Pacific Journal of Mathematics

ORTHOGONALLY ADDITIVE AND ORTHOGONALLY INCREASING FUNCTIONS ON VECTOR SPACES

STANLEY P. GUDDER AND D. STRAWTHER

Vol. 58, No. 2 April 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

ORTHOGONALLY ADDITIVE AND ORTHOGONALLY INCREASING FUNCTIONS ON VECTOR SPACES

S. GUDDER AND D. STRAWTHER

A real-valued function $f\colon X \to R$ on an inner product space X is *orthogonally additive* if $f(x+y)=f(x)+f(y)$ whenever $x \perp y$. We extend this concept to more general spaces called orthogonality vector spaces. If X is an orthogonality vector space and if there exists an orthogonally additive function on X which satisfies certain natural conditions then there is an inner product on X which is equivalent to the original orthogonality and $f(x) = \pm \|x\|^2$ for all $x \in X$. We next consider a normed space X with James' orthogonality. A function $f\colon X \to R$ is *orthogonally increasing* if $f(x+y) \geqq f(x)$ whenever $x \perp y$. Orthogonally increasing functions on normed spaces are characterized.

1. Pythagoras' theorem. Pythagoras' theorem states that the function $f(x)=\|x\|^2$ is orthogonally additive, that is $f(x+y)=f(x)+f(y)$ whenever $x \perp y$ where x, y are vectors in the plane. One of the concerns of this paper is a converse of Pythagoras' theorem on an inner product space X. That is, if $f\colon X \to R$ is orthogonally additive, is $f(x)=c\|x\|^2$ for some $c \in R$? As it stands, the answer is no, since any linear functional is orthogonally additive.

Some natural additional conditions on f are:

(1) $f(x) \geqq 0$, *nonnegativity*;

(2) $f(x)=f(-x)$, *evenness*;

(3) $\lambda_i \to \lambda$ implies $f(\lambda_i x) \to f(\lambda x)$ for all $x \in X$, *hemicontinuity*.

We shall show that orthogonal additivity along with (1), or with (2) and (3) imply $f(x)=c\|x\|^2$ for some $c \in R$.

2. Orthogonality vector spaces. In this paper, vector spaces will be real and of dimension $\geqq 2$. In Theorem 2.2 we shall prove that Pythagoras' theorem characterizes inner product spaces in a certain sense.

A vector space X is an *orthogonality vector space* if there is a relation $x \perp y$ on X such that

(01) $x \perp 0$, $0 \perp x$ for all $x \in X$;

(02) if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(03) if $x \perp y$, then $ax \perp by$ for all $a, b \in R$;

(04) if P is a two-dimensional subspace of X, then for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;

(05) if P is a two-dimensional subspace of X, then there exist nonzero vectors $u, v \in P$ such that $u \perp v$ and $u+v \perp u-v$.

Any vector space can be made into an orthogonality vector space if we define $x \perp 0$, $0 \perp x$ for all x, and for nonzero vectors x, y define $x \perp y$ iff x, y are linearly independent. Also an inner product space is such a space; we shall see that a normed space is one also with James' definition of orthogonality.

LEMMA 2.1. *Let $(X, \perp)$ be an orthogonality vector space and let $f\colon X \to R$ be orthogonally additive and hemi-continuous.* (a) *If f is odd, then f is linear.* (b) *If f is even, then $f(\alpha x) = \alpha^2 f(x)$ for all $\alpha \in R$, $x \in X$ and if $x \perp y$ and $x+y \perp x-y$, then $f(x) = f(y)$.*

Proof. Same as in [**2**; Lemmas 2, 3].

REMARK. The referee has pointed out to us that there is a mistake in the proof of Lemma 2 [**2**]. In that proof it is incorrectly stated that $F(2^r u) = 2^r F(u)$ for all rational r when, in fact, this is only proved for integral r. However, it is easily seen that $F(3^r u) = 3^r F(u)$ for all integral r. Indeed, in the notation of that proof

$$F(3u) - F(v) = F(3u - v) = F(u + v + 2u - 2v)$$
$$= F(u+v) + F(2(u-v)) = 3F(u) - F(v).$$

Hence, by induction $F(2^p 3^q u) = 2^p 3^q F(u)$ for all integral p and q. Since these scalars $2^p 3^q$ are dense, continuity implies $F(\alpha u) = \alpha F(u)$.

