

# Pacific Journal of Mathematics

## **CHARACTERIZING LOCAL CONNECTEDNESS IN INVERSE LIMITS**

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## CHARACTERIZING LOCAL CONNECTEDNESS IN INVERSE LIMITS

G. R. GORDH, JR. AND SIBE MARDEŠIĆ

Let  $X$  denote the limit of an inverse system  $\underline{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$  of locally connected Hausdorff continua. The main purpose of this paper is to define a notion of local connectedness for inverse systems, and to prove that if  $\underline{X}$  is locally connected, then so is the limit  $X$ . If the bonding maps  $p_{\alpha\alpha'}$  are surjections, then  $X$  is locally connected if and only if  $\underline{X}$  is. The following corollaries are obtained. (1) If  $\underline{X}$  is  $\sigma$ -directed and surjective, then  $X$  is locally connected. (2) If  $\underline{X}$  is well-ordered, surjective, and  $\text{weight}(X_\alpha) \leq \lambda$  for each  $\alpha$  in  $A$ , then either  $\text{weight}(X) \leq \lambda$ , or  $X$  is locally connected. (3) If  $\underline{X}$  is  $\sigma$ -directed and the factor spaces  $X_\alpha$  are trees (generalized arcs), then  $X$  is a tree (generalized arc). (4) If  $\underline{X}$  is well-ordered and the factor spaces  $X_\alpha$  are dendrites (arcs), then either  $X$  is metrizable, or  $X$  is a tree (generalized arc).

**1. Introduction.** By a continuum we mean a compact connected Hausdorff space. Let  $X$  denote the limit of an inverse system  $\underline{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$  where the factor spaces  $X_\alpha$  are locally connected continua, and  $A$  is an arbitrary directed set. It is well-known that every continuum  $X$  can be obtained as the limit of such a system where the factor spaces are polyhedra (see Theorem 10.1, p. 284, [2]). Hence local connectedness of the factor spaces  $X_\alpha$  does not imply local connectedness of the limit  $X$ . It is the main purpose of this paper to introduce a notion of *local connectedness* for inverse systems, and to prove that for such systems  $\underline{X}$  the limit space  $X$  is locally connected (see Theorem 1). The converse holds if  $\underline{X}$  is a surjective system, i.e., if the bonding maps  $p_{\alpha\alpha'}$  are surjections. An immediate corollary is the known result that if  $\underline{X}$  is a monotone inverse system, then  $X$  is locally connected [1].

In §3 the main theorem is applied to well-ordered and  $\sigma$ -directed inverse systems, i.e., systems in which every countable subset of the index set is bounded above. The following somewhat surprising results are obtained. (1) If the inverse system  $\underline{X}$  is  $\sigma$ -directed and surjective, then the limit  $X$  is locally connected. (2) If  $\underline{X}$  is well-ordered, surjective, and  $\text{weight}(X_\alpha) \leq \lambda$  for each  $\alpha$  in  $A$ , then  $\text{weight}(X) \leq \lambda$  or  $X$  is locally connected.

Section 4 contains similar results about well-ordered and  $\sigma$ -directed inverse systems of trees (i.e., locally connected, hereditarily uncoherent continua [9]) and generalized arcs (i.e., ordered continua).

For example, the limit of a  $\sigma$ -directed inverse system of trees (generalized arcs) is a tree (generalized arc).<sup>1</sup>

The problem of characterizing locally connected inverse limits has been studied from a different point of view in [3].

The reader is referred to [1] for basic results concerning inverse limits of compact Hausdorff spaces.

**2. Locally connected inverse systems.** A continuum  $X$  has *property S* if given any open cover  $\mathcal{U}$  of  $X$ , there exists a finite cover  $\mathcal{C}$  of  $X$  which refines  $\mathcal{U}$  and consists of connected subsets of  $X$ . A continuum is locally connected if and only if it has property *S* (e.g., Chapter IV, Theorem 3.7, p. 106, [11]).

**DEFINITION** Let  $f: X \rightarrow Y$  be a mapping of locally connected continua, and let  $F \subset U \subset Y$  where  $F$  is closed and  $U$  is open. We define the *splitting number*  $s(f, U, F)$  of the triple  $(f, U, F)$  to be the number of components of  $f^{-1}(U)$  which meet  $f^{-1}(F)$ .

**LEMMA 1.** *The splitting number  $s(f, U, F)$  is finite.*

*Proof.* Since  $X$  is locally connected, the components of  $f^{-1}(U)$  are open sets. By compactness, only finitely many components of  $f^{-1}(U)$  can meet the closed set  $f^{-1}(F)$ .

