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**ORTHOGONALLY ADDITIVE AND ORTHOGONALLY  
INCREASING FUNCTIONS ON VECTOR SPACES**

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## ORTHOGONALLY ADDITIVE AND ORTHOGONALLY INCREASING FUNCTIONS ON VECTOR SPACES

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A real-valued function  $f: X \rightarrow R$  on an inner product space  $X$  is *orthogonally additive* if  $f(x + y) = f(x) + f(y)$  whenever  $x \perp y$ . We extend this concept to more general spaces called orthogonality vector spaces. If  $X$  is an orthogonality vector space and if there exists an orthogonally additive function on  $X$  which satisfies certain natural conditions then there is an inner product on  $X$  which is equivalent to the original orthogonality and  $f(x) = \pm \|x\|^2$  for all  $x \in X$ . We next consider a normed space  $X$  with James' orthogonality. A function  $f: X \rightarrow R$  is *orthogonally increasing* if  $f(x + y) \geq f(x)$  whenever  $x \perp y$ . Orthogonally increasing functions on normed spaces are characterized.

**1. Pythagoras' theorem.** Pythagoras' theorem states that the function  $f(x) = \|x\|^2$  is orthogonally additive, that is  $f(x + y) = f(x) + f(y)$  whenever  $x \perp y$  where  $x, y$  are vectors in the plane. One of the concerns of this paper is a converse of Pythagoras' theorem on an inner product space  $X$ . That is, if  $f: X \rightarrow R$  is orthogonally additive, is  $f(x) = c \|x\|^2$  for some  $c \in R$ ? As it stands, the answer is no, since any linear functional is orthogonally additive.

Some natural additional conditions on  $f$  are:

- (1)  $f(x) \geq 0$ , *nonnegativity*;
- (2)  $f(x) = f(-x)$ , *evenness*;
- (3)  $\lambda_i \rightarrow \lambda$  implies  $f(\lambda_i x) \rightarrow f(\lambda x)$  for all  $x \in X$ , *hemicontinuity*.

We shall show that orthogonal additivity along with (1), or with (2) and (3) imply  $f(x) = c \|x\|^2$  for some  $c \in R$ .

**2. Orthogonality vector spaces.** In this paper, vector spaces will be real and of dimension  $\geq 2$ . In Theorem 2.2 we shall prove that Pythagoras' theorem characterizes inner product spaces in a certain sense.

A vector space  $X$  is an *orthogonality vector space* if there is a relation  $x \perp y$  on  $X$  such that

- (01)  $x \perp 0, 0 \perp x$  for all  $x \in X$ ;
- (02) if  $x \perp y$  and  $x, y \neq 0$ , then  $x, y$  are linearly independent;
- (03) if  $x \perp y$ , then  $ax \perp by$  for all  $a, b \in R$ ;
- (04) if  $P$  is a two-dimensional subspace of  $X$ , then for every  $x \in P$  there exists  $0 \neq y \in P$  such that  $x \perp y$ ;

(05) if  $P$  is a two-dimensional subspace of  $X$ , then there exist nonzero vectors  $u, v \in P$  such that  $u \perp v$  and  $u + v \perp u - v$ .

Any vector space can be made into an orthogonality vector space if we define  $x \perp 0, 0 \perp x$  for all  $x$ , and for nonzero vectors  $x, y$  define  $x \perp y$  iff  $x, y$  are linearly independent. Also an inner product space is such a space; we shall see that a normed space is one also with James' definition of orthogonality.

LEMMA 2.1. *Let  $(X, \perp)$  be an orthogonality vector space and let  $f: X \rightarrow R$  be orthogonally additive and hemi-continuous. (a) If  $f$  is odd, then  $f$  is linear. (b) If  $f$  is even, then  $f(\alpha x) = \alpha^2 f(x)$  for all  $\alpha \in R, x \in X$  and if  $x \perp y$  and  $x + y \perp x - y$ , then  $f(x) = f(y)$ .*

*Proof.* Same as in [2; Lemmas 2, 3].