An inner product $\langle \cdot, \cdot \rangle$ on $(X, \perp)$ is $\perp$-*equivalent* when $x \perp y$ iff $\langle x, y \rangle = 0$.

THEOREM 2.2. *If there exists an $f\colon (X, \perp) \to R$ which is orthogonally additive, even, hemicontinuous, and not identically* 0, *then there is a $\perp$-equivalent inner product $\langle \cdot, \cdot \rangle$ on $(X, \perp)$. In fact, $\langle x, y \rangle = \frac{1}{4}[f(x+y) - f(x-y)]$ and the induced norm satisfies $\|x\|^2 = f(x)$ for all $x \in X$, or $\|x\|^2 = -f(x)$ for all $x \in X$. Moreover, if $\langle \cdot, \cdot \rangle_1$ is another $\perp$-equivalent inner product on $(X, \perp)$, then there is a nonzero $c \in R$ such that $\langle \cdot, \cdot \rangle_1 = c\langle \cdot, \cdot \rangle$.*

Proof. We first show that f has constant sign. Let $0 \neq x \in X$ and suppose $f(x) > 0$. Let $0 \neq y \in X$. If $y = \alpha x$, then $f(y) = \alpha^2 f(x) > 0$. If y, x are linearly independent, let P be the generated 2-dimensional subspace. Then there exist $u, v \in X$ satisfying (05) and

(02). Hence $y = au + bv$, $x = cu + dv$ for $a, b, c, d \in R$. By Lemma 2.1 (b), $f(y) = (a^2 + b^2)f(u)$, $f(x) = (c^2 + d^2)f(u)$ so $f(y) > 0$. Similarly, $f(x) < 0$ implies $f(y) < 0$. For concreteness, suppose $f(x) \geqq 0$ for all $x \in X$. One can now show that $f(x)^{1/2}$ is a norm on X which satisfies the parallelogram law so X is an inner product space. If $x \perp y$ then $f(x+y) = f(x) + f(y)$ and so $\langle x, y\rangle = 0$. Conversely, suppose $x, y \neq 0$ and $\langle x, y\rangle = 0$. By (04) there is a $z \neq 0$ in the span of $\{x, y\}$ such that $x \perp z$. Hence $\langle x, z\rangle = 0$ and by (02) $y = ax + bz$ for some $a, b \in R$. From $\langle x, y\rangle = 0$ it follows that $a = 0$ so $x \perp y$. Corollary 3.4 concludes the proof.

If X is a normed linear space, James **[1]** defines $x \perp y$ iff $\|x + ky\| \geqq \|x\|$ for all $k \in R$. With this definition of $\perp$, $(X, \perp)$ is an orthogonality vector space. Indeed, (01), (02), (03) follows easily, (04) follows from [**1**; Corollary 2.3] and (05) follows from [**2**; Lemma 1].

The next result generalizes to inner product spaces a result of Sundaresan **[2]** whose proof relies on the completeness of Hilbert space.

COROLLARY 2.3. *Let X be a normed space and let $f: X \to R$ be an orthogonally additive, even, hemicontinuous function.* (a) *If X is not an inner product space, then $f \equiv 0$.* (b) *If X is an inner product space, then there is a $c \in R$ such that $f(x) = c\|x\|^2$ for all $x \in X$.*

We next prove a generalization of the Riesz representation theorem.

COROLLARY 2.4. *Let X be an inner product space and let $f: X \to R$ be orthogonally additive and satisfy $|f(x)| \leqq M\|x\|$ for all $x \in X$. Then f is a continuous linear functional and hence, if X is a Hilbert space, $f(x) = \langle x, z\rangle$ for some $z \in X$.*

Proof. We can assume $M > 0$. Clearly f is continuous at 0. Let $x \neq 0$. We first show that $\beta \to 1$ implies $f(\beta x) \to f(x)$. Let $\beta > 1$, $y \perp x$, $\|y\| = 1$ and $u = x + (\beta - 1)^{1/2}\|x\| y$. Then $(u - x) \perp x$ and $(u - \beta x) \perp u$. Thus $f(u) - f(x) = f(u - x)$ and $f(\beta x) - f(u) = f(\beta x - u)$. Hence

$$\begin{aligned} |f(x) - f(\beta x)| &\leqq |f(x) - f(u)| + |f(u) - f(\beta x)| \\ &\leqq M\|x\|[2(\beta - 1)^{1/2} + (\beta - 1)]. \end{aligned}$$

