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**BIRNBAUM-ORLICZ SPACES OF FUNCTIONS ON GROUPS**

IRACEMA M. BUND

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**It is natural to ask how far the theory of closed invariant subspaces for  $\mathcal{L}_p(G)$  can be extended to Birnbaum-Orlicz spaces  $\mathcal{L}_A(G)$ . If  $G$  is a compact group and  $A$  satisfies the  $\Delta_2$ -condition for  $u \geq u_0 \geq 0$ , the class of all closed invariant subspaces of  $\mathcal{L}_A(G)$  is exactly the family  $\{(\mathcal{L}_A)_P: P \subset \Sigma\}$  where  $\Sigma$  is the dual object of  $G$ . Distinct subsets of  $\Sigma$  engender distinct subspaces.**

The generalization of the classical  $\mathcal{L}_p$ -spaces foreshadowed by Z. W. Birnbaum in 1930 [1] was the subject of a long article by Z. W. Birnbaum and W. Orlicz [2]. In the next four decades their theory has been extended by many writers, among them G. Weiss [9] and W. Luxemburg who invented convenient new definitions. More recently M. Jodeit and A. Torchinsky [7] introduced a generalization of the concept of Young's function which we adopt here.

The essential introductory definitions and theorems are stated in §1; proofs may be found in [3], [8] and [9]. In §2 we show that if  $G$  is a locally compact group, the Birnbaum-Orlicz space  $\mathcal{L}_A(G)$  is a left Banach  $\mathcal{L}_1$ -module and a right Banach  $(\mathcal{L}_1 \cap \mathcal{L}_1^*)$ -module. Finally in §3 we establish the result stated in the synopsis. Our notation is as in [4], [5] and [6].

**1. Preliminaries.** (1.1) A function  $A$  on  $[0, \infty[$  will be called a generalized Young's function if it is left continuous on  $]0, \infty[$ ,  $A(u)/u$  is nondecreasing for  $u > 0$ , and  $A(0) = 0$ . It easily follows that

$$(i) \quad A(\alpha u) \leq \alpha A(u) \quad \text{for } 0 \leq \alpha \leq 1 \quad \text{and} \quad 0 \leq u < \infty.$$

The zero function and the function  $A(u) = \infty \cdot \xi_{]0, \infty[}(u)$  are trivial generalized Young's functions. Throughout the remaining of this work the letter  $A$  will denote a nontrivial generalized Young's function. We also fix  $a = \sup\{u: A(u) = 0\}$ .

A Young's function  $A_0$  is associated to  $A$  by the equality  $A_0(u) = \int_0^u A(t)/t \, dt$ .

(1.2) Let  $(X, \mathcal{M}, \mu)$  be an arbitrary measure space. The set  $\mathcal{L}_A(X, \mathcal{M}, \mu)$  of all complex-valued,  $\mathcal{M}$ -measurable functions defined  $\mu$ -a.e. on  $X$ , such that  $\int_X A(\alpha |f|) \, d\mu < \infty$  for some positive number  $\alpha$  is

called a Birnbaum-Orlicz space. Where no confusion seems possible, we will write  $\mathfrak{L}_A(X)$  for  $\mathfrak{L}_A(X, \mathcal{M}, \mu)$ .

The equality

$$(i) \quad p_A(f) = \inf\{k \in ]0, \infty[ : \int_X A(|f|/k) d\mu \leq 1\}$$

defines a nonnegative finite-valued function on  $\mathfrak{L}_A(X)$  which is a norm in case  $A$  is convex. This suggests that we define a norm on  $\mathfrak{L}_A(X)$  by the equality  $\|f\|_A = p_A(f)$ . With this norm,  $\mathfrak{L}_A(X)$  is a Banach space.