**DEFINITION.** Let  $\underline{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$  be an inverse system of continua over an arbitrary directed set  $A$ . We say that the system  $\underline{X}$  is *locally connected* if (1) the factor spaces  $X_\alpha$  are locally connected; and (2) whenever  $F_\alpha \subset U_\alpha \subset X_\alpha$ , where  $F_\alpha$  is closed and  $U_\alpha$  is open, there exists an  $\alpha' \cong \alpha$  in  $A$  such that the splitting number  $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$  agrees with  $s(p_{\alpha\alpha''}, U_\alpha, F_\alpha)$  for every  $\alpha'' \cong \alpha'$ .

**THEOREM. 1.** *The limit of a locally connected inverse system is locally connected.*

*Proof.* Let  $\underline{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$  be a locally connected inverse system with limit  $X$  and projections  $p_\alpha: X \rightarrow X_\alpha$ . We shall prove that  $X$  has property *S*. Let  $\mathcal{U}$  be any open cover of  $X$ . There exists an  $\alpha \in A$  and a finite open cover  $\mathcal{U}_\alpha = (U_1, \dots, U_n)$  of  $X_\alpha$  such that  $\{p_\alpha^{-1}(U_i)\}_{i=1}^n$  refines  $\mathcal{U}$  (e.g., Lemma 3.7, p. 263, [2]). Choose open covers  $\mathcal{U}'_\alpha = (U'_1, \dots, U'_n)$  and  $\mathcal{U}''_\alpha = (U''_1, \dots, U''_n)$  of  $X_\alpha$  such that  $U'_i \subset \text{cl}(U''_i) \subset U'_i \subset \text{cl}(U'_i) \subset U_i$ . Let  $F_i = \text{cl}(U''_i)$  and consider the pairs  $(U'_i, F_i)$ . Since the system  $\underline{X}$  is locally connected, there exists an  $\alpha' \in A$  such that for  $\alpha'' \cong \alpha'$  we have  $s(p_{\alpha\alpha'}, U'_i, F_i) = s(p_{\alpha\alpha''}, U'_i, F_i)$  for  $1 \leq i \leq n$ . Let  $s_i$  denote the splitting number  $s(p_{\alpha\alpha'}, U'_i, F_i)$ . For  $\alpha' \in A$  as above, let

<sup>1</sup> M. Smith has announced results similar to Corollary 5 and Theorem 6 at the Topology Conference held at the University of North Carolina at Charlotte, March, 1974.

$\{V_{\alpha'j}^i\}_{j=1}^{s_i}$  denote the collection of components of  $p_{\alpha\alpha'}^{-1}(U_i)$  which intersect  $p_{\alpha\alpha'}^{-1}(F_i)$ . For  $\alpha'' \cong \alpha'$  there are also  $s_i$  components of  $p_{\alpha\alpha''}^{-1}(U_i)$  which intersect  $p_{\alpha\alpha''}^{-1}(F_i)$ . Denote these components by  $\{V_{\alpha''j}^i\}_{j=1}^{s_i}$ , and assume that they are labelled so that  $p_{\alpha'\alpha''}(V_{\alpha''j}^i) \subset V_{\alpha'j}^i$ . Define  $C_{\alpha''j}^i = \text{cl}(V_{\alpha''j}^i)$  for all  $\alpha'' \cong \alpha'$ , and let

$$C_j^i = \text{inv lim} \{C_{\alpha''j}^i; \alpha'' \cong \alpha'\}.$$

Since  $\{F_i\}$  covers  $X_\alpha$ , it follows that  $\{C_{\alpha''j}^i\}$  covers  $X_\alpha$  for each  $\alpha'' \cong \alpha'$ . To every  $x \in X$  one can assign a pair  $(i, j)$  such that  $p_\alpha(x) \in C_{\alpha''j}^i$ . Since  $i$  and  $j$  vary through a finite set, some pair  $(i, j)$  occurs cofinally often; and consequently  $x \in C_j^i$ . Consequently,  $\{C_j^i\}_{i,j}$  covers  $X$  and refines  $\{p_\alpha^{-1}(U_i)\}_{i=1}^n$  which refines  $\mathcal{U}$ . Since each  $C_j^i$  is a subcontinuum of  $X$ , it follows that  $X$  has property  $S$ .

The next theorem provides a converse to Theorem 1 for inverse systems with surjective bounding maps.

**THEOREM 2.** *Let  $X = \text{inv lim } \underline{X}$  where  $\underline{X}$  is a surjective inverse system of continua. If  $X$  is locally connected, then the system  $\underline{X}$  is locally connected.*

The proof of Theorem 2 depends on two simple lemmas.