Now let $0 < \beta < 1$, $y \perp x$, $\|y\| = 1$ and

$$u = \beta x + (1 - \beta)^{1/2}\beta^{1/2}\|x\| y.$$

Then $(u-\beta x)\perp\beta x$ and $(x-u)\perp u$. Again $f(u)-f(\beta x)=f(u-\beta x)$, and $f(x)-f(u)=f(x-u)$, so that

$$\begin{aligned}|f(x)-f(\beta x)| &\leqq |f(x-u)|+|f(u-\beta x)|\\ &\leqq M\|x\|[(1-\beta)+2(1-\beta)^{1/2}\beta^{1/2}].\end{aligned}$$

It follows that $f(\beta x)\to f(x)$ as $\beta\to 1$. We now show that f is norm continuous. If $x_i\to x$, there exist $y_i\perp x$ such that $x_i=\alpha_i x+y_i$. Taking the inner product with x, we see that $\alpha_i\to 1$ and hence $y_i\to 0$. Since $f(x_i)=f(\alpha_i x+y_i)=f(\alpha_i x)+f(y)$, we have $f(x_i)\to f(x)$ as $x_i\to x$ and f is norm continuous. Applying Corollary 2.3 and Lemma 2.1(a), there is a continuous linear functional f_2 such that $f(x)=c\|x\|^2+f_2(x)$. Hence $|c|\|x\|\leqq M+\|f_2\|$ for all $x\in X$, which implies $c=0$.

3. Orthogonally increasing functions. In this section orthogonality on a normed space X will always be defined according to James' definition (see §2). A function $f\colon X\to R$ is *orthogonally increasing* iff $x\perp y$ implies $f(x+y)\geqq f(x)$. We shall later define other types of increasing functions.

In the last section we characterized orthogonally additive, hemicontinuous functions. We saw that they formed a very restricted class, being the sum of a linear functional and a constant times the norm squared. The orthogonally increasing functions form a much larger class. Indeed, if $g\colon R^+\to R$, where $R^+=$ nonnegative reals, is any nondecreasing function then $f(x)=g(\|x\|)$ is orthogonally increasing since $x\perp y$ implies $f(x+y)=g(\|x+y\|)\geqq g(\|x\|)=f(x)$. The main result of this section characterizes orthogonally increasing functions on a normed space and shows that they are essentially of this form.

Let X be a normed space. A function $f\colon X\to R$ is *radially increasing* if $\alpha>1$ implies $f(\alpha x)\geqq f(x)\,\forall x\in X$, and f is *spherically increasing* if $\|x\|>\|y\|$ implies $f(x)\geqq f(y)$. It is clear that spherically increasing implies radially increasing and simple examples show that the converse need not hold. In a strictly convex (rotund) normed space, spherically increasing implies orthogonally increasing. Indeed, let f be a spherically increasing function on such a space and let $x\perp y$. Then $\|x+y\|\geqq\|x\|$. If $\|x+y\|>\|x\|$, then by spherical increasing

$$f(x+y)\geqq f(x).$$

Now suppose $\|x+y\|=\|x\|$. Then

$$\|x+\tfrac{1}{2}y\|=\|\tfrac{1}{2}(x+y)+\tfrac{1}{2}x\|\leqq\tfrac{1}{2}\|x+y\|+\tfrac{1}{2}\|x\|=\|x\|.$$

Since $x \perp y$, $\|x + \frac{1}{2}y\| \geqq \|x\|$ so $\|x + \frac{1}{2}y\| = \|x\|$. But a normed space is strictly convex if and only if $\|u\| = \|v\| = \|\frac{1}{2}(u+v)\|$ implies $u = v$, and so $\|x+y\| = \|x\| = \|x + \frac{1}{2}y\|$ implies $y = 0$. Hence $f(x+y) \geqq f(x)$ and f is orthogonally increasing. It is well known that any uniformly convex space is strictly convex, in particular an inner product space is strictly convex.