If  $f \in \mathfrak{L}_A(G)$  the following hold:

$$(ii) \quad \|f\|_A \leq p_A(f) \leq 2\|f\|_A ;$$

$$(iii) \quad \int_X A(|f|/p_A(f)) d\mu \leq 1, \text{ provided that } p_A(f) > 0.$$

Denoting the Young's complement of  $A$  by  $\bar{A}$ , for  $f$  in  $\mathfrak{L}_A(X)$  and  $g$  in  $\mathfrak{L}_{\bar{A}}(X)$  we obtain

$$(iv) \quad \int_X |fg| d\mu \leq 2p_A(f)p_{\bar{A}}(g).$$

If  $\mu(X)$  is finite,  $\mathfrak{L}_A(X)$  is contained in  $\mathfrak{L}_1(X)$  and for  $f \in \mathfrak{L}_A(X)$  we have

$$(v) \quad \|f\|_1 \leq [4/(\bar{A})^{-1}(1/\mu(X))]\|f\|_A,$$

where  $(\bar{A})^{-1}$  denotes the right inverse of  $\bar{A}$ .

(1.3) THEOREM. *Let  $f$  be a complex-valued measurable function vanishing outside of a  $\sigma$ -finite set. Suppose that*

$$N_A(f) = \sup\left\{\int_X |fg| d\mu : g \in \mathfrak{L}_{\bar{A}}(X), p_{\bar{A}}(g) \leq 1\right\} < \infty.$$

*Then  $f \in \mathfrak{L}_A(X)$  and we have  $\|f\|_A \leq N_A(f)$ .*

(1.4) THEOREM. *Let  $X$  be a locally compact Hausdorff space. Let  $\mu$  be a measure obtained from a nonnegative linear functional on  $\mathfrak{C}_{00}(X)$ , and let  $\mathcal{M}$  be the  $\sigma$ -algebra of all  $\mu$ -measurable subsets of  $X$ . Then each function  $f$  in  $\mathfrak{L}_A(X)$  can be written as  $f_1 + f_2$ , where  $f_1 = f \xi_F$  for some  $\sigma$ -compact set  $F$ , and  $|f_2| \leq ap_A(f)$   $\mu$ -a.e. on  $X$ . In particular, if  $a = 0$ , then  $f$  vanishes  $\mu$ -a.e. outside of a  $\sigma$ -compact set.*

**2. Birnbaum-Orlicz spaces of functions on groups.** From here on we consider spaces  $\mathfrak{L}_A(G, \mathcal{M}, \lambda)$ , where  $G$  is a locally compact group,  $\lambda$  is a left Haar measure on  $G$ , and  $\mathcal{M}$  is the

$\sigma$ -algebra of  $\lambda$ -measurable subsets of  $G$ . We will often write  $\int_G f d\lambda$  as

$$\int_G f(x) dx.$$

Our first theorem follows easily from (20.2) in [4], and the fact that  $\mathfrak{L}_1(G, \mathcal{M}, \max\{1, 1/\Delta\}\lambda)$  is complete.

(2.1) THEOREM. *A complex-valued measurable function  $f$  belongs to  $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$  if and only if  $\max\{1, 1/\Delta\}f \in \mathfrak{L}_1(G)$ . The equalities*

(i)  $\|f\| = \|f\|_1 + \|(1/\Delta)f\|_1,$

and

(ii)  $\| \|f\| \| = \|\max\{1, 1/\Delta\}f\|_1$

define equivalent norms on the linear space  $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ . Precisely, we have

(iii)  $\| \|f\| \| \leq \|f\| \leq 2 \| \|f\| \|$  for all  $f \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ .

With either of these two norms,  $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$  is a Banach space.

(2.2) THEOREM. *Let  $f$  be a function in  $\mathfrak{L}_A(G)$  and let  $s$  be an arbitrary element of  $G$ . Then the functions  $f_s$  and  $f_s$  belong to  $\mathfrak{L}_A(G)$  and we have:*

(i)  $p_A(f_s) = p_A(f);$

(ii)  $p_A(f_s) \leq \max\{1, \Delta(s^{-1})\}p_A(f).$

*Proof.* It is clear that  $f_s$  and  $f_s$  are  $\lambda$ -measurable. Relations (i) and (ii) trivially become equalities if  $p_A(f) = 0$ . Suppose that  $p_A(f) > 0$ .

