfrak{L}_1(G)$. The equalities*

(i) $\|f\| = \|f\|_1 + \|(1/\Delta)f\|_1,$

and

(ii) $\| \|f\| \| = \|\max\{1, 1/\Delta\}f\|_1$

define equivalent norms on the linear space $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$. Precisely, we have

(iii) $\| \|f\| \| \leq \|f\| \leq 2 \| \|f\| \|$ for all $f \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$.

With either of these two norms, $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ is a Banach space.

(2.2) THEOREM. *Let f be a function in $\mathfrak{L}_A(G)$ and let s be an arbitrary element of G . Then the functions ${}_s f$ and f_s belong to $\mathfrak{L}_A(G)$ and we have:*

(i) $p_A({}_s f) = p_A(f);$

(ii) $p_A(f_s) \leq \max\{1, \Delta(s^{-1})\}p_A(f).$

Proof. It is clear that ${}_s f$ and f_s are λ -measurable. Relations (i) and (ii) trivially become equalities if $p_A(f) = 0$. Suppose that $p_A(f) > 0$.

Theorem (20.1.i) in [4], and (1.2.iii) yield the inequality $p_A({}_s f) \leq p_A(f)$, from which (i) easily follows. Using (20.1.ii) in [4], and once again (1.2.iii) we write

$$(1) \quad \int_G A(|f_s|/p_A(f)) d\lambda \leq \Delta(s^{-1}),$$

which establishes (ii) in case $\Delta(s^{-1}) \leq 1$. For $\Delta(s^{-1}) > 1$, use (1) and (1.1.i).

The following result is part of (20.7) in the Russian edition of Hewitt and Ross "Abstract Harmonic Analysis", to be published.

(2.3) LEMMA. *Let f be a λ -measurable function on G . The following functions are $\lambda \times \lambda$ -measurable on $G \times G$:*

$$(x, y) \rightarrow f(xy^{-1}), \quad (x, y) \rightarrow f(y^{-1}x), \quad (x, y) \rightarrow f(x),$$

$$(x, y) \rightarrow f(x^{-1}), \quad (x, y) \rightarrow f(y), \quad (x, y) \rightarrow f(y^{-1}).$$

(2.4) THEOREM *Let f be a function in $\mathfrak{L}_A(G)$ vanishing outside of a σ -compact set F and let g be a function in $\mathfrak{L}_1(G)$. The integral*

(i) $g * f(x) = \int_G f(y^{-1}x)g(y)dy$

exists and is finite for almost all x in G . The function $g * f$ is in $\mathfrak{L}_A(G)$ and we have

$$(ii) \quad \|g * f\|_A \leq 4 \|f\|_A \|g\|_1.$$

If $g \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$, the integral

$$(iii) \quad f * g(x) = \int_G \Delta(y^{-1}) f(xy^{-1}) g(y) dy$$

exists and is finite for λ -almost all x in G . The function $f * g$ is in $\mathfrak{L}_A(G)$ and we have

$$(iv) \quad \|f * g\|_A \leq 4 \|f\|_A \|g\|,$$

where $\|\cdot\|$ is as in (2.1.i).

Proof. We may suppose that g vanishes outside of a σ -compact set E . Thus the function $(x, y) \rightarrow f(y^{-1}x)g(y)$ vanishes outside of the σ -compact set $(EF) \times E$.

Let v be an arbitrary function in $\mathfrak{L}_A(G)$. From (2.3) we know that the mapping $(x, y) \rightarrow v(x)f(y^{-1}x)g(y)$ is $\lambda \times \lambda$ -measurable. Plainly this function vanishes outside of $(EF) \times E$.

Recalling (1.2.iv) and (2.2.i), we obtain

$$(1) \quad \int_G \int_G |v(x)f(y^{-1}x)g(y)| dx dy \\ \leq 2p_A(f) p_{\bar{A}}(v) \|g\|_1.$$

Thus we may apply (13.10) of [4] to conclude that

$$(2) \quad \int_G \int_G |v(x)f(y^{-1}x)g(y)| dy dx \\ = \int_G \int_G |v(x)f(y^{-1}x)g(y)| dx dy.$$

From (13.10) and (13.8) in [4], we see that the integral $\int_G v(x)f(y^{-1}x)g(y)dy$ exists and is finite for λ -almost all x in G , and that

$$(3) \quad x \rightarrow v(x) \int_G f(y^{-1}x)g(y)dy.$$

is a function in $\mathfrak{L}_1(G)$; in particular it is a λ -measurable function.

We define $g * f(x)$ by the equality (i), provided the integral exists, and put $g * f(x) = 0$, otherwise. It is easy to see that $g * f(x)$ is finite λ -a.e. on G .

In (3) we may take v to be any function in $\mathfrak{C}_{00}(G)$. Recalling (11.42) in [4], we see that $g * f$ is λ -measurable.