In a general normed space, spherically increasing need not imply orthogonally increasing. Indeed, let $X = (R^2, \|\cdot\|_\infty)$; that is, $X = R^2$ with $\|(x_1, x_2)\| = \max(|x_1|, |x_2|)$. Note that X is not strictly convex. Let $f\colon X \to R$ be defined as follows: $f(x) = \|x\|$ if $0 \leqq \|x\| < 1$, $f(x) = 2\|x\|$ if $\|x\| > 1$, $f(x) = 1$ if $\|x\| = 1$ and $x \neq (1, 0)$, and $f((1,0)) = 2$. It is easy to check that f is spherically increasing. If $x = (1, 0)$ and $y = (0, 1)$ then $x \perp y$ but $f(x+y) = f((1,1)) = 1 < 2 = f(x)$. Hence f is not orthogonally increasing. The next theorem shows that orthogonally increasing implies spherically increasing.

THEOREM 3.1. *Let X be a normed space with* $\dim X \geqq 2$ *and let $f\colon X \to R$ be orthogonally increasing. Then f is spherically increasing and there exists a countable number of spheres $S_1, S_2, \cdots$ such that f is norm continuous at w iff $w \notin \cup S_i$. Furthermore, there exists a nondecreasing function $g\colon R^+ \to R$ such that $f(w) = g(\|w\|)$ for every $w \notin \cup S_i$.*

Proof. We first show that f is radially increasing. Let $0 \neq y \in X$ and let $\alpha > 1$. By a modification of the proof of Lemma 1 **[2]** there exists $0 \neq x \in X$ such that $y \perp x$ and $(y + x) \perp [(\alpha - 1)y - x]$. Hence

$$f(\alpha y) = f[y + x + (\alpha - 1)y - x] \geqq f(y + x) \geqq f(y)$$

and f is radially increasing. We now show that f is norm continuous on a dense subset of X. Let $\|x_0\| = 1$ and let $V = \{\lambda x_0 \colon \lambda \in R^+\}$. Then f restricted to V is an increasing function and hence is continuous in V on a dense subset B of V. We shall show that f is norm continuous on $B - \{0\}$. Let $0 \neq x \in B$ and let $x_i \to x$. Now there exists y_i such that $x \perp y_i$ and $x_i = \alpha_i x + y_i$. Since

$$\|x_i - x\| = \|(\alpha_i - 1)x + y_i\| \geqq |\alpha_i - 1|\,\|x\|$$

we have $\alpha_i \to 1$. By the Hahn-Banach theorem, there exist continuous linear functionals f_{x_i} on X such that $f_{x_i}(x_i) = \|x_i\|^2$ and $\|f_{x_i}\| = \|x_i\|$. Now

$$|f_{x_i}(x_i) - f_{x_i}(x)| = |f_{x_i}(x_i - x)| \leqq \|x_i\|\,\|x_i - x\|$$

so $f_{x_i}(x) \to \|x\|^2$. Letting $k_i = \|x_i\|^2 / f_{x_i}(x)$ we see that $k_i \to 1$. Furthermore, for every $\alpha \in R$ we have

$$\|x_i + \alpha(k_i x - x_i)\| \geqq f_{x_i}[(1-\alpha)x_i + \alpha k_i x]/\|f_{x_i}\|$$
$$= [(1-\alpha)\|x_i\|^2 + \alpha k_i f_{x_i}(x)]/\|f_{x_i}\| = \|x_i\|.$$

Hence $x_i \perp (k_i x - x_i)$. Thus

$$f(k_i x) = f(x_i + k_i x - x_i) \geqq f(x_i) = f(\alpha_i x + y_i) \geqq f(\alpha_i x).$$

Since α_i, $k_i \to 1$ we have $f(k_i x)$, $f(\alpha_i x) \to f(x)$ so $f(x_i) \to f(x)$ and f is norm continuous on a dense subset of X. We next show that f is spherically increasing. Let $x, y \in X$ and suppose $\|y\| > \|x\|$. We shall show there exists $\lambda > 1$ and $x = x_0, x_1, \cdots, x_n \in X$ such that $y = \lambda x_n$ and $x_{i-1} \perp (x_i - x_{i-1})$, $i = 1, \cdots, n$. It would then follow that

$$f(y) = f(\lambda x_n) \geqq f(x_n) = f(x_{n-1} + x_n - x_{n-1}) \geqq f(x_{n-1}) \geqq \cdots \geqq f(x_0) = f(x).$$

