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RADICALS OF SUPPLEMENTARY SEMILATTICE SUMS OF ASSOCIATIVE RINGS

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This paper deals with the effect of radicals (in the Kurosh-Amitsur sense) on supplementary semilattice sums of rings as defined by J. Weissglass (Proc. Amer. Math. Soc., 39 (1973), 471–473). It is shown that if $\mathscr{R}$ is a strict, hereditary radical class, then $\mathscr{R}(R) = \Sigma_{\alpha\in\Omega}\mathscr{R}(R_\alpha)$ for every supplementary semilattice sum $R = \Sigma_{\alpha\in\Omega}R_\alpha$ with finite Ω. If $\mathscr{R}$ is an A-radical class or the generalized nil radical class, the same conclusion holds with the finiteness restriction removed. On the other hand, if $\mathscr{R}(\Sigma_{\alpha\in\Omega}R_\alpha) = \Sigma_{\alpha\in\Omega}\mathscr{R}(R_\alpha)$ for all finite Ω, then $\mathscr{R}$ is strict and satisfies

(*) $R \in \mathscr{R} \Rightarrow$ the zeroring on the additive group of R belongs to $\mathscr{R}$,

a condition satisfied by both hereditary strict and A-radical classes.

Introduction. Semilattice sums of rings were introduced by Weissglass [11]. Let Ω be a *semilattice*, a commutative semigroup in which all elements are idempotent. A ring $R = \Sigma_{\alpha\in\Omega}R_\alpha$ is a *supplementary semilattice sum* of its subrings R_α if (i) $R^+ = \bigoplus_{\alpha\in\Omega}R_\alpha^+$ (here $(\ )^+$ denotes the additive group) i.e., R is a *supplementary sum* in the language of [3], and (ii) $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all $\alpha, \beta \in \Omega$. Examples include direct sums, polynomial rings and semigroup rings over semilattices.

In [11], Weissglass considered the inheritance of properties by a supplementary semilattice sum $R = \Sigma_{\alpha\in\Omega}R_\alpha$ from its subrings R_α. In [8], Janeski and Weissglass proved that R is regular if and only if each R_α is. Their arguments need minimal modification to obtain corresponding results in which regularity is replaced by various other hereditary radical properties, including quasi-regularity, nilness and local nilpotence.

We shall be concerned with a stronger condition on a radical class $\mathscr{R}$: $\mathscr{R}(\Sigma_{\alpha\in\Omega}R_\alpha) = \Sigma_{\alpha\in\Omega}\mathscr{R}(R_\alpha)$ (supplementary semilattice sum) for all (finite) supplementary semilattice sums $\Sigma_{\alpha\in\Omega}R_\alpha$.

For general information about radical classes the reader is referred to [3]. A radical class $\mathscr{R}$ is *strict* if every $\mathscr{R}$-subring S of a ring R is contained in $\mathscr{R}(R)$, or equivalently every subring of an $\mathscr{R}$-semi-simple ring is $\mathscr{R}$-semi-simple. See [9] for further details. An *A-radical* class [5] is one which contains with any ring R all ring S with $S^+ \cong R^+$. We denote the additive group of a ring by $(\ )^+$, the zeroring on an abelian group by $(\ )^0$; $<$ signifies an ideal. All rings considered are associative.

The results.

THEOREM 1. *Let $\mathcal{R}$ be a radical class.*

(i) *If $\mathcal{R}$ is strict and hereditary, then for any supplementary semilattice sum $R = \Sigma_{\alpha\in\Omega} R_\alpha$, where Ω is finite, the rings $\mathcal{R}(R_\alpha)$ form a supplementary semilattice sum and $\mathcal{R}(R) = \Sigma_{\alpha\in\Omega}\mathcal{R}(R_\alpha)$.*

(ii) *If $\mathcal{R}$ is an A-radical class the same is true without the finiteness restriction on Ω.*

Proof. (i) We prove this by induction on $|\Omega|$, making use of the construction described in Lemmas 2 and 3 of **[8]**. Suppose firstly that $\Omega = \{\alpha, \beta\}$, $\alpha\beta = \beta$ and thus $R_\beta < R$. Then

$$\mathcal{R}(R_\alpha)\mathcal{R}(R_\beta) \subseteq R\mathcal{R}(R_\beta) \subseteq \mathcal{R}(R_\beta)$$

since $\mathcal{R}(R_\beta) < R$ **[1]**. Similarly $\mathcal{R}(R_\beta)\mathcal{R}(R_\alpha) \subseteq \mathcal{R}(R_\beta)$ and so $\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta)$ is a supplementary semilattice sum.