REMARK. The referee has pointed out to us that there is a mistake in the proof of Lemma 2 [2]. In that proof it is incorrectly stated that  $F(2^r u) = 2^r F(u)$  for all rational  $r$  when, in fact, this is only proved for integral  $r$ . However, it is easily seen that  $F(3^r u) = 3^r F(u)$  for all integral  $r$ . Indeed, in the notation of that proof

$$\begin{aligned} F(3u) - F(v) &= F(3u - v) = F(u + v + 2u - 2v) \\ &= F(u + v) + F(2(u - v)) = 3F(u) - F(v). \end{aligned}$$

Hence, by induction  $F(2^p 3^q u) = 2^p 3^q F(u)$  for all integral  $p$  and  $q$ . Since these scalars  $2^p 3^q$  are dense, continuity implies  $F(\alpha u) = \alpha F(u)$ .

An inner product  $\langle \cdot, \cdot \rangle$  on  $(X, \perp)$  is  $\perp$ -equivalent when  $x \perp y$  iff  $\langle x, y \rangle = 0$ .

THEOREM 2.2. *If there exists an  $f: (X, \perp) \rightarrow R$  which is orthogonally additive, even, hemicontinuous, and not identically 0, then there is a  $\perp$ -equivalent inner product  $\langle \cdot, \cdot \rangle$  on  $(X, \perp)$ . In fact,  $\langle x, y \rangle = \frac{1}{4}[f(x + y) - f(x - y)]$  and the induced norm satisfies  $\|x\|^2 = f(x)$  for all  $x \in X$ , or  $\|x\|^2 = -f(x)$  for all  $x \in X$ . Moreover, if  $\langle \cdot, \cdot \rangle_1$  is another  $\perp$ -equivalent inner product on  $(X, \perp)$ , then there is a nonzero  $c \in R$  such that  $\langle \cdot, \cdot \rangle_1 = c \langle \cdot, \cdot \rangle$ .*

*Proof.* We first show that  $f$  has constant sign. Let  $0 \neq x \in X$  and suppose  $f(x) > 0$ . Let  $0 \neq y \in X$ . If  $y = \alpha x$ , then  $f(y) = \alpha^2 f(x) > 0$ . If  $y, x$  are linearly independent, let  $P$  be the generated 2-dimensional subspace. Then there exist  $u, v \in X$  satisfying (05) and

(02). Hence  $y = au + bv, x = cu + dv$  for  $a, b, c, d \in R$ . By Lemma 2.1 (b),  $f(y) = (a^2 + b^2)f(u), f(x) = (c^2 + d^2)f(u)$  so  $f(y) > 0$ . Similarly,  $f(x) < 0$  implies  $f(y) < 0$ . For concreteness, suppose  $f(x) \geq 0$  for all  $x \in X$ . One can now show that  $f(x)^{1/2}$  is a norm on  $X$  which satisfies the parallelogram law so  $X$  is an inner product space. If  $x \perp y$  then  $f(x + y) = f(x) + f(y)$  and so  $\langle x, y \rangle = 0$ . Conversely, suppose  $x, y \neq 0$  and  $\langle x, y \rangle = 0$ . By (04) there is a  $z \neq 0$  in the span of  $\{x, y\}$  such that  $x \perp z$ . Hence  $\langle x, z \rangle = 0$  and by (02)  $y = ax + bz$  for some  $a, b \in R$ . From  $\langle x, y \rangle = 0$  it follows that  $a = 0$  so  $x \perp y$ . Corollary 3.4 concludes the proof.

If  $X$  is a normed linear space, James [1] defines  $x \perp y$  iff  $\|x + ky\| \geq \|x\|$  for all  $k \in R$ . With this definition of  $\perp, (X, \perp)$  is an orthogonality vector space. Indeed, (01), (02), (03) follows easily, (04) follows from [1; Corollary 2.3] and (05) follows from [2; Lemma 1].

The next result generalizes to inner product spaces a result of Sundaresan [2] whose proof relies on the completeness of Hilbert space.

**COROLLARY 2.3.** *Let  $X$  be a normed space and let  $f: X \rightarrow R$  be an orthogonally additive, even, hemicontinuous function. (a) If  $X$  is not an inner product space, then  $f \equiv 0$ . (b) If  $X$  is an inner product space, then there is a  $c \in R$  such that  $f(x) = c \|x\|^2$  for all  $x \in X$ .*

We next prove a generalization of the Riesz representation theorem.