Theorem (20.1.i) in [4], and (1.2.iii) yield the inequality  $p_A(f_s) \leq p_A(f)$ , from which (i) easily follows. Using (20.1.ii) in [4], and once again (1.2.iii) we write

(1) 
$$\int_G A(|f_s|/p_A(f)) d\lambda \leq \Delta(s^{-1}),$$

which establishes (ii) in case  $\Delta(s^{-1}) \leq 1$ . For  $\Delta(s^{-1}) > 1$ , use (1) and (1.1.i).

The following result is part of (20.7) in the Russian edition of Hewitt and Ross "Abstract Harmonic Analysis", to be published.

(2.3) LEMMA. *Let  $f$  be a  $\lambda$ -measurable function on  $G$ . The following functions are  $\lambda \times \lambda$ -measurable on  $G \times G$ :*

$(x, y) \rightarrow f(xy^{-1}), \quad (x, y) \rightarrow f(y^{-1}x), \quad (x, y) \rightarrow f(x),$

$(x, y) \rightarrow f(x^{-1}), \quad (x, y) \rightarrow f(y), \quad (x, y) \rightarrow f(y^{-1}).$

(2.4) THEOREM *Let  $f$  be a function in  $\mathfrak{L}_A(G)$  vanishing outside of a  $\sigma$ -compact set  $F$  and let  $g$  be a function in  $\mathfrak{L}_1(G)$ . The integral*

(i)  $g * f(x) = \int_G f(y^{-1}x)g(y)dy$

exists and is finite for almost all  $x$  in  $G$ . The function  $g*f$  is in  $\mathfrak{L}_A(G)$  and we have

(ii)  $\|g * f\|_A \leq 4\|f\|_A \|g\|_1.$

If  $g \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ , the integral

(iii)  $f * g(x) = \int_G \Delta(y^{-1})f(xy^{-1})g(y) dy$

exists and is finite for  $\lambda$ -almost all  $x$  in  $G$ . The function  $f * g$  is in  $\mathfrak{L}_A(G)$  and we have

(iv)  $\|f * g\|_A \leq 4\|f\|_A \|g\|,$

where  $\|\cdot\|$  is as in (2.1.i).

*Proof.* We may suppose that  $g$  vanishes outside of a  $\sigma$ -compact set  $E$ . Thus the function  $(x, y) \rightarrow f(y^{-1}x)g(y)$  vanishes outside of the  $\sigma$ -compact set  $(EF) \times E$ .

Let  $v$  be an arbitrary function in  $\mathfrak{L}_A(G)$ . From (2.3) we know that the mapping  $(x, y) \rightarrow v(x)f(y^{-1}x)g(y)$  is  $\lambda \times \lambda$ -measurable. Plainly this function vanishes outside of  $(EF) \times E$ .

Recalling (1.2.iv) and (2.2.i), we obtain

$$(1) \quad \int_G \int_G |v(x)f(y^{-1}x)g(y)| dx dy \leq 2p_A(f) p_{\bar{A}}(v) \|g\|_1.$$

Thus we may apply (13.10) of [4] to conclude that

$$(2) \quad \int_G \int_G |v(x)f(y^{-1}x)g(y)| dy dx = \int_G \int_G |v(x)f(y^{-1}x)g(y)| dx dy.$$

From (13.10) and (13.8) in [4], we see that the integral  $\int_G v(x)f(y^{-1}x)g(y)dy$  exists and is finite for  $\lambda$ -almost all  $x$  in  $G$ , and that

$$(3) \quad x \rightarrow v(x) \int_G f(y^{-1}x)g(y) dy.$$

is a function in  $\mathfrak{L}_1(G)$ ; in particular it is a  $\lambda$ -measurable function.