Consider v in $\mathfrak{L}_{\bar{\lambda}}(G)$ with $p_{\bar{\lambda}}(v) \leq 1$. Taking account of (1) and (2), we obtain

$$\begin{aligned} \oint_G |v(x)(g * f)(x)| dx &\leq \int_G \int_G |v(x)f(y^{-1}x)g(y)| dy dx \\ &= \int_G \int_G |v(x)f(y^{-1}x)g(y)| dx dy \leq 2p_{\lambda}(f) \|g\|_1. \end{aligned}$$

This implies that

$$(4) \quad N_{\lambda}(g * f) \leq 2p_{\lambda}(f) \|g\|_1.$$

Now we observe that $g * f(x) = 0$ for x outside of the σ -compact set EF . Thus from (4) and (1.3), we conclude that $g * f \in \mathfrak{L}_{\lambda}(G)$ and that $\|g * f\|_{\lambda} \leq 2p_{\lambda}(f) \|g\|_1$. Applying (1.2.ii) to this last inequality, we obtain (ii).

Next suppose that $g \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$. Consider the function

$$(5) \quad (x, y) \rightarrow v(x)f(xy^{-1})g(y)\Delta(y^{-1}),$$

where v is an arbitrary function in $\mathfrak{L}_{\bar{\lambda}}(G)$. As in the previous case, we see that the function (5) is $\lambda \times \lambda$ -measurable and vanishes outside of the σ -compact set $(FE) \times E$. From (1.2.iv) and (2.2.ii) we obtain

$$\int_G |v(x)f(xy^{-1})| dx \leq 2 \max\{1, \Delta(y)\} p_{\lambda}(f) p_{\bar{\lambda}}(v).$$

Thus we have

$$\begin{aligned} &\int_G \int_G |v(x)f(xy^{-1})g(y)\Delta(y^{-1})| dx dy \\ &\leq 2p_{\lambda}(f) p_{\bar{\lambda}}(v) \int_G \max\{1, \Delta(y^{-1})\} |g(y)| dy \\ &= 2p_{\lambda}(f) p_{\bar{\lambda}}(v) \|\max\{1, 1/\Delta\}g\|_1 \\ &\leq 2p_{\lambda}(f) p_{\bar{\lambda}}(v) \|g\|, \end{aligned}$$

the last inequality being a consequence of (2.1.ii) and (2.1.iii).

From this point on the proof is completely analogous to that presented above for $g * f$ and we omit it.

Theorem (2.4) serves as a lemma for the following general result.

(2.5) THEOREM. *Suppose that $f \in \mathfrak{L}_A(G)$ and $g \in \mathfrak{L}_1(G)$. Then the integral*

$$(i) \quad g * f(x) = \int_G f(y^{-1}x)g(y)dy$$

*exists and is finite for λ -almost all x in G . The function $g * f$ is in $\mathfrak{L}_A(G)$ and we have*

$$(ii) \quad \|g * f\|_A \leq k \|f\|_A \|g\|_1,$$

where $k = 4$ if $a = 0$ or if G is σ -compact, and $k = 6$ otherwise.

If $g \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^(G)$, the integral*

$$(iii) \quad f * g(x) = \int_G \Delta(y^{-1})g(y)dy$$

*exists and is finite for λ -almost all x in G . The function $f * g$ is in $\mathfrak{L}_A(G)$ and we have*

$$(iv) \quad \|f * g\|_A \leq k \|f\|_A \|g\|,$$

where k is as above and $\|\cdot\|$ is as in (2.1.i).

Proof. If G is σ -compact, the assertion follows immediately from (2.4). If $a = 0$, it follows from (1.4) and (2.4). Thus we may suppose that $a > 0$ and that G fails to be σ -compact.

Using (1.4), we may write $f = f_1 + f_2$, where $f_1 = f\xi_F$ for some σ -compact set F , and $|f_2| \leq ap_A(f)$. It follows that

$$(1) \quad \int_G |f_2(y^{-1}x)g(y)| dy \leq ap_A(f) \|g\|_1$$

for all x in G , and hence that $g * f_2(x)$ exists and is finite for all x in G . A short computation, in which we use (1), gives us

$$g * f_2(x) \|g * f_2\|_A \leq p_A(g * f_2) \leq a^{-1} \|g * f_2\|_\infty \leq 2 \|f\|_A \|g\|_1.$$

Applying (2.4.i) to f_1 , we conclude that

$$\int_G f_1(y^{-1}x)g(y)dy + \int_G f_2(y^{-1}x)g(y)dy$$

exists and is finite for λ -almost all x in G . Hence the same is true of $g * f(x)$.

Inequality (ii) follows from (2) and (2.4.ii) applied to f_1 . The remaining assertions are similarly established.

(2.6) THEOREM. *The space $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ is a Banach algebra.*

Proof. For f and g in $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ we obtain

$$(1) \quad ((1/\Delta)g) * ((1/\Delta)f) = (1/\Delta)(g * f).$$

Thus (2.1) and (2.5.i) tell us that $g * f \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$. We use (1) to prove that $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$, with the norm $\|\cdot\|$ defined in (2.1.i), is a normed algebra:

$$\|g * f\| \leq \|g\|_1 \|f\|_1 + \|(1/\Delta)g\|_1 \|(1/\Delta)f\|_1 \leq \|g\| \|f\|.$$

(2.7) **THEOREM.** *The space $\mathfrak{L}_A(G)$ is a left Banach \mathfrak{L}_1 -module and a right Banach $(\mathfrak{L}_1 \cap \mathfrak{L}_1^*)$ -module.*

Proof. For g in $\mathfrak{L}_1(G)$ and f in $\mathfrak{L}_A(G)$, (2.5.ii) tells us that there is a positive number k such that $\|g * f\|_A \leq k \|f\|_A \|g\|_1$.