**COROLLARY 2.4.** *Let  $X$  be an inner product space and let  $f: X \rightarrow R$  be orthogonally additive and satisfy  $|f(x)| \leq M \|x\|$  for all  $x \in X$ . Then  $f$  is a continuous linear functional and hence, if  $X$  is a Hilbert space,  $f(x) = \langle x, z \rangle$  for some  $z \in X$ .*

*Proof.* We can assume  $M > 0$ . Clearly  $f$  is continuous at 0. Let  $x \neq 0$ . We first show that  $\beta \rightarrow 1$  implies  $f(\beta x) \rightarrow f(x)$ . Let  $\beta > 1, y \perp x, \|y\| = 1$  and  $u = x + (\beta - 1)^{1/2} \|x\| y$ . Then  $(u - x) \perp x$  and  $(u - \beta x) \perp u$ . Thus  $f(u) - f(x) = f(u - x)$  and  $f(\beta x) - f(u) = f(\beta x - u)$ . Hence

$$\begin{aligned} |f(x) - f(\beta x)| &\leq |f(x) - f(u)| + |f(u) - f(\beta x)| \\ &\leq M \|x\| [2(\beta - 1)^{1/2} + (\beta - 1)]. \end{aligned}$$

Now let  $0 < \beta < 1, y \perp x, \|y\| = 1$  and

$$u = \beta x + (1 - \beta)^{1/2} \beta^{1/2} \|x\| y.$$

Then  $(u - \beta x) \perp \beta x$  and  $(x - u) \perp u$ . Again  $f(u) - f(\beta x) = f(u - \beta x)$ , and  $f(x) - f(u) = f(x - u)$ , so that

$$\begin{aligned} |f(x) - f(\beta x)| &\leq |f(x - u)| + |f(u - \beta x)| \\ &\leq M \|x\| [(1 - \beta) + 2(1 - \beta)^{1/2} \beta^{1/2}]. \end{aligned}$$

It follows that  $f(\beta x) \rightarrow f(x)$  as  $\beta \rightarrow 1$ . We now show that  $f$  is norm continuous. If  $x_i \rightarrow x$ , there exist  $y_i \perp x$  such that  $x_i = \alpha_i x + y_i$ . Taking the inner product with  $x$ , we see that  $\alpha_i \rightarrow 1$  and hence  $y_i \rightarrow 0$ . Since  $f(x_i) = f(\alpha_i x + y_i) = f(\alpha_i x) + f(y_i)$ , we have  $f(x_i) \rightarrow f(x)$  as  $x_i \rightarrow x$  and  $f$  is norm continuous. Applying Corollary 2.3 and Lemma 2.1(a), there is a continuous linear functional  $f_2$  such that  $f(x) = c \|x\|^2 + f_2(x)$ . Hence  $|c| \|x\| \leq M + \|f_2\|$  for all  $x \in X$ , which implies  $c = 0$ .

**3. Orthogonally increasing functions.** In this section orthogonality on a normed space  $X$  will always be defined according to James' definition (see §2). A function  $f: X \rightarrow R$  is *orthogonally increasing* iff  $x \perp y$  implies  $f(x + y) \geq f(x)$ . We shall later define other types of increasing functions.

In the last section we characterized orthogonally additive, hemicontinuous functions. We saw that they formed a very restricted class, being the sum of a linear functional and a constant times the norm squared. The orthogonally increasing functions form a much larger class. Indeed, if  $g: R^+ \rightarrow R$ , where  $R^+ =$  nonnegative reals, is any nondecreasing function then  $f(x) = g(\|x\|)$  is orthogonally increasing since  $x \perp y$  implies  $f(x + y) = g(\|x + y\|) \geq g(\|x\|) = f(x)$ . The main result of this section characterizes orthogonally increasing functions on a normed space and shows that they are essentially of this form.