We define  $g * f(x)$  by the equality (i), provided the integral exists, and put  $g * f(x) = 0$ , otherwise. It is easy to see that  $g * f(x)$  is finite  $\lambda$ -a.e. on  $G$ .

In (3) we may take  $v$  to be any function in  $\mathfrak{C}_{00}(G)$ . Recalling (11.42) in [4], we see that  $g * f$  is  $\lambda$ -measurable.

Consider  $v$  in  $\mathfrak{L}_{\bar{A}}(G)$  with  $p_{\bar{A}}(v) \leq 1$ . Taking account of (1) and (2), we obtain

$$\begin{aligned} \oint_G |v(x)(g * f)(x)| dx &\leq \int_G \int_G |v(x)f(y^{-1}x)g(y)| dy dx \\ &= \int_G \int_G |v(x)f(y^{-1}x)g(y)| dx dy \leq 2p_A(f) \|g\|_1. \end{aligned}$$

This implies that

$$(4) \quad N_A(g * f) \leq 2p_A(f) \|g\|_1.$$

Now we observe that  $g * f(x) = 0$  for  $x$  outside of the  $\sigma$ -compact set  $EF$ . Thus from (4) and (1.3), we conclude that  $g * f \in \mathfrak{L}_A(G)$  and that  $\|g * f\|_A \leq 2p_A(f) \|g\|_1$ . Applying (1.2.ii) to this last inequality, we obtain (ii).

Next suppose that  $g \in \mathfrak{L}_i(G) \cap \mathfrak{L}_i^*(G)$ . Consider the function

$$(5) \quad (x, y) \rightarrow v(x)f(xy^{-1})g(y)\Delta(y^{-1}),$$

where  $v$  is an arbitrary function in  $\mathfrak{L}_{\bar{A}}(G)$ . As in the previous case, we see that the function (5) is  $\lambda \times \lambda$ -measurable and vanishes outside of the  $\sigma$ -compact set  $(FE) \times E$ . From (1.2.iv) and (2.2.ii) we obtain

$$\int_G |v(x)f(xy^{-1})| dx \leq 2 \max\{1, \Delta(y)\} p_A(f) p_{\bar{A}}(v).$$

Thus we have

$$\begin{aligned} &\int_G \int_G |v(x)f(xy^{-1})g(y)\Delta(y^{-1})| dx dy \\ &\leq 2p_A(f) p_{\bar{A}}(v) \int_G \max\{1, \Delta(y^{-1})\} |g(y)| dy \\ &= 2p_A(f) p_{\bar{A}}(v) \|\max\{1, 1/\Delta\} g\|_1 \\ &\leq 2p_A(f) p_{\bar{A}}(v) \|g\|, \end{aligned}$$

the last inequality being a consequence of (2.1.ii) and (2.1.iii).

From this point on the proof is completely analogous to that presented above for  $g * f$  and we omit it.

Theorem (2.4) serves as a lemma for the following general result.

(2.5) THEOREM. *Suppose that  $f \in \mathfrak{L}_A(G)$  and  $g \in \mathfrak{L}_1(G)$ . Then the integral*

$$(i) \quad g * f(x) = \int_G f(y^{-1}x)g(y)dy$$

*exists and is finite for  $\lambda$ -almost all  $x$  in  $G$ . The function  $g * f$  is in  $\mathfrak{L}_A(G)$  and we have*

$$(ii) \quad \|g * f\|_A \leq k \|f\|_A \|g\|_1,$$

*where  $k = 4$  if  $a = 0$  or if  $G$  is  $\sigma$ -compact, and  $k = 6$  otherwise.*

*If  $g \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ , the integral*

$$(iii) \quad f * g(x) = \int_G \Delta(y^{-1})g(y)dy$$

*exists and is finite for  $\lambda$ -almost all  $x$  in  $G$ . The function  $f * g$  is in  $\mathfrak{L}_A(G)$  and we have*

$$(iv) \quad \|f * g\|_A \leq k \|f\|_A \|g\|,$$

*where  $k$  is as above and  $\|\cdot\|$  is as in (2.1.i).*

*Proof.* If  $G$  is  $\sigma$ -compact, the assertion follows immediately from (2.4). If  $a = 0$ , it follows from (1.4) and (2.4). Thus we may suppose that  $a > 0$  and that  $G$  fails to be  $\sigma$ -compact.