**LEMMA 2.** *Let  $X_1, X_2$  and  $Y$  be locally connected continua and suppose that  $f_i: X_i \rightarrow Y$  ( $i = 1, 2$ ) and  $g: X_2 \rightarrow X_1$  are continuous surjections such that  $f_2 = f_1g$ . Let  $F \subset U \subset Y$  where  $F$  is closed and  $U$  is open. Then  $s(f_1, U, F) \cong s(f_2, U, F)$ .*

*Proof.* Let  $s_1 = s(f_1, U, F)$ , and let  $V_1, \dots, V_{s_1}$  denoted the components of  $f_1^{-1}(U)$  which meet  $f_1^{-1}(F)$ . For each  $i \leq s_1$ , at least one component of  $g^{-1}(V_i)$  meets  $g^{-1}(f_1^{-1}(F)) = f_2^{-1}(F)$ . Since each component of  $g^{-1}(V_i)$  is a component of  $f_2^{-1}(U)$ , at least  $s_1$  components of  $f_2^{-1}(U)$  meet  $f_2^{-1}(F)$ . Thus  $s_1 \leq s(f_2, U, F)$ .

**LEMMA 3.** *Let  $A$  be a directed set and  $N$  the set of natural numbers. If  $\pi: A \rightarrow N$  is an order preserving bounded function, then  $\pi$  is eventually constant.*

*Proof.* Let  $m = \max \pi(A)$ , and choose  $\alpha \in A$  such that  $\pi(\alpha) = m$ . Thus for  $\alpha' \cong \alpha$ ,  $\pi(\alpha') = m$ .

*Proof of Theorem 2.* Let  $\underline{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$  be a surjective system of continua with locally connected limit  $X$  and projections  $p_\alpha: X \rightarrow X_\alpha$ . Since the projections  $p_\alpha$  are surjections (e.g., Theorem 2.6, [1]), each

factor space  $X_\alpha$  is the image of a locally connected continuum; hence each  $X_\alpha$  is locally connected (e.g., Theorem 3-22, p. 126, [5]). Given  $\alpha \in A$ , let  $A(\alpha) = \{\alpha' \in A \mid \alpha' \cong \alpha\}$ , and let  $F_\alpha \subset U_\alpha \subset X_\alpha$  where  $F_\alpha$  is closed and  $U_\alpha$  is open. Define  $\pi: A(\alpha) \rightarrow N$  by  $\pi(\alpha') = s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$ . Lemma 2 implies that  $\pi$  is order preserving and bounded by  $s(p_\alpha, U_\alpha, F_\alpha)$ . By Lemma 3, there exists  $\alpha' \in A(\alpha)$  such that for all  $\alpha'' \cong \alpha'$ ,  $\pi(\alpha') = \pi(\alpha'')$ ; i.e.,  $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha) = s(p_{\alpha\alpha''}, U_\alpha, F_\alpha)$ .

**COROLLARY 1.** *Let  $\underline{X}$  be a surjective inverse system of locally connected continua with limit  $X$ . Then  $X$  is locally connected if and only if  $\underline{X}$  is locally connected.*

A surjective continuous function  $f: X \rightarrow Y$  between continua is *monotone* if  $f^{-1}(y)$  is a continuum for each  $y \in Y$ . An inverse system of continua is *monotone* if each bonding map is monotone.

**COROLLARY 2.** (Capel [1]). *The limit of a monotone inverse system of locally connected continua is locally connected.*

*Proof.* Let  $\{X_\alpha; p_{\alpha\alpha'}; A\}$  be a monotone inverse system of locally connected continua. Let  $F_\alpha \subset U_\alpha \subset X_\alpha$  where  $F_\alpha$  is closed and  $U_\alpha$  is open in  $X_\alpha$ . If  $\alpha' \cong \alpha$ , then since  $p_{\alpha\alpha'}$  is monotone, the splitting number  $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$  is precisely the number of components of  $U_\alpha$  which meet  $F_\alpha$ . Thus, for  $\alpha' \cong \alpha$  the splitting number  $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$  is independent of  $\alpha'$ , and so the inverse system is locally connected. By Theorem 1, the limit of the system is locally connected.

**3. Well-ordered and  $\sigma$ -directed inverse systems of locally connected continua.** We say that a quasi-ordered set  $A$  is  $\sigma$ -directed (directed) if every countable (finite) subset of  $A$  is bounded above. Thus every bounded quasi-ordered set is  $\sigma$ -directed. Clearly, an unbounded well-ordered set is  $\sigma$ -directed if and only if it contains no cofinal sequence. Another example of a  $\sigma$ -directed set is the collection of all countable subsets of a given set, ordered by inclusion. An inverse system is said to be  $\sigma$ -directed (*well-ordered*) if its index set is  $\sigma$ -directed (well-ordered).