Let  $X$  be a normed space. A function  $f: X \rightarrow R$  is *radially increasing* if  $\alpha > 1$  implies  $f(\alpha x) \geq f(x) \forall x \in X$ , and  $f$  is *spherically increasing* if  $\|x\| > \|y\|$  implies  $f(x) \geq f(y)$ . It is clear that spherically increasing implies radially increasing and simple examples show that the converse need not hold. In a strictly convex (rotund) normed space, spherically increasing implies orthogonally increasing. Indeed, let  $f$  be a spherically increasing function on such a space and let  $x \perp y$ . Then  $\|x + y\| \geq \|x\|$ . If  $\|x + y\| > \|x\|$ , then by spherical increasing

$$f(x + y) \geq f(x).$$

Now suppose  $\|x + y\| = \|x\|$ . Then

$$\|x + \frac{1}{2}y\| = \|\frac{1}{2}(x + y) + \frac{1}{2}x\| \leq \frac{1}{2}\|x + y\| + \frac{1}{2}\|x\| = \|x\|.$$

Since  $x \perp y$ ,  $\|x + \frac{1}{2}y\| \cong \|x\|$  so  $\|x + \frac{1}{2}y\| = \|x\|$ . But a normed space is strictly convex if and only if  $\|u\| = \|v\| = \|\frac{1}{2}(u + v)\|$  implies  $u = v$ , and so  $\|x + y\| = \|x\| = \|x + \frac{1}{2}y\|$  implies  $y = 0$ . Hence  $f(x + y) \cong f(x)$  and  $f$  is orthogonally increasing. It is well known that any uniformly convex space is strictly convex, in particular an inner product space is strictly convex.

In a general normed space, spherically increasing need not imply orthogonally increasing. Indeed, let  $X = (R^2, \|\cdot\|_\infty)$ ; that is,  $X = R^2$  with  $\|(x_1, x_2)\| = \max(|x_1|, |x_2|)$ . Note that  $X$  is not strictly convex. Let  $f: X \rightarrow R$  be defined as follows:  $f(x) = \|x\|$  if  $0 \leq \|x\| < 1$ ,  $f(x) = 2\|x\|$  if  $\|x\| > 1$ ,  $f(x) = 1$  if  $\|x\| = 1$  and  $x \neq (1, 0)$ , and  $f((1, 0)) = 2$ . It is easy to check that  $f$  is spherically increasing. If  $x = (1, 0)$  and  $y = (0, 1)$  then  $x \perp y$  but  $f(x + y) = f((1, 1)) = 1 < 2 = f(x)$ . Hence  $f$  is not orthogonally increasing. The next theorem shows that orthogonally increasing implies spherically increasing.

**THEOREM 3.1.** *Let  $X$  be a normed space with  $\dim X \geq 2$  and let  $f: X \rightarrow R$  be orthogonally increasing. Then  $f$  is spherically increasing and there exists a countable number of spheres  $S_1, S_2, \dots$  such that  $f$  is norm continuous at  $w$  iff  $w \notin \cup S_i$ . Furthermore, there exists a nondecreasing function  $g: R^+ \rightarrow R$  such that  $f(w) = g(\|w\|)$  for every  $w \notin \cup S_i$ .*

*Proof.* We first show that  $f$  is radially increasing. Let  $0 \neq y \in X$  and let  $\alpha > 1$ . By a modification of the proof of Lemma 1 [2] there exists  $0 \neq x \in X$  such that  $y \perp x$  and  $(y + x) \perp [(\alpha - 1)y - x]$ . Hence

$$f(\alpha y) = f[y + x + (\alpha - 1)y - x] \cong f(y + x) \cong f(y)$$

and  $f$  is radially increasing. We now show that  $f$  is norm continuous on a dense subset of  $X$ . Let  $\|x_0\| = 1$  and let  $V = \{\lambda x_0: \lambda \in R^+\}$ . Then  $f$  restricted to  $V$  is an increasing function and hence is continuous in  $V$  on a dense subset  $B$  of  $V$ . We shall show that  $f$  is norm continuous on  $B - \{0\}$ . Let  $0 \neq x \in B$  and let  $x_i \rightarrow x$ . Now there exists  $y_i$  such that  $x \perp y_i$  and  $x_i = \alpha_i x + y_i$ . Since

