BIRNBAUM-ORLICZ SPACES OF FUNCTIONS ON GROUPS

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It is natural to ask how far the theory of closed invariant subspaces for $L_p(G)$ can be extended to Birnbaum-Orlicz spaces $L_A(G)$. If $G$ is a compact group and $A$ satisfies the $\Delta_2$-condition for $u \geq u_0 \geq 0$, the class of all closed invariant subspaces of $L_A(G)$ is exactly the family $\{(L_A)_P : P \subset \Sigma\}$ where $\Sigma$ is the dual object of $G$. Distinct subsets of $\Sigma$ engender distinct subspaces.

The generalization of the classical $L_p$-spaces foreshadowed by Z. W. Birnbaum in 1930 [1] was the subject of a long article by Z. W. Birnbaum and W. Orlicz [2]. In the next four decades their theory has been extended by many writers, among them G. Weiss [9] and W. Luxemburg who invented convenient new definitions. More recently M. Jodeit and A. Torchinsky [7] introduced a generalization of the concept of Young's function which we adopt here.

The essential introductory definitions and theorems are stated in §1; proofs may be found in [3], [8] and [9]. In §2 we show that if $G$ is a locally compact group, the Birnbaum-Orlicz space $L_A(G)$ is a left Banach $L_1$-module and a right Banach ($L_1 \cap L_1^*$)-module. Finally in §3 we establish the result stated in the synopsis. Our notation is as in [4], [5] and [6].

1. Preliminaries. (1.1) A function $A$ on $[0,\infty]$ will be called a generalized Young's function if it is left continuous on $]0,\infty[$, $A(u)/u$ is nondecreasing for $u > 0$, and $A(0) = 0$. It easily follows that

\[(i) \quad A(\alpha u) \leq \alpha A(u) \quad \text{for} \quad 0 \leq \alpha \leq 1 \quad \text{and} \quad 0 \leq u < \infty.\]

The zero function and the function $A(u) = \infty \cdot \xi_{10,\infty}(u)$ are trivial generalized Young's functions. Throughout the remaining of this work the letter $A$ will denote a nontrivial generalized Young's function. We also fix $a = \sup\{u : A(u) = 0\}$.

A Young's function $A_0$ is associated to $A$ by the equality $A_0(u) = \int_0^u A(t)/t \, dt$.

(1.2) Let $(X,\mathcal{M},\mu)$ be an arbitrary measure space. The set $L_A(X,\mathcal{M},\mu)$ of all complex-valued, $\mathcal{M}$-measurable functions defined $\mu$-a.e. on $X$, such that $\int_X A(\alpha |f|) \, d\mu < \infty$ for some positive number $\alpha$ is
called a Birnbaum-Orlicz space. Where no confusion seems possible, we will write $\mathcal{L}_A(X)$ for $\mathcal{L}_A(X,\mathcal{M},\mu)$.

The equality

\[(i)\quad p_A(f) = \inf\{k \in ]0,\infty[: \int_X A(|f|/k)\,d\mu \leq 1\}\]

defines a nonnegative finite-valued function on $\mathcal{L}_A(X)$ which is a norm in case $A$ is convex. This suggests that we define a norm on $\mathcal{L}_A(X)$ by the equality $\|f\|_A = p_A(f)$. With this norm, $\mathcal{L}_A(X)$ is a Banach space.

If $f \in \mathcal{L}_A(G)$ the following hold:

\[(ii)\quad \|f\|_A \leq p_A(f) \leq 2\|f\|_A;\]

\[(iii)\quad \int_X A(|f|/p_A(f))\,d\mu \leq 1, \text{ provided that } p_A(f) > 0.\]

Denoting the Young's complement of $A$ by $\tilde{A}$, for $f$ in $\mathcal{L}_A(X)$ and $g$ in $\mathcal{L}_{\tilde{A}}(X)$ we obtain

\[(iv)\quad \int_X |fg|\,d\mu \leq 2p_A(f)p_{\tilde{A}}(g).\]

If $\mu(X)$ is finite, $\mathcal{L}_A(X)$ is contained in $\mathcal{L}_{\tilde{A}}(X)$ and for $f \in \mathcal{L}_A(X)$ we have

\[(v)\quad \|f\|_A \leq [4/(\tilde{A})^{-1}(1/\mu(X))]\|f\|_{\tilde{A}},\]

where $(\tilde{A})^{-1}$ denotes the right inverse of $\tilde{A}$.