Using (1.4), we may write  $f = f_1 + f_2$ , where  $f_1 = f\xi_F$  for some  $\sigma$ -compact set  $F$ , and  $|f_2| \leq ap_A(f)$ . It follows that

$$(1) \quad \int_G |f_2(y^{-1}x)g(y)| dy \leq ap_A(f) \|g\|_1$$

for all  $x$  in  $G$ , and hence that  $g * f_2(x)$  exists and is finite for all  $x$  in  $G$ . A short computation, in which we use (1), gives us

$$g * f_2(x) \|g * f_2\|_A \leq p_A(g * f_2) \leq a^{-1} \|g * f_2\|_\infty \leq 2 \|f\|_A \|g\|_1.$$

Applying (2.4.i) to  $f_1$ , we conclude that

$$\int_G f_1(y^{-1}x)g(y)dy + \int_G f_2(y^{-1}x)g(y)dy$$

exists and is finite for  $\lambda$ -almost all  $x$  in  $G$ . Hence the same is true of  $g * f(x)$ .

Inequality (ii) follows from (2) and (2.4.ii) applied to  $f_1$ . The remaining assertions are similarly established.

(2.6) THEOREM. *The space  $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$  is a Banach algebra.*

*Proof.* For  $f$  and  $g$  in  $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$  we obtain

$$(1) \quad ((1/\Delta)g) * ((1/\Delta)f) = (1/\Delta)(g * f).$$

Thus (2.1) and (2.5.i) tell us that  $g * f \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ . We use (1) to prove that  $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$ , with the norm  $\|\cdot\|$  defined in (2.1.i), is a normed algebra:

$$\|g * f\| \leq \|g\|_1 \|f\|_1 + \|(1/\Delta)g\|_1 \|(1/\Delta)f\|_1 \leq \|g\| \|f\|.$$

(2.7) THEOREM. *The space  $\mathfrak{L}_A(G)$  is a left Banach  $\mathfrak{L}_1$ -module and a right Banach  $(\mathfrak{L}_1 \cap \mathfrak{L}_1^*)$ -module.*

*Proof.* For  $g$  in  $\mathfrak{L}_1(G)$  and  $f$  in  $\mathfrak{L}_A(G)$ , (2.5.ii) tells us that there is a positive number  $k$  such that  $\|g * f\|_A \leq k \|f\|_A \|g\|_1$ .

Next we show that, for  $f$  as above, and  $g_1$  and  $g_2$  in  $\mathfrak{L}_1(G)$ , we have  $g_1 * (g_2 * f) = (g_1 * g_2) * f$ . Using (20.1) of [4], we obtain the equality

$$\int_G f(v^{-1}y^{-1}x)g_2(v)dv = \int_G f(v^{-1}x)g_2(y^{-1}v)dv,$$

which implies that

$$(1) \quad g_1 * (g_2 * f)(x) = \int_G \int_G f(v^{-1}x)g_2(y^{-1}v)g_1(y)dvdy.$$

By (2.5.i),  $g_1 * (g_2 * f)$  is in  $\mathfrak{L}_A(G)$ , and hence the integral in (1) exists and is finite  $\lambda$ -almost everywhere in  $G$ . From (1.4) we know that  $g_1$  and  $g_2$  vanish outside of  $\sigma$ -compact sets  $E_1$  and  $E_2$ , respectively. Thus the function  $(v, y) \rightarrow f(v^{-1}x)g_2(y^{-1}v)g_1(y)$  vanishes outside of the  $\sigma$ -compact set  $(E_1E_2) \times E_1$ . By (2.3) this function is  $\lambda \times \lambda$ -measurable.