$$\|x_i - x\| = \|(\alpha_i - 1)x + y_i\| \cong |\alpha_i - 1| \|x\|$$

we have  $\alpha_i \rightarrow 1$ . By the Hahn-Banach theorem, there exist continuous linear functionals  $f_{x_i}$  on  $X$  such that  $f_{x_i}(x_i) = \|x_i\|^2$  and  $\|f_{x_i}\| = \|x_i\|$ . Now

$$|f_{x_i}(x_i) - f_{x_i}(x)| = |f_{x_i}(x_i - x)| \leq \|x_i\| \|x_i - x\|$$

so  $f_{x_i}(x) \rightarrow \|x\|^2$ . Letting  $k_i = \|x_i\|^2 / f_{x_i}(x)$  we see that  $k_i \rightarrow 1$ . Furthermore, for every  $\alpha \in R$  we have

$$\begin{aligned} \|x_i + \alpha(k_i x - x_i)\| &\geq f_{x_i}[(1 - \alpha)x_i + \alpha k_i x] / \|f_{x_i}\| \\ &= [(1 - \alpha)\|x_i\|^2 + \alpha k_i f_{x_i}(x)] / \|f_{x_i}\| = \|x_i\|. \end{aligned}$$

Hence  $x_i \perp (k_i x - x_i)$ . Thus

$$f(k_i x) = f(x_i + k_i x - x_i) \geq f(x_i) = f(\alpha_i x + y_i) \geq f(\alpha_i x).$$

Since  $\alpha_i, k_i \rightarrow 1$  we have  $f(k_i x), f(\alpha_i x) \rightarrow f(x)$  so  $f(x_i) \rightarrow f(x)$  and  $f$  is norm continuous on a dense subset of  $X$ . We next show that  $f$  is spherically increasing. Let  $x, y \in X$  and suppose  $\|y\| > \|x\|$ . We shall show there exists  $\lambda > 1$  and  $x = x_0, x_1, \dots, x_n \in X$  such that  $y = \lambda x_n$  and  $x_{i-1} \perp (x_i - x_{i-1})$ ,  $i = 1, \dots, n$ . It would then follow that

$$f(y) = f(\lambda x_n) \geq f(x_n) = f(x_{n-1} + x_n - x_{n-1}) \geq f(x_{n-1}) \geq \dots \geq f(x_0) = f(x).$$

To show such  $\lambda$  and  $x_i$  exist we proceed as follows. We can assume without loss of generality that  $\|x\| = 1$ , that  $x$  and  $y$  are linearly independent, and that the 2-dimensional subspace generated by  $\{x, y\}$  is  $R^2$  with  $x = (1, 0)$ . Let  $S$  be the unit sphere in  $R^2$  corresponding to the unit sphere in  $X$ . Since the norm is a convex function, using polar coordinates, we can assume that  $S$  is given by  $\rho = F(\theta)$  where  $F$  is a continuous function on  $[0, 2\pi]$ , which is periodic of period  $\pi$ , the right-hand derivative  $F'$  exists everywhere, and  $F'$  is bounded. Let  $S_0$  be a unit sphere obtained by reflecting  $S$  about the  $x$ -axis. Then, in polar coordinates,  $S_0$  is given by  $\rho_0 = F_0(\theta)$  where  $F_0(\theta) = F(2\pi - \theta)$ . Denote orthogonality with respect to  $S$  and  $S_0$  by  $\perp$  and  $\perp_0$  respectively, and the norm with respect to  $S$  and  $S_0$  by  $\|\cdot\|$  and  $\|\cdot\|_0$  respectively. We now construct a polygonal path  $P$  starting at  $x$  and sweeping twice around the origin with vertices  $x_0 = x, x_1, x_2, \dots, x_{2n}$  as follows. The angle between  $x_{i-1}$  and  $x_i$  is  $2\pi/n$ ,  $x_{i-1} \perp (x_i - x_{i-1})$  for  $i = 1, 2, \dots, n$ , and  $x_{i-1} \perp_0 (x_i - x_{i-1})$  for  $i = n + 1, n + 2, \dots, 2n$ . Now

$$\|x_{2n}\|_0 \geq \|x_{2n-1}\|_0 \geq \dots \geq \|x_n\|_0 = \|x_n\| \geq \|x_{n-1}\| \geq \dots \geq \|x\|.$$