Next we show that, for f as above, and g_1 and g_2 in $\mathfrak{L}_1(G)$, we have $g_1 * (g_2 * f) = (g_1 * g_2) * f$. Using (20.1) of [4], we obtain the equality

$$\int_G f(v^{-1}y^{-1}x)g_2(v)dv = \int_G f(v^{-1}x)g_2(y^{-1}v)dv,$$

which implies that

$$(1) \quad g_1 * (g_2 * f)(x) = \int_G \int_G f(v^{-1}x)g_2(y^{-1}v)g_1(y)dvdy.$$

By (2.5.i), $g_1 * (g_2 * f)$ is in $\mathfrak{L}_A(G)$, and hence the integral in (1) exists and is finite λ -almost everywhere in G . From (1.4) we know that g_1 and g_2 vanish outside of σ -compact sets E_1 and E_2 , respectively. Thus the function $(v, y) \rightarrow f(v^{-1}x)g_2(y^{-1}v)g_1(y)$ vanishes outside of the σ -compact set $(E_1E_2) \times E_1$. By (2.3) this function is $\lambda \times \lambda$ -measurable.

We apply (13.10) in [4] to conclude that for λ -almost all x in G we have

$$\begin{aligned} g_1 * (g_2 * f)(x) &= \int_G \int_G f(v^{-1}x)g_2(y^{-1}v)g_1(y)dydv \\ &= \int_G f(v^{-1}x)(g_1 * g_2)(v)dv = (g_1 * g_2) * f(x). \end{aligned}$$

It is now clear that $\mathfrak{L}_A(G)$ is a left Banach \mathfrak{L}_1 -module. The proof that $\mathfrak{L}_A(G)$ is a right Banach $(\mathfrak{L}_1 \cap \mathfrak{L}_1^*)$ -module is similar and we omit it.

3. Closed ideals in $\mathfrak{L}_A(G)$ for G a compact group. Throughout this section we suppose that G is compact and that $\lambda(G) = 1$.

(3.1) THEOREM. *If f and g are in $\mathfrak{L}_A(G)$ the equality $g * f(x) = \int_G f(y^{-1}x)g(y)dy$ defines a function in $\mathfrak{L}_A(G)$. We have*

$$(i) \quad \|g * f\|_A \leq (16/(\bar{A})^{-1}(1))\|f\|_A \|g\|_A.$$

Proof. Follows from (2.5.i), (1.2.v) and (2.5.ii).

(3.2) THEOREM. *The Birnbaum-Orlicz space $\mathfrak{L}_A(G)$ is a Banach algebra under a norm which is a positive constant times $\|\cdot\|_A$.*

Proof. Define $n_A(f) = (16/(\bar{A})^{-1}(1))\|f\|_A$ and use (3.1).

(3.3) THEOREM. *Suppose that A satisfies the Δ_2 -condition for $u \geq u_0 \geq 0$. Then the space $\mathfrak{T}(G)$ of trigonometric polynomials on G is $\|\cdot\|_A$ -dense in $L_A(G)$.*

Proof. Our hypothesis imply that $\mathfrak{C}(G)$ is $\|\cdot\|_A$ -dense in $\mathfrak{L}_A(G)$: see [3] or [8]. Theorem (27.39.ii) of [5] tells us that $\mathfrak{T}(G)$ is uniformly dense in $\mathfrak{C}(G)$, and it is easy to see that $\mathfrak{T}(G)$ is also $\|\cdot\|_A$ -dense in $\mathfrak{C}(G)$.

(3.4) THEOREM. *Let A be as in (3.3). Suppose that S is a closed linear subspace of $\mathfrak{L}_A(G)$. Then S is a left [right] ideal in $\mathfrak{L}_A(G)$ if and only if S is closed under the formation of left [right] translates.*

Proof. Since G is unimodular, it follows from (2.1) and (2.7) that $\mathfrak{L}_A(G)$ is a Banach \mathfrak{L}_1 -module with respect to convolution. From (3.2) we know that $\mathfrak{L}_A(G)$ is a subalgebra of $\mathfrak{L}_1(G)$ which is a Banach algebra with the norm n_A . Taking (3.3) into account, we see that $L_A(G)$ has the properties stated in (38.6.a) in [5]. Thus the theorem follows immediately from (38.22.b) of [5].

(3.5) THEOREM. *Let A be as in (3.3). Then the class of all closed two-sided ideals in $\mathfrak{L}_A(G)$ is exactly the family $\{(\mathfrak{L}_A)_P: P \subset \Sigma\}$. Distinct subsets of Σ engender distinct ideals.*

Proofs. This is a direct application of (38.7) in [5].

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