\[\text{(1.3) Theorem.} \quad \text{Let } f \text{ be a complex-valued measurable function vanishing outside of a } \sigma\text{-finite set. Suppose that } \]

\[N_A(f) = \sup\left\{ \int_X |fg|\,d\mu : g \in \mathcal{L}_{\tilde{A}}(X), p_{\tilde{A}}(g) \leq 1 \right\} < \infty.\]

Then $f \in \mathcal{L}_A(X)$ and we have $\|f\|_A \leq N_A(f)$.

\[\text{(1.4) Theorem.} \quad \text{Let } X \text{ be a locally compact Hausdorff space. Let } \mu \text{ be a measure obtained from a nonnegative linear functional on } \mathcal{L}_0(X), \text{ and let } \mathcal{M} \text{ be the } \sigma\text{-algebra of all } \mu\text{-measurable subsets of } X. \text{ Then each function } f \text{ in } \mathcal{L}_A(X) \text{ can be written as } f = f_1 + f_2, \text{ where } f_1 = f \xi_F \text{ for some } \sigma\text{-compact set } F, \text{ and } |f_2| \leq a p_A(f) \mu\text{-a.e. on } X. \text{ In particular, if } a = 0, \text{ then } f \text{ vanishes } \mu\text{-a.e. outside of a } \sigma\text{-compact set.}\]

\[\text{2. Birnbaum-Orlicz spaces of functions on groups.} \quad \text{From here on we consider spaces } \mathcal{L}_A(G,\mathcal{M},\lambda), \text{ where } G \text{ is a locally compact group, } \lambda \text{ is a left Haar measure on } G, \text{ and } \mathcal{M} \text{ is the } \sigma\text{-algebra of } \lambda\text{-measurable subsets of } G. \text{ We will often write } \int_G fd\lambda \text{ as } \int_G f(x)dx.\]

Our first theorem follows easily from (20.2) in [4], and the fact that $\mathcal{L}_{\lambda}(G,\mathcal{M},\max\{1,1/\Delta\}\lambda)$ is complete.
(2.1) **Theorem.** A complex-valued measurable function \( f \) belongs to \( \mathcal{L}_1(G) \cap \mathcal{L}^*_1(G) \) if and only if \( \max\{1, 1/\Delta\} f \in \mathcal{L}_1(G) \). The equalities

(i) \( \| f \| = \| f \|_1 + \| (1/\Delta) f \|_1 \),

and

(ii) \( \| f \| = \| \max\{1, 1/\Delta\} f \|_1 \)

define equivalent norms on the linear space \( \mathcal{L}_1(G) \cap \mathcal{L}^*_1(G) \). Precisely, we have

(iii) \( \| f \| \leq \| f \|_1 \leq 2 \| f \| \) for all \( f \in \mathcal{L}_1(G) \cap \mathcal{L}^*_1(G) \).

With either of these two norms, \( \mathcal{L}_1(G) \cap \mathcal{L}^*_1(G) \) is a Banach space.

(2.2) **Theorem.** Let \( f \) be a function in \( \mathcal{L}_A(G) \) and let \( s \) be an arbitrary element of \( G \). Then the functions \( f_s \) and \( f_s \) belong to \( \mathcal{L}_A(G) \) and we have:

(i) \( p_A(f_s) = p_A(f) \);

(ii) \( p_A(f_s) \leq \max\{1, \Delta(\Delta^{-1})\} p_A(f) \).

**Proof.** It is clear that \( f_s \) and \( f_s \) are \( \lambda \)-measurable. Relations (i) and (ii) trivially become equalities if \( p_A(f) = 0 \). Suppose that \( p_A(f) > 0 \).

Theorem (20.1.i) in [4], and (1.2.iii) yield the inequality \( p_A(f_s) \leq p_A(f) \), from which (i) easily follows. Using (20.1.ii) in [4], and once again (1.2.iii) we write

\[
(1) \quad \int_G A(|f_s|/p_A(f)) d\lambda \leq \Delta(\Delta^{-1}),
\]

which establishes (ii) in case \( \Delta(\Delta^{-1}) \leq 1 \). For \( \Delta(\Delta^{-1}) > 1 \), use (1) and (1.1.i).

The following result is part of (20.7) in the Russian edition of Hewitt and Ross “Abstract Harmonic Analysis”, to be published.