Indeed, since  $x_{2n} = x_{2n-1} + (x_{2n} - x_{2n-1})$  we have  $\|x_{2n}\|_0 \geq \|x_{2n-1}\|_0$  and the others follow in a similar way. Furthermore,  $\|x_n\| \geq \|w\|$  for any  $w \in P$  which precedes  $x_n$ . Indeed, if  $w$  is on the edge with vertices  $x_n$  and  $x_{n-1}$  then  $w = \lambda x_n + (1 - \lambda)x_{n-1}$  for some  $0 \leq \lambda \leq 1$  and hence  $\|w\| \leq \lambda \|x_n\| + (1 - \lambda)\|x_{n-1}\| \leq \|x_n\|$ . A similar argument holds for other  $w \in P$ . Hence, if we can show that  $\lim_{n \rightarrow \infty} \|x_{2n}\|_0 = 1$  we will be finished with this part of the proof. A simple calculation shows that the slope of  $S$  in the forward direction at angle  $\theta$  is

$$[F(\theta) \cos \theta + F'(\theta) \sin \theta] / [F'(\theta) \cos \theta - F(\theta) \sin \theta].$$

Since  $x \perp (x_1 - x)$  it follows that the slope of  $x_1 - x$  equals the slope of  $S$  in the forward direction at  $\theta = 0$ . Letting  $\rho_1$  be the  $\rho$  coordinate of  $x_1$  we have

$$\rho_1 \sin(2\pi/n) / [\rho_1 \cos(2\pi/n) - 1] = [F'(0)]^{-1}.$$

Hence

$$\rho_1 = [\cos(2\pi/n) - F'(0) \sin(2\pi/n)]^{-1}$$

and this formula holds even if  $F'(0) = 0$ . In a similar way, a straightforward calculation gives

$$\rho_i = \rho_{i-1} \{ \cos(2\pi/n) - [F'(2\pi i/n) / F(2\pi i/n)] \sin(2\pi/n) \}^{-1},$$

$i = 2, 3, \dots, n$ . A similar formula holds for  $\rho_{0i}$ ,  $i = n + 1, n + 2, \dots, 2n$ . Using the fact that  $F_0(2\pi i/n) = F[2\pi(n - i)/n]$  and  $F'_0(2\pi i/n) = -F'[2\pi(n - i)/n]$  we obtain

$$\begin{aligned} \rho_{02n} &= \{ \cos^2(2\pi/n) - [F'(0)]^2 \sin^2(2\pi/n) \}^{-1} \\ &\times \{ \cos^2(2\pi/n) - [F'(2\pi/n) / F(2\pi/n)]^2 \sin^2(2\pi/n) \}^{-1} \\ &\times \dots \times \{ \cos^2(2\pi/n) - [F'((n - 1)2\pi/n) / F((n - 1)2\pi/n)]^2 \sin^2(2\pi/n) \}^{-1}. \end{aligned}$$

Letting  $M = \sup[F'(\theta) / F(\theta)]^2$  we have

$$\lim_{n \rightarrow \infty} \rho_{02n} \leq \lim_{n \rightarrow \infty} [\cos^2(2\pi/n) - M \sin^2(2\pi/n)]^{-n}.$$

But L'Hospital's rule shows that

$$\lim_{x \rightarrow 0} (2\pi/x) \log [\cos^2 x - M \sin^2 x] = 0$$

so

$$\lim_{n \rightarrow \infty} \rho_{02n} = 1. \quad \text{Hence} \quad \lim_{n \rightarrow \infty} \|x_{2n}\|_0 = 1.$$