**LEMMA 4.** *Let  $A$  be a  $\sigma$ -directed set and let  $N$  denote the set of natural numbers. If  $\pi: A \rightarrow N$  is an order preserving function, then  $\pi$  is eventually constant.*

*Proof.* If  $\pi$  is not eventually constant, then there exists an increasing sequence  $\{\alpha_i\}_{i=1}^\infty$  in  $A$  such that  $\{\pi(\alpha_i)\}_{i=1}^\infty$  is cofinal in  $N$ .

Since  $A$  is  $\sigma$ -directed, there exists  $\alpha \in A$  such that  $\alpha_i \leq \alpha$  for every  $i \in N$ . Thus  $\pi(\alpha_i) \leq \pi(\alpha)$  for every  $i$ , which is a contradiction.

**THEOREM 3.** *The limit of a  $\sigma$ -directed surjective inverse system of locally connected continua is locally connected.*

*Proof.* Let  $\underline{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$  be a  $\sigma$ -directed surjective inverse system of locally connected continua. According to Theorem 1, it suffices to show that  $\underline{X}$  is a locally connected system. Let  $F_\alpha \subset U_\alpha \subset X_\alpha$  where  $F_\alpha$  is closed and  $U_\alpha$  is open. Let  $A(\alpha) = \{\alpha' \in A \mid \alpha' \geq \alpha\}$  and note that  $A(\alpha)$  is a  $\sigma$ -directed set. We define a function  $\pi: A(\alpha) \rightarrow N$  by  $\pi(\alpha') = s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$ . By Lemma 2,  $\pi$  is an increasing function. Thus, by Lemma 4,  $\pi$  is eventually constant, and there exists  $\alpha' \in A(\alpha)$  such that  $\pi(\alpha') = \pi(\alpha'')$  whenever  $\alpha' \leq \alpha''$ . Thus for  $\alpha' \leq \alpha''$  we have  $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha) = s(p_{\alpha\alpha''}, U_\alpha, F_\alpha)$ , and  $\underline{X}$  is locally connected.

**COROLLARY 3.** *If  $X$  is the limit of a  $\sigma$ -directed inverse system of hereditarily locally connected continua, then  $X$  is hereditarily locally connected.*

*Proof.* Let  $X = \text{inv lim} \{X_\alpha; p_{\alpha\alpha'}; A\}$  where  $A$  is  $\sigma$ -directed and the factor spaces  $X_\alpha$  are hereditarily locally connected continua. Let  $Y$  be any subcontinuum of  $X$ . Then  $\{p_\alpha(Y); p_{\alpha\alpha'} \mid p_{\alpha'}(Y); A\}$  is a  $\sigma$ -directed surjective inverse system of locally connected continua with limit  $Y$  (see [1]). By Theorem 3,  $Y$  is locally connected.

The *weight* of a topological space  $X$ , denoted  $w(X)$ , is the smallest cardinal number  $\lambda$  such that  $X$  admits a basis for its topology of cardinality  $\lambda$ .

**THEOREM 4.** *Let  $X$  be the limit of a well-ordered surjective inverse system  $\underline{X}$  of locally connected continua  $X_\alpha$  such that  $w(X_\alpha) \leq \lambda$  for each  $X_\alpha$ . Then, either  $w(X) \leq \lambda$ , or  $X$  is locally connected. In particular, if the factor spaces  $X_\alpha$  are metrizable, then either  $X$  is metrizable, or  $X$  is locally connected.*

*Proof.* Let  $A$  denote the well-ordered index set for the system  $\underline{X}$ . If  $A$  contains a cofinal sequence, then  $X$  is the limit of an inverse sequence of continua  $X_n$  such that  $w(X_n) \leq \lambda$ ; hence  $w(X) \leq \lambda$ . Otherwise,  $A$  is  $\sigma$ -directed and  $X$  is locally connected by Theorem 3.

REMARK. Suppose that the nonmetrizable continuum  $X$  is the limit of a well-ordered surjective inverse system of metric continua  $X_\alpha$ . If  $X$  is non-locally connected, then by Theorem 4 the factor spaces  $X_\alpha$  are eventually nonlocally connected as well. This remark applies to all continua of weight  $\aleph_1$ , since such continua are known to be limits of well-ordered surjective inverse systems of metric continua [7].