(2.3) **Lemma.** Let \( f \) be a \( \lambda \)-measurable function on \( G \). The following functions are \( \lambda \times \lambda \)-measurable on \( G \times G \):

\[
(x, y) \to f(xy^{-1}), \quad (x, y) \to f(y^{-1}x), \quad (x, y) \to f(x),
\]

\[
(x, y) \to f(x^{-1}), \quad (x, y) \to f(y), \quad (x, y) \to f(y^{-1}).
\]

(2.4) **Theorem.** Let \( f \) be a function in \( \mathcal{L}_A(G) \) vanishing outside of a \( \sigma \)-compact set \( F \) and let \( g \) be a function in \( \mathcal{L}_1(G) \). The integral

(i) \( g \ast f(x) = \int_G f(y^{-1}x) g(y) dy \)

exists and is finite for almost all \( x \) in \( G \). The function \( g \ast f \) is in \( \mathcal{L}_A(G) \) and we have
(ii) \( \|g * f\|_A \leq 4\|f\|_A \|g\|_1. \)

If \( g \in L^1(G) \cap L^2(G) \), the integral

\[
(f * g)(x) = \int_G \Delta(y^{-1})f(xy^{-1})g(y)\,dy
\]

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \). The function \( f * g \) is in \( L^1(G) \) and we have

(iv) \( \|f * g\|_A \leq 4\|f\|_A \|g\| \),

where \( \| \cdot \| \) is as in (2.1.i).

**Proof.** We may suppose that \( g \) vanishes outside of a \( \sigma \)-compact set \( E \). Thus the function \( (x, y) \mapsto f(y^{-1}x)g(y) \) vanishes outside of the \( \sigma \)-compact set \( (EF) \times E \).

Let \( v \) be an arbitrary function in \( L^1(G) \). From (2.3) we know that the mapping \( (x, y) \mapsto v(x)f(y^{-1}x)g(y) \) is \( \lambda \times \lambda \)-measurable. Plainly this function vanishes outside of \( (EF) \times E \).

Recalling (1.2.iv) and (2.2.i), we obtain

\[
\int_G \int_G |v(x)f(y^{-1}x)g(y)|\,dx\,dy
\]

\[
\leq 2p_A(f)p_A(v)\|g\|_1.
\]

Thus we may apply (13.10) of [4] to conclude that

\[
\int_G \int_G |v(x)f(y^{-1}x)g(y)|\,dy\,dx
\]

\[
= \int_G \int_G |v(x)f(y^{-1}x)g(y)|\,dx\,dy.
\]

From (13.10) and (13.8) in [4], we see that the integral

\[
\int_G v(x)f(y^{-1}x)g(y)\,dy
\]

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \), and that

\[
x \mapsto v(x) \int_G f(y^{-1}x)g(y)\,dy.
\]

is a function in \( L^1(G) \); in particular it is a \( \lambda \)-measurable function.

We define \( g * f(x) \) by the equality (i), provided the integral exists, and put \( g * f(x) = 0 \), otherwise. It is easy to see that \( g * f(x) \) is finite \( \lambda \)-a.e. on \( G \).

In (3) we may take \( v \) to be any function in \( C_0(G) \). Recalling (11.42) in [4], we see that \( g * f \) is \( \lambda \)-measurable.
Consider \( v \) in \( \mathcal{L}_A(G) \) with \( p_A(v) \leq 1 \). Taking account of (1) and (2), we obtain
\[
\int_G |v(x)(g*f)(x)| \, dx \leq \int_G \int_G |v(x)f(y^{-1}x)g(y)| \, dy \, dx
\]
\[
= \int_G \int_G |v(x)f(y^{-1}x)g(y)| \, dx \, dy \leq 2p_A(f)\|g\|.
\]
This implies that
\[
(4) \quad N_A(g*f) \leq 2p_A(f)\|g\|.
\]

Now we observe that \( g*f(x) = 0 \) for \( x \) outside of the \( \sigma \)-compact set \( EF \). Thus from (4) and (1.3), we conclude that \( g*f \in \mathcal{L}_A(G) \) and that \( \|g*f\|_A \leq 2p_A(f)\|g\| \). Applying (1.2.ii) to this last inequality, we obtain (ii).

Next suppose that \( g \in \mathcal{L}_1(G) \cap \mathcal{L}_\ast(G) \). Consider the function
\[
(5) \quad (x, y) \rightarrow v(x)f(xy^{-1})g(y)\Delta(y^{-1}),
\]
where \( v \) is an arbitrary function in \( \mathcal{L}_A(G) \). As in the previous case, we see that the function (5) is \( \lambda \times \lambda \)-measurable and vanishes outside of the \( \sigma \)-compact set \((FE) \times E\). From (1.2.iv) and (2.2.ii) we obtain
\[
\int_G |v(x)f(xy^{-1})| \, dx \leq 2\max\{1, \Delta(y)\} p_A(f)p_A(v).
\]
Thus we have
\[
\int_G \int_G |v(x)f(xy^{-1})g(y)\Delta(y^{-1})| \, dx \, dy
\]
\[
\leq 2p_A(f)p_A(v) \int_G \max\{1, \Delta(y^{-1})\}|g(y)| \, dy
\]
\[
= 2p_A(f)p_A(v)\|\max\{1, 1/\Delta\}g\|,
\]
the last inequality being a consequence of (2.1.ii) and (2.1.iii).