We next show that  $f$  is norm continuous except on a countable set of spheres. Let  $\|x_0\| = 1$ . Then from the above,  $f$  is norm continuous at  $\delta x_0$  except for countably many  $\delta$ 's, say  $\delta_1, \delta_2, \dots$ . Suppose  $f$  is continuous at  $x = \delta x_0$  and  $\|y\| = \|x\|$ . If  $\lambda > 1$  then  $f(\lambda x) \geq f(y)$ , so



letting  $\lambda \rightarrow 1$  we have  $f(x) \geq f(y)$  and in a similar way we show that  $f(x) \leq f(y)$  so  $f(x) = f(y)$ . To show  $f$  is continuous at  $y$ , let  $y_i \rightarrow y$ . As  $\|y\| = \|x\| > 0$ , it is possible, for  $i$  sufficiently large, to find a sequence  $a_i \in R$  such that  $a_i \rightarrow 0$ ,  $a_i > 0$  and  $\|y_i\| - a_i > 0$ . Let  $x_i = (\|y_i\| + a_i)x/\|y\|$  and  $z_i = (\|y_i\| - a_i)x/\|y\|$ . Then  $\|x_i\| > \|y_i\| > \|z_i\|$  so  $f(z_i) \leq f(y_i) \leq f(x_i)$ . Now  $x_i \rightarrow x$ ,  $z_i \rightarrow x$  and since  $f$  is continuous at  $x$  we have  $f(y_i) \rightarrow f(x) = f(y)$ . Hence  $f$  is continuous at  $y$ . If  $S_i = \{x \in X; \|x\| = \delta_i\}$ , it follows that  $f$  is continuous at  $w$  iff  $w \notin \cup S_i$ . Define  $g: R^+ \rightarrow R$  by  $g(\alpha) = f(\alpha x_0)$ . Then  $g$  is a nondecreasing function and if  $w \notin \cup S_i$  we have  $f(w) = f(\|w\| x_0) = g(\|w\|)$ .

Using Theorem 3.1 we can prove a result similar to Corollary 2.3 concerning nonnegative orthogonally additive functions.

**COROLLARY 3.2.** *Let  $X$  be a normed space with  $\dim X \geq 2$  and let  $f: X \rightarrow R^+$  be orthogonally additive. (a) If  $X$  is not an inner product space, then  $f \equiv 0$ . (b) If  $X$  is an inner product space, then there is a  $c \in R^+$  such that  $f(x) = c \|x\|^2$  for all  $x \in X$ .*

In the rest of this section  $X$  will denote an inner product space with  $\dim X \geq 2$  and inner product  $\langle \cdot, \cdot \rangle$ .

**COROLLARY 3.3.** *If  $f: X \rightarrow R^+$  is orthogonally additive, then there is a  $c \in R^+$  with  $f(x) = c \|x\|^2$ .*

**COROLLARY 3.4.** *Let  $\langle \cdot, \cdot \rangle_1$  be another inner product on  $X$ . If  $x \perp y$  implies  $x \perp_1 y$ , then there is a  $c > 0$  such that  $\langle u, v \rangle_1 = c \langle u, v \rangle$  for all  $u, v \in X$ .*

*Proof.* Let  $g(w) = \|w\|_1$ . If  $x \perp y$  then  $x \perp_1 y$  so  $g^2(x+y) = g^2(x) + g^2(y)$ . Hence  $g^2$  is orthogonally additive so there is a  $c > 0$  with  $\|w\|_1 = g(w) = c \|w\|$ . Hence

$$\begin{aligned} \langle u, v \rangle_1 &= [ \|u+v\|_1^2 - \|u-v\|_1^2 ] / 4 = c^2 [ \|u+v\|^2 - \|u-v\|^2 ] / 4 \\ &= c^2 \langle u, v \rangle. \end{aligned}$$

**COROLLARY 3.5.** *If  $f: X \rightarrow R$  is orthogonally additive and  $f(x) \geq -M \|x\|^2$  for all  $x \in X$  for some  $M \geq 0$ , then there is an  $\alpha \in R$  such that  $f(x) = \alpha \|x\|^2$ .*

*Proof.* If  $g(x) = f(x) + M \|x\|^2$ , then  $g: X \rightarrow R^+$  is orthogonally additive. Hence there is a  $c \geq 0$  such that  $g(x) = c \|x\|^2$ . Hence  $f(x) = (c - M) \|x\|^2$ .