We next show that $\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta) < R$. Now $R_\alpha\mathcal{R}(R_\alpha) \subseteq \mathcal{R}(R_\alpha)$, while $\mathcal{R}(R_\alpha) \subseteq \mathcal{R}(R)$ ($\mathcal{R}$ is strict) whence $R_\beta\mathcal{R}(R_\alpha) \subset \mathcal{R}(R) \cap R_\beta = \mathcal{R}(R_\beta)$ ($\mathcal{R}$ is hereditary). Thus $R\mathcal{R}(R_\alpha) \subseteq \mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta)$. Since $\mathcal{R}(R_\beta) < R$, we have $R[\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta)] \subseteq \mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta)$ and a similar argument on the right completes the proof that $\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta) < R$.

Since $\mathcal{R}(R_\beta) < \mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta)$ and

$$[\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta)]/\mathcal{R}(R_\beta) \cong \mathcal{R}(R_\alpha)/[\mathcal{R}(R_\alpha) \cap \mathcal{R}(R_\beta)] \cong \mathcal{R}(R_\alpha),$$

it follows that $\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta) \in \mathcal{R}$.

Since the sum $R_\alpha + R_\beta$ is supplementary we have isomorphisms

$$R_\alpha/\mathcal{R}(R_\alpha) \cong [R/R_\beta]/[[R_\beta + \mathcal{R}(R_\alpha)]/R_\beta] \cong R/[R_\beta + \mathcal{R}(R_\alpha)].$$

It therefore follows from the exact sequence

$$\begin{array}{ccccc}
0 \to [R_\beta + \mathcal{R}(R_\alpha)]/[\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta)] & \to & R/[\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta)] & & \\
\wr\Vert & & & & \\
R_\beta/\mathcal{R}(R_\beta) & & \to R/[R_\beta + \mathcal{R}(R_\alpha)] \to 0 & & \\
 & & \wr\Vert & & \\
 & & R_\alpha/\mathcal{R}(R_\alpha) & &
\end{array}$$

that $R/[\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta)]$ is $\mathcal{R}$-semi-simple.

Thus $\mathcal{R}(R_\alpha) + \mathcal{R}(R_\beta) = \mathcal{R}(R)$ and we have proved (i) for $|\Omega| = 2$.

Now consider $|\Omega| = k$ and assume (i) for smaller semilattices. Let

$\Omega = \Lambda \cup \theta$ where Λ and θ are disjoint subsemilattices and θ is an ideal of Ω, as in **[8]**, Lemma 3. Then

$$\mathcal{R}(R) = \mathcal{R}(R_\Lambda) + \mathcal{R}(R_\theta); \ \mathcal{R}(R_\Lambda) = \sum_{\alpha \in \Lambda} \mathcal{R}(R_\alpha); \ \mathcal{R}(R_\theta) = \sum_{\alpha \in \theta} \mathcal{R}(R_\alpha)$$

and all these sums are supplementary semilattice sums. It follows that $\mathcal{R}(R_\Lambda) = \mathcal{R}(R) \cap R_\Lambda$ and

$$\mathcal{R}(R_\alpha) = \mathcal{R}(R_\Lambda) \cap R_\alpha = \mathcal{R}(R) \cap R_\Lambda \cap R_\alpha = \mathcal{R}(R) \cap R_\alpha$$