From this point on the proof is completely analogous to that presented above for \( g*f \) and we omit it.
Theorem (2.4) serves as a lemma for the following general result.

(2.5) THEOREM. Suppose that \( f \in \mathcal{L}_\Lambda(G) \) and \( g \in \mathcal{L}_\ast(G) \). Then the integral

\[
\text{(i) } g \ast f(x) = \int_G f(y^{-1}x)g(y)dy
\]

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \). The function \( g \ast f \) is in \( \mathcal{L}_\Lambda(G) \) and we have

\[
\text{(ii) } \| g \ast f \|_\Lambda \leq k \| f \|_\Lambda \| g \|,
\]

where \( k = 4 \) if \( a = 0 \) or if \( G \) is \( \sigma \)-compact, and \( k = 6 \) otherwise. If \( g \in \mathcal{L}_\Lambda(G) \cap \mathcal{L}_\ast(G) \), the integral

\[
\text{(iii) } f \ast g(x) = \int_G \Delta(y^{-1})g(y)dy
\]

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \). The function \( f \ast g \) is in \( \mathcal{L}_\Lambda(G) \) and we have

\[
\text{(iv) } \| f \ast g \|_\Lambda \leq k \| f \|_\Lambda \| g \|,
\]

where \( k \) is as above and \( \| \cdot \| \) is as in (2.1.i).

Proof. If \( G \) is \( \sigma \)-compact, the assertion follows immediately from (2.4). If \( a = 0 \), it follows from (1.4) and (2.4). Thus we may suppose that \( a > 0 \) and that \( G \) fails to be \( \sigma \)-compact.

Using (1.4), we may write \( f = f_1 + f_2 \), where \( f_1 = f \xi_F \) for some \( \sigma \)-compact set \( F \), and \( |f_2| \leq ap_\Lambda(f) \). It follows that

\[
\int_G |f_2(y^{-1}x)g(y)|dy \leq ap_\Lambda(f)\|g\|,
\]

for all \( x \) in \( G \), and hence that \( g \ast f_2(x) \) exists and is finite for all \( x \) in \( G \). A short computation, in which we use (1), gives us

\[
g \ast f_2(x)\|g \ast f_2\|_\Lambda \leq p_\Lambda(g \ast f_2) \leq a^{-1}\|g \ast f_2\|_\infty \leq 2\|f\|_\Lambda \|g\|.
\]

Applying (2.4.i) to \( f_1 \), we conclude that

\[
\int_G f_1(y^{-1}x)g(y)dy + \int_G f_2(y^{-1}x)g(y)dy
\]

exists and is finite for \( \lambda \)-almost all \( x \) in \( G \). Hence the same is true of \( g \ast f(x) \).

Inequality (ii) follows from (2) and (2.4.ii) applied to \( f_1 \). The remaining assertions are similarly established.

(2.6) THEOREM. The space \( \mathcal{L}_\Lambda(G) \cap \mathcal{L}_\ast(G) \) is a Banach algebra.
Proof. For \( f \) and \( g \) in \( \mathcal{L}_1(G) \cap \mathcal{L}_1^*(G) \) we obtain

\[
((1/\Delta)g)*(1/\Delta)f) = (1/\Delta)(g*f).
\]

Thus (2.1) and (2.5.i) tell us that \( g*f \in \mathcal{L}_1(G) \cap \mathcal{L}_1^*(G) \). We use (1) to prove that \( \mathcal{L}_1(G) \cap \mathcal{L}_1^*(G) \), with the norm \( \| \cdot \| \) defined in (2.1.i), is a normed algebra:

\[
\| g*f \| \leq \| g \| \| f \| + \| (1/\Delta)g \| \| (1/\Delta)f \| \leq \| g \| \| f \|.
\]

(2.7) THEOREM. The space \( \mathcal{L}_A(G) \) is a left Banach \( \mathcal{L}_1 \)-module and a right Banach \( (\mathcal{L}_1 \cap \mathcal{L}_1^*) \)-module.

Proof. For \( g \) in \( \mathcal{L}_1(G) \) and \( f \) in \( \mathcal{L}_A(G) \), (2.5.ii) tells us that there is a positive number \( k \) such that \( \| g*f \|_A \leq k \| f \|_A \| g \|_A \).