To show such λ and x_i exist we proceed as follows. We can assume without loss of generality that $\|x\| = 1$, that x and y are linearly independent, and that the 2-dimensional subspace generated by $\{x, y\}$ is R^2 with $x = (1, 0)$. Let S be the unit sphere in R^2 corresponding to the unit sphere in X. Since the norm is a convex function, using polar coordinates, we can assume that S is given by $\rho = F(\theta)$ where F is a continuous function on $[0, 2\pi]$, which is periodic of period π, the right-hand derivative F' exists everywhere, and F' is bounded. Let S_0 be a unit sphere obtained by reflecting S about the x-axis. Then, in polar coordinates, S_0 is given by $\rho_0 = F_0(\theta)$ where $F_0(\theta) = F(2\pi - \theta)$. Denote orthogonality with respect to S and S_0 by $\perp$ and $\perp_0$ respectively, and the norm with respect to S and S_0 by $\|\cdot\|$ and $\|\cdot\|_0$ respectively. We now construct a polygonal path P starting at x and sweeping twice around the origin with vertices $x_0 = x, x_1, x_2, \cdots, x_{2n}$ as follows. The angle between x_{i-1} and x_i is $2\pi/n$, $x_{i-1} \perp (x_i - x_{i-1})$ for $i = 1, 2, \cdots, n$, and $x_{i-1} \perp_0 (x_i - x_{i-1})$ for $i = n+1, n+2, \cdots, 2n$. Now

$$\|x_{2n}\|_0 \geqq \|x_{2n-1}\|_0 \geqq \cdots \geqq \|x_n\|_0 = \|x_n\| \geqq \|x_{n-1}\| \geqq \cdots \geqq \|x\|.$$

Indeed, since $x_{2n} = x_{2n-1} + (x_{2n} - x_{2n-1})$ we have $\|x_{2n}\|_0 \geqq \|x_{2n-1}\|_0$ and the others follow in a similar way. Furthermore, $\|x_n\| \geqq \|w\|$ for any $w \in P$ which precedes x_n. Indeed, if w is on the edge with vertices x_n and x_{n-1} then $w = \lambda x_n + (1-\lambda)x_{n-1}$ for some $0 \leqq \lambda \leqq 1$ and hence $\|w\| \leqq \lambda\|x_n\| + (1-\lambda)\|x_{n-1}\| \leqq \|x_n\|$. A similar argument holds for other $w \in P$. Hence, if we can show that $\lim_{n\to\infty} \|x_{2n}\|_0 = 1$ we will be finished with this part of the proof. A simple calculation shows that the slope of S in the forward direction at angle θ is

$$[F(\theta)\cos\theta + F'(\theta)\sin\theta]/[F'(\theta)\cos\theta - F(\theta)\sin\theta].$$

Since $x \perp (x_1 - x)$ it follows that the slope of $x_1 - x$ equals the slope of S in the forward direction at $\theta = 0$. Letting ρ_1 be the ρ coordinate of x_1 we have

$$\rho_1 \sin(2\pi/n)/[\rho_1 \cos(2\pi/n) - 1] = [F'(0)]^{-1}.$$

Hence

$$\rho_1 = [\cos(2\pi/n) - F'(0)\sin(2\pi/n)]^{-1}$$

and this formula holds even if $F'(0) = 0$. In a similar way, a straightforward calculation gives

$$\rho_i = \rho_{i-1}\{\cos(2\pi/n) - [F'(2\pi i/n)/F(2\pi i/n)]\sin(2\pi/n)\}^{-1},$$

$i = 2, 3, \cdots, n$. A similar formula holds for ρ_{0i}, $i = n+1, n+2, \cdots, 2n$. Using the fact that $F_0(2\pi i/n) = F[2\pi(n-i)/n]$ and $F_0'(2\pi i/n) = -F'[2\pi(n-i)/n]$ we obtain

$$\begin{aligned}\rho_{02n} = {} & \{\cos^2(2\pi/n) - [F'(0)]^2\sin^2(2\pi/n)\}^{-1} \\ & \times \{\cos^2(2\pi/n) - [F'(2\pi/n)/F(2\pi/n)]^2\sin^2(2\pi/n)\}^{-1} \\ & \times \cdots \times \{\cos^2(2\pi/n) - [F'((n-1)2\pi/n)/F((n \\ & \qquad - 1)2\pi/n)]^2\sin^2(2\pi/n)\}^{-1}.\end{aligned}$$

Letting $M = \sup[F'(\theta)/F(\theta)]^2$ we have

$$\lim_{n\to\infty} \rho_{02n} \leqq \lim_{n\to\infty} [\cos^2(2\pi/n) - M\sin^2(2\pi/n)]^{-n}.$$