for every $\alpha \in \Lambda$, $\mathcal{R}(R_\theta) = \mathcal{R}(R) \cap R_\theta$ and $\mathcal{R}(R_\alpha) = \mathcal{R}(R) \cap R_\alpha$ for $\alpha \in \theta$. If $\alpha, \beta \in \Lambda$, then $\mathcal{R}(R_\alpha)\mathcal{R}(R_\beta) \subseteq \mathcal{R}(R_{\alpha\beta})$ and the same conclusion is true if $\alpha, \beta \in \theta$. If $\alpha \in \Lambda$ and $\beta \in \theta$, then $\alpha\beta \in \theta$ and $\mathcal{R}(R_\alpha)\mathcal{R}(R_\beta) \subseteq R_\alpha R_\beta \subseteq R_{\alpha\beta}$. Also, $\mathcal{R}(R_\alpha) \subseteq \mathcal{R}(R)$, so $\mathcal{R}(R_\alpha)\mathcal{R}(R_\beta) \subseteq \mathcal{R}(R)$. Thus

$$\mathcal{R}(R_\alpha)\mathcal{R}(R_\beta) \subseteq \mathcal{R}(R) \cap R_{\alpha\beta} = \mathcal{R}(R_{\alpha\beta}).$$

Similarly $\mathcal{R}(R_\beta)\mathcal{R}(R_\alpha) \subseteq \mathcal{R}(R_{\alpha\beta})$, so $\Sigma_{\alpha \in \Omega} \mathcal{R}(R_\alpha)$ is a supplementary semilattice sum.

Finally,

$$\mathcal{R}(R) = \mathcal{R}(R_\Lambda) + \mathcal{R}(R_\theta) = \Sigma_{\alpha \in \Lambda} \mathcal{R}(R_\alpha) + \Sigma_{\alpha \in \theta} \mathcal{R}(R_\alpha) = \Sigma_{\alpha \in \Omega} \mathcal{R}(R_\alpha).$$

(ii) Since $R^+ = \bigoplus_{\alpha \in \Omega} R_\alpha^+$, we have $\mathcal{R}(R) = \Sigma_{\alpha \in \Omega} \mathcal{R}(R_\alpha)$ (cf. the proof of Proposition 1.5 in **[5]**). The sum is a supplementary semilattice sum, because $\mathcal{R}(R_\alpha)\mathcal{R}(R_\beta) \subseteq \mathcal{R}(R) \cap R_{\alpha\beta} = \mathcal{R}(R_{\alpha\beta})$ for any α, $\beta \in \Omega$.

Whether or not hereditary strict radicals commute with formation of supplementary semilattice sums in general remains an open question. Some information is given by our next result, in the course of the proof of which we show that strict radical properties satisfy Weissglass' condition (F) **[11]**.

PROPOSITION 2. *Let $\mathcal{R}$ be a hereditary, strict radical class, $R = \Sigma_{\alpha \in \Omega} R_\alpha$ a supplementary semilattice sum. Then $\Sigma_{\alpha \in \Omega} \mathcal{R}(R_\alpha)$ is a supplementary semilattice sum and an $\mathcal{R}$-ideal of R. Furthermore, $R / \Sigma_{\alpha \in \Omega} \mathcal{R}(R_\alpha) \cong \Sigma_{\alpha \in \Omega} R_\alpha / \mathcal{R}(R_\alpha)$ where the latter is a supplementary semilattice sum.*

Proof. For any finite subsemilattice Ω' of Ω, we have $\mathcal{R}(\Sigma_{\alpha \in \Omega'} R_\alpha) = \Sigma_{\alpha \in \Omega'} \mathcal{R}(R_\alpha)$ by Theorem 1. Thus $\Sigma_{\alpha \in \Omega'} \mathcal{R}(R_\alpha) \in \mathcal{R}$ for every such Ω'. But then each such $\Sigma_{\alpha \in \Omega'} \mathcal{R}(R_\alpha) \subseteq \mathcal{R}(\Sigma_{\alpha \in \Omega} \mathcal{R}(R_\alpha))$, so $\mathcal{R}$ contains $\Sigma_{\alpha \in \Omega} \mathcal{R}(R_\alpha)$ (not assumed to be a semilattice sum at this stage).