We apply (13.10) in [4] to conclude that for  $\lambda$ -almost all  $x$  in  $G$  we have

$$\begin{aligned} g_1 * (g_2 * f)(x) &= \int_G \int_G f(v^{-1}x)g_2(y^{-1}v)g_1(y)dydv \\ &= \int_G f(v^{-1}x)(g_1 * g_2)(v)dv = (g_1 * g_2) * f(x). \end{aligned}$$

It is now clear that  $\mathfrak{L}_A(G)$  is a left Banach  $\mathfrak{L}_1$ -module. The proof that  $\mathfrak{L}_A(G)$  is a right Banach  $(\mathfrak{L}_1 \cap \mathfrak{L}_1^*)$ -module is similar and we omit it.

**3. Closed ideals in  $\mathfrak{L}_A(G)$  for  $G$  a compact group.** Throughout this section we suppose that  $G$  is compact and that  $\lambda(G) = 1$ .



(3.1) THEOREM. *If  $f$  and  $g$  are in  $\mathfrak{L}_A(G)$  the equality  $g * f(x) = \int_G f(y^{-1}x)g(y)dy$  defines a function in  $\mathfrak{L}_A(G)$ . We have*

$$(i) \quad \|g * f\|_A \leq (16/(\bar{A})^{-1}(1))\|f\|_A \|g\|_A.$$

*Proof.* Follows from (2.5.i), (1.2.v) and (2.5.ii).

(3.2) THEOREM. *The Birnbaum-Orlicz space  $\mathfrak{L}_A(G)$  is a Banach algebra under a norm which is a positive constant times  $\|\cdot\|_A$ .*

*Proof.* Define  $n_A(f) = (16/(\bar{A})^{-1}(1))\|f\|_A$  and use (3.1).

(3.3) THEOREM. *Suppose that  $A$  satisfies the  $\Delta_2$ -condition for  $u \geq u_0 \geq 0$ . Then the space  $\mathfrak{T}(G)$  of trigonometric polynomials on  $G$  is  $\|\cdot\|_A$ -dense in  $L_A(G)$ .*

*Proof.* Our hypothesis imply that  $\mathfrak{C}(G)$  is  $\|\cdot\|_A$ -dense in  $\mathfrak{L}_A(G)$ : see [3] or [8]. Theorem (27.39.ii) of [5] tells us that  $\mathfrak{T}(G)$  is uniformly dense in  $\mathfrak{C}(G)$ , and it is easy to see that  $\mathfrak{T}(G)$  is also  $\|\cdot\|_A$ -dense in  $\mathfrak{C}(G)$ .

(3.4) THEOREM. *Let  $A$  be as in (3.3). Suppose that  $S$  is a closed linear subspace of  $\mathfrak{L}_A(G)$ . Then  $S$  is a left [right] ideal in  $\mathfrak{L}_A(G)$  if and only if  $S$  is closed under the formation of left [right] translates.*

*Proof.* Since  $G$  is unimodular, it follows from (2.1) and (2.7) that  $\mathfrak{L}_A(G)$  is a Banach  $\mathfrak{L}_1$ -module with respect to convolution. From (3.2) we know that  $\mathfrak{L}_A(G)$  is a subalgebra of  $\mathfrak{L}(G)$  which is a Banach algebra with the norm  $n_A$ . Taking (3.3) into account, we see that  $L_A(G)$  has the properties stated in (38.6.a) in [5]. Thus the theorem follows immediately from (38.22.b) of [5].

(3.5) THEOREM. *Let  $A$  be as in (3.3). Then the class of all closed two-sided ideals in  $\mathfrak{L}_A(G)$  is exactly the family  $\{(\mathfrak{L}_A)_P: P \subset \Sigma\}$ . Distinct subsets of  $\Sigma$  engender distinct ideals.*

*Proofs.* This is a direct application of (38.7) in [5].

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