Next we show that, for \( f \) as above, and \( g_1 \) and \( g_2 \) in \( \mathcal{L}_1(G) \), we have \( g_1*(g_2*f) = (g_1*g_2)*f \). Using (20.1) of [4], we obtain the equality

\[
\int_G f(v^{-1}y^{-1}x)g_2(v)dv = \int_G f(v^{-1}x)g_2(y^{-1}v)dv,
\]

which implies that

\[
g_1*(g_2*f)(x) = \int_G \int_G f(v^{-1}x)g_2(y^{-1}v)g_1(y)dvdy.
\]

By (2.5.i), \( g_1*(g_2*f) \) is in \( \mathcal{L}_A(G) \), and hence the integral in (1) exists and is finite \( \lambda \)-almost everywhere in \( G \). From (1.4) we know that \( g_1 \) and \( g_2 \) vanish outside of \( \sigma \)-compact sets \( E_1 \) and \( E_2 \), respectively. Thus the function \( (v,y) \rightarrow f(v^{-1}x)g_2(y^{-1}v)g_1(y) \) vanishes outside of the \( \sigma \)-compact set \( (E_1E_2) \times E_1 \). By (2.3) this function is \( \lambda \times \lambda \)-measurable.

We apply (13.10) in [4] to conclude that for \( \lambda \)-almost all \( x \) in \( G \) we have

\[
g_1*(g_2*f)(x) = \int_G \int_G f(v^{-1}x)g_2(y^{-1}v)g_1(y)dydv
\]

\[=
\int_G f(v^{-1}x)(g_1*g_2)(v)dv = (g_1*g_2)*f(x).
\]

It is now clear that \( \mathcal{L}_A(G) \) is a left Banach \( \mathcal{L}_1 \)-module. The proof that \( \mathcal{L}_A(G) \) is a right Banach \( (\mathcal{L}_1 \cap L_1^*) \)-module is similar and we omit it.

3. Closed ideals in \( \mathcal{L}_A(G) \) for \( G \) a compact group. Throughout this section we suppose that \( G \) is compact and that \( \lambda(G) = 1 \).
(3.1) **Theorem.** If $f$ and $g$ are in $L_A(G)$ the equality $g * f(x) = \int_G f(y^{-1}x)g(y)dy$ defines a function in $L_A(G)$. We have

(i) $\|g * f\|_A \leq (16/(\bar{A})^{-1}(1))\|f\|_A \|g\|_A$

**Proof.** Follows from (2.5.i), (1.2.v) and (2.5.ii).

(3.2) **Theorem.** The Birnbaum-Orlicz space $L_A(G)$ is a Banach algebra under a norm which is a positive constant times $\|\cdot\|_A$.

**Proof.** Define $n_A(f) = (16/(\bar{A})^{-1}(1))\|f\|_A$ and use (3.1).

(3.3) **Theorem.** Suppose that $A$ satisfies the $\Delta_2$-condition for $u \geq u_0 \geq 0$. Then the space $\mathcal{E}(G)$ of trigonometric polynomials on $G$ is $\|\cdot\|_A$-dense in $L_A(G)$.

**Proof.** Our hypothesis imply that $\mathcal{C}(G)$ is $\|\cdot\|_A$-dense in $L_A(G)$: see [3] or [8]. Theorem (27.39.ii) of [5] tells us that $\mathcal{E}(G)$ is uniformly dense in $\mathcal{C}(G)$, and it is easy to see that $\mathcal{E}(G)$ is also $\|\cdot\|_A$-dense in $\mathcal{C}(G)$.

(3.4) **Theorem.** Let $A$ be as in (3.3). Suppose that $S$ is a closed linear subspace of $L_A(G)$. Then $S$ is a left [right] ideal in $L_A(G)$ if and only if $S$ is closed under the formation of left [right] translates.

**Proof.** Since $G$ is unimodular, it follows from (2.1) and (2.7) that $L_A(G)$ is a Banach $L_1$-module with respect to convolution. From (3.2) we know that $L_A(G)$ is a subalgebra of $L_1(G)$ which is a Banach algebra with the norm $n_A$. Taking (3.3) into account, we see that $L_A(G)$ has the properties stated in (38.6.a) in [5]. Thus the theorem follows immediately from (38.22.b) of [5].

(3.5) **Theorem.** Let $A$ be as in (3.3). Then the class of all closed two-sided ideals in $L_A(G)$ is exactly the family $\{L_A_P: P \subseteq \Sigma\}$. Distinct subsets of $\Sigma$ engender distinct ideals.

**Proofs.** This is a direct application of (38.7) in [5].

**References**


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