Let $\beta, \gamma \in \Omega$ generate the semilattice Γ. Then Γ is finite so by Theorem 1, $\mathcal{R}(\Sigma_{\alpha\in\Gamma} R_\alpha) = \Sigma_{\alpha\in\Gamma}\mathcal{R}(R_\alpha)$ (supplementary semilattice sum). Hence

$$R_\beta\mathcal{R}(R_\gamma) \subseteq \mathcal{R}\Big(\sum_{\alpha\in\Gamma} R_\alpha\Big) \cap R_{\beta\gamma} = \mathcal{R}(R_{\beta\gamma})$$

and similarly $\mathcal{R}(R_\gamma)R_\beta \subseteq \mathcal{R}(R_{\beta\gamma})$. This shows that $\Sigma_{\alpha\in\Omega}\mathcal{R}(R_\alpha)$ is both a supplementary semilattice sum and an ideal of R.

Now for any $\beta \in \Omega$, we have

$$\Big[R_\beta + \sum_{\alpha\in\Omega}\mathcal{R}(R_\alpha)\Big]\Big/\sum_{\alpha\in\Omega}\mathcal{R}(R_\alpha) \cong R_\beta\Big/\Big[R_\beta \cap \sum_{\alpha\in\Omega}\mathcal{R}(R_\alpha)\Big] = R_\beta\Big/\mathcal{R}(R_\beta).$$

Let $I = \Sigma_{\alpha\in\Omega}\mathcal{R}(R_\alpha)$. It is a routine matter to show that the supplementary semilattice sum structure on R induces a similar one on

$$\sum_{\alpha\in\Omega}(R_\alpha + I)/I = R/I.$$

Two facts about strict hereditary radical classes may help to put the foregoing results into perspective:

(i) Such a class $\mathcal{R}$ (if nontrivial) must contain some zerorings.

Proof. If $\mathcal{R}$ doesn't contain zerorings, then every ring in $\mathcal{R}$ is idempotent. The class $\mathcal{R}^{(1)} = \{R \mid R[x] \in \mathcal{R}\}$ is also a radical class (**[4]**, Theorem 1). Theorem 10 of **[4]** implies that $\mathcal{R}^{(1)} = \{0\}$, but Proposition 3.1 of **[9]** implies that $\mathcal{R}^{(1)} = \mathcal{R}$.

(ii) If $\mathcal{R}$ contains all zerorings, then $\mathcal{N}_g \subseteq \mathcal{R}$, where $\mathcal{N}_g$ is the *generalized nil radical class* of Andrunakievič and Rjabuhin **[2]** and Thierrin **[10]** (cf. Theorem 3.7 of **[6]**).

We can improve on Proposition 2 for $\mathcal{R} = \mathcal{N}_g$.

PROPOSITION 3. *Let $R = \Sigma_{\alpha\in\Omega} R_\alpha$ be a supplementary semilattice sum. Then $\mathcal{N}_g(R) = \Sigma_{\alpha\in\Omega}\mathcal{N}_g(R_\alpha)$* (supplementary semilattice sum).

Proof. By Proposition 2, $I = \Sigma_{\alpha\in\Omega}\mathcal{N}_g(R_\alpha)$ is a supplementary semilattice sum and an $\mathcal{N}_g$-ideal of R and R/I is isomorphic to a supplementary semilatice sum $\Sigma_{\alpha\in\Omega} I_\alpha$ of $\mathcal{N}_g$-semi-simple rings. The $\mathcal{N}_g$-semi-simple rings are those without nonzero nilpotent elements. Suppose $x \in \Sigma_{\alpha\in\Omega} I_\alpha$ is a nonzero nilpotent element. Let Γ be the (finite) subsemilattice of Ω generated by the α appearing in the representation of x. Then $x \in \Sigma_{\alpha\in\Gamma} I_\alpha$, so $x \in \Sigma_{\alpha\in\Gamma}\mathcal{N}_g(I_\alpha) = \mathcal{N}_g(\Sigma_{\alpha\in\Gamma} I_\alpha)$, contradicting the $\mathcal{N}_g$-semi-simplicity of the I_α. Hence R/I is $\mathcal{N}_g$-semi-simple, so $I = \mathcal{N}_g(R)$.