In a similar way, Corollary 3.5 holds if  $f(x) \leq M \|x\|^2$ , for all  $x \in X$ .

Let  $x_0 \in X$ ,  $c, d \in \mathbb{R}^+$  and define  $f(x) = c \|x - x_0\|^2 + d$ . Then  $f(x) \geq f(x_0)$  and if  $x \perp y$  we have

$$\begin{aligned} f(x + y) &= c \|x - x_0\|^2 - 2c \langle y, x_0 \rangle + c \|y\|^2 + d \\ &= c \|x - x_0\|^2 + c \|y - x_0\|^2 - c \|x_0\|^2 + d \\ &= f(x) + f(y) - d - c \|x_0\|^2 = f(x) + f(y) - f(0). \end{aligned}$$

We now show that the converse holds.

**COROLLARY 3.6.** *Let  $f: X \rightarrow \mathbb{R}$  satisfy: (a) there is an  $x_0 \in X$  such that  $f(x) \geq f(x_0)$  for all  $x \in X$ , (b) if  $x \perp y$  then*

$$f(x + y) = f(x) + f(y) - f(0).$$

*Then is a  $c \geq 0$  such that  $f(x) = c \|x - x_0\|^2 + f(x_0)$  and if  $c \neq 0$ ,  $x_0$  is unique.*

*Proof.* Let  $g(x) = f(x + x_0) - f(x_0)$ . Then  $g: X \rightarrow \mathbb{R}^+$ . Let  $x \perp y$  and write  $x = x_1 + x_2 + x_3$  where  $x_1$  is a multiple of  $x$ ,  $x_2$  is a multiple of  $y$  and  $x_3$  is orthogonal to  $x$  and  $y$ . Then  $g(x + y) = g(x) + g(y)$ . Hence  $g(x) = c \|x\|^2$  for some  $c \geq 0$  and  $f(x + x_0) = c \|x\|^2 + f(x_0)$ . Hence  $f(x) = c \|x - x_0\|^2 + f(x_0)$ . If  $c \neq 0$  and  $f(x) \geq f(y_0)$  for all  $x \in X$  then  $f(y_0) = f(x_0)$  and  $f(y_0) = c \|y_0 - x_0\|^2 + f(x_0)$ . Thus  $\|y_0 - x_0\| = 0$  so  $y_0 = x_0$ .

**COROLLARY 3.7.** *Let  $f: X \rightarrow \mathbb{R}$  be orthogonally additive. If there is an  $x_0 \in X$  such that  $f(x_0) = \|x_0\|^2$  and  $|f(x)| \leq \|x_0\| \|x\|$  for all  $x \in X$ , then  $f(x) = \langle x, x_0 \rangle$  for all  $x \in X$ .*

*Proof.* Let  $g(x) = \|x\|^2 - 2f(x) + \|x_0\|^2$ . Then

$$g(x) \geq \|x\|^2 - 2\|x\| \|x_0\| + \|x_0\|^2 = (\|x\| + \|x_0\|)^2 \geq 0 = g(x_0).$$

Also  $x \perp y$  implies  $g(x + y) = g(x) + g(y) - g(0)$ . Hence by Corollary 3.6 there is a  $c \geq 0$  such that  $g(x) = c \|x - x_0\|^2$ . Therefore

$$\begin{aligned} 2f(x) &= \|x\|^2 + \|x_0\|^2 - c \|x - x_0\|^2 = (1 - c) \|x\|^2 \\ &\quad + (1 - c) \|x_0\|^2 + 2c \langle x, x_0 \rangle. \end{aligned}$$

Since  $f(0) = 0$  we have  $(1 - c) \|x_0\|^2 = 0$ . Thus either  $c = 1$  or  $x_0 = 0$ . If  $x_0 = 0$  then  $|1 - c| \|x\|^2 = 2|f(x)| \leq 0$  for all  $x \in X$  so again  $c = 1$ . Hence  $f(x) = \langle x, x_0 \rangle$ .

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#### REFERENCES

1. R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc., **61** (1947), 265–292.
2. K. Sundaresan, *Orthogonality and nonlinear functionals on Banach spaces*, Proc. Amer. Math. Soc., **34** (1972), 187–190.

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