COROLLARY 4. *Let  $X$  be the limit of a well-ordered inverse system  $X$  of hereditarily locally connected continua  $X_\alpha$  such that  $w(X_\alpha) \leq \lambda$  for each  $\alpha \in A$ . Then either  $w(X) \leq \lambda$ , or  $X$  is hereditarily locally connected.*

**4. Well-ordered and  $\sigma$ -directed inverse systems of trees and generalized arcs.** A continuum  $X$  is a *tree* [9] if each pair of points is separated by a third point. A continuum  $X$  with precisely two nonseparating points is called a *generalized arc* (or an *ordered continuum*). According to [9], a continuum  $X$  is a tree if and only if  $X$  is locally connected and hereditarily unicoherent. Clearly every subcontinuum of a tree  $X$  is a tree, and consequently  $X$  is hereditarily locally connected. It follows immediately from Theorem 4.1(3) of [4] that a tree is a generalized arc if and only if it is atriodic.

It is known that the limit of a monotone inverse system of trees is a tree (see the proof of Theorem 4.2 in [4]); and that the limit of a monotone inverse system of generalized arcs is a generalized arc (Lemma 4.7 of [1], or [8]). We shall obtain the same conclusions for  $\sigma$ -directed inverse systems of trees and generalized arcs without any assumptions about the bonding maps.

LEMMA 5. *Suppose that  $X$  is the limit of an arbitrary inverse system of trees (generalized arcs). If  $X$  is locally connected, then  $X$  is a tree (generalized arc).*

*Proof.* Since the factor spaces are hereditarily unicoherent,  $X$  is also hereditarily unicoherent by a routine application of ((2.9), p. 235, [1]). Consequently,  $X$  is a tree. If the factor spaces are generalized arcs, then  $X$  is chainable (e.g., [6]). Since chainable continua are atriodic,  $X$  is an atriodic tree; i.e., a generalized arc.

REMARK. The proof of Lemma 5 can be modified to show that a locally connected tree-like (arc-like, i.e., chainable) continuum is a tree (generalized arc). If  $X$  is tree-like, then  $X$  is hereditarily unicoherent. Consequently, if  $X$  is locally connected, then  $X$  is a tree. If, in addition,  $X$  is arc-like, then  $X$  is atriodic; hence  $X$  is a generalized arc (see [8] for a different proof).

**THEOREM 5.** *If  $X$  is the limit of a  $\sigma$ -directed inverse system of trees (generalized arcs), then  $X$  is a tree (generalized arc).*

*Proof.* Apply Corollary 3 and Lemma 5.

**THEOREM 6.** *Let  $X$  be the limit of a well-ordered inverse system of trees (generalized arcs)  $X_\alpha$  such that  $w(X_\alpha) \leq \lambda$  for each  $X_\alpha$ . Then, either  $w(X) \leq \lambda$ , or  $X$  is a tree (generalized arc).*

*Proof.* Apply Corollary 4 and Lemma 5.

**COROLLARY 5.** *Let  $X$  be the limit of a well-ordered inverse system of dendrites (arcs). Then, either  $X$  is metrizable, or  $X$  is a tree (generalized arc).*

*Proof.* A dendrite (arc) is a metrizable tree (generalized arc) (see (1.1), p. 88 and Theorem (6.2), p. 54 of [10]). Thus the desired conclusion follows from Theorem 6.

**REMARK.** The limit of a well-ordered inverse system of arcs need not be metrizable. For example, the long line (p. 55, [5]) is the limit of a well-ordered monotone inverse system of arcs.

#### REFERENCES

1. C. E. Capel, *Inverse limit spaces*, Duke Math. J., **21** (1954), 233–246.
2. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N.J., 1952.
3. M. K. Fort, Jr. and J. Segal, *Local connectedness of inverse limit spaces*, Duke Math. J., **28** (1961), 253–260.
4. G. R. Gordh, Jr., *Monotone decompositions of irreducible Hausdorff continua*, Pacific J. Math., **36** (1971), 647–658.
5. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961.
6. S. Mardešić, *Chainable continua and inverse limits*, Glasnik Mat. Fiz. Astr., **14** (1959), 219–232.
7. ———, *On covering dimension and inverse limits of compact spaces*, Illinois J. Math., **4** (1960), 278–291.
8. ———, *Locally connected, ordered and chainable continua*, Rad Jugoslav. Akad. Znan. Umjetn., **319** (1960), 147–166.
9. L. E. Ward, Jr., *Mobs, trees and fixed points*, Proc. Amer. Math. Soc., **8** (1957), 798–804.
10. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications 28, Providence, 1942.
11. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 32, Providence, 1949.

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