COROLLARY 4. *A supplementary semilattice sum* $\Sigma_{\alpha\in\Omega} R_\alpha$ *has no nonzero nilpotent elements if* (and clearly only if) *each* R_α *has none.*

A special case of this corollary is given in **[11]**.

Proposition 3 holds for strict, hereditary radical classes $\mathcal{R}$ such that the property of $\mathcal{R}$-semi-simplicity satisfies condition (F) of **[11]**.

Our final result is to some extent a converse to Theorem 1.

THEOREM 5. *Let* $\mathcal{R}$ *be a radical class such that*

$$\mathcal{R}\Big(\sum_{\alpha\in\Omega} R_\alpha\Big) = \sum_{\alpha\in\Omega} \mathcal{R}(R_\alpha)$$

(supplementary semilattice sum) *for every supplementary semilattice sum* $\Sigma_{\alpha\in\Omega} R_\alpha$ *with finite* Ω. *Then*

(i) $\mathcal{R}$ *is strict.*

(ii) $\mathcal{R}$ *satisfies*

$$(*) \qquad A \in \mathcal{R} \Rightarrow (A^+)^0 \in \mathcal{R}.$$

(iii) *If in addition* $\mathcal{R}$ *contains all zerorings, then* $\mathcal{R}$ *is hereditary.*

Proof. (i) Suppose $\mathcal{R}$ is not strict. Then there is a ring Y with a subring $X \neq 0$ such that $X \in \mathcal{R}$ and $\mathcal{R}(Y) = 0$. Define a ring R by

$$R^+ = Y^+ \oplus X^+$$

$$(a, b)(c, d) = (ac + ad + bc, bd).$$

Then R is a supplementary semilattice sum of Y and X, while $\mathcal{R}(Y) + \mathcal{R}(X) = X$ is not an ideal of R unless $XY = 0 = YX$. But this would imply that $X \lhd Y$, whence $\mathcal{R}(X) = 0$. Not being an ideal, $\mathcal{R}(Y) + \mathcal{R}(X)$ cannot coincide with $\mathcal{R}(R)$.

(ii) If $\mathcal{R}$ does not satisfy (*), let $A \in \mathcal{R}$, $(A^+)^0 \notin \mathcal{R}$. Then $\mathcal{R}[(A^+)^0]^+$ is the additive group of an ideal I of A (cf. Propositions 1.1 and 1.3 of **[5]**) and $(A^+)^0/\mathcal{R}[(A^+)^0] = [(A/I)^+]^0$ and $0 \neq A/I \in \mathcal{R}$. Thus we may assume that $(A^+)^0$ is $\mathcal{R}$-semi-simple. The ring R defined by

$$R^+ = A^+ \oplus A^+$$

$$(a, b)(c, d) = (ad + bc, bd)$$

is a supplementary semilattice sum of $(A^+)^0$ and A, with $(A^+)^0$ an ideal. Now $\mathcal{R}(A)+\mathcal{R}[(A^+)^0]=A$, while since $A^2\neq 0$, A is not an ideal of R. Hence $A\neq\mathcal{R}(R)$.

(iii) Suppose $\mathcal{R}$ contains all zerorings, but is not hereditary. Then $\mathcal{R}$ contains a ring Y with a nonzero ideal $X\notin\mathcal{R}$. Since $\mathcal{R}(X)\lhd Y$ [1], we have $0\neq X/\mathcal{R}(X)\lhd Y/\mathcal{R}(X)\in\mathcal{R}$, so it may be assumed that $\mathcal{R}(X)=0$. Define the ring R by

$$R^+=X^+\oplus Y^+$$

$$(a,b)(c,d)=(ac+ad+bc,bd).$$

This makes R a supplementary semilattice sum of X and Y, where $\mathcal{R}(X)+\mathcal{R}(Y)=Y$. But Y is not an ideal of R unless $XY=0=YX$. But then $X^2=0$, so $X\in\mathcal{R}$.

Both A-radical classes (clearly) and hereditary, strict radical classes [7] satisfy (*), but no example of a strict radical class satisfying (*) which is neither hereditary nor an A-radical class is known to the author.

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