But L'Hospital's rule shows that

$$\lim_{x\to 0} (2\pi/x)\log[\cos^2 x - M\sin^2 x] = 0$$

so

$$\lim_{n\to\infty} \rho_{02n} = 1. \quad \text{Hence} \quad \lim_{n\to\infty} \|x_{2n}\|_0 = 1.$$

We next show that f is norm continuous except on a countable set of spheres. Let $\|x_0\| = 1$. Then from the above, f is norm continuous at δx_0 except for countably many δ's, say $\delta_1, \delta_2, \cdots$. Suppose f is continuous at $x = \delta x_0$ and $\|y\| = \|x\|$. If $\lambda > 1$ then $f(\lambda x) \geqq f(y)$, so letting $\lambda \to 1$ we have $f(x) \geqq f(y)$ and in a similar way we show that $f(x) \geqq f(y)$ so $f(x) = f(y)$. To show f is continuous at y, let $y_i \to y$. As $\|y\| = \|x\| > 0$, it is possible, for i sufficiently large, to find a sequence $a_i \in R$ such that $a_i \to 0$, $a_i > 0$ and $\|y_i\| - a_i > 0$. Let $x_i = (\|y_i\| + a_i)x/\|y\|$ and $z_i = (\|y_i\| - a_i)x/\|y\|$. Then $\|x_i\| > \|y_i\| > \|z_i\|$ so $f(z_i) \leqq f(y_i) \leqq f(x_i)$. Now $x_i \to x$, $z_i \to x$ and since f is continuous at x we have $f(y_i) \to f(x) = f(y)$. Hence f is continuous at y. If $S_i = \{x \in X; \|x\| = \delta_i\}$, it follows that f is continuous at w iff $w \notin \cup S_i$. Define $g: R^+ \to R$ by $g(\alpha) = f(\alpha x_0)$. Then g is a nondecreasing function and if $w \notin \cup S_i$ we have $f(w) = f(\|w\| x_0) = g(\|w\|)$.

Using Theorem 3.1 we can prove a result similar to Corollary 2.3 concerning nonnegative orthogonally additive functions.

COROLLARY 3.2. *Let X be a normed space with* $\dim X \geqq 2$ *and let $f: X \to R^+$ be orthogonally additive.* (a) *If X is not an inner product space, then $f \equiv 0$.* (b) *If X is an inner product space, then there is a $c \in R^+$ such that $f(x) = c\|x\|^2$ for all $x \in X$.*

In the rest of this section X will denote an inner product space with $\dim X \geqq 2$ and inner product $\langle \cdot, \cdot \rangle$.

COROLLARY 3.3. *If $f: X \to R^+$ is orthogonally additive, then there is a $c \in R^+$ with $f(x) = c\|x\|^2$.*

COROLLARY 3.4. *Let $\langle \cdot, \cdot \rangle_1$ be another inner product on X. If $x \perp y$ implies $x \perp_1 y$, then there is a $c > 0$ such that $\langle u, v \rangle_1 = c\langle u, v \rangle$ for all $u, v \in X$.*

Proof. Let $g(w) = \|w\|_1$. If $x \perp y$ then $x \perp_1 y$ so $g^2(x+y) = g^2(x) + g^2(y)$. Hence g^2 is orthogonally additive so there is a $c > 0$ with $\|w\|_1 = g(w) = c\|w\|$. Hence

$$\begin{aligned}\langle u, v \rangle_1 &= [\|u+v\|_1^2 - \|u-v\|_1^2]/4 = c^2[\|u+v\|^2 - \|u-v\|^2]/4 \\ &= c^2\langle u, v \rangle.\end{aligned}$$

COROLLARY 3.5. *If $f: X \to R$ is orthogonally additive and $f(x) \geqq -M\|x\|^2$ for all $x \in X$ for some $M \geqq 0$, then there is an $\alpha \in R$ such that $f(x) = \alpha\|x\|^2$.*

Proof. If $g(x) = f(x) + M\|x\|^2$, then $g: X \to R^+$ is orthogonally additive. Hence there is a $c \geqq 0$ such that $g(x) = c\|x\|^2$. Hence $f(x) = (c - M)\|x\|^2$.

In a similar way, Corollary 3.5 holds if $f(x) \leqq M\|x\|^2$, for all $x \in X$.

Let $x_0 \in X$, c, $d \in R^+$ and define $f(x) = c\,\|x - x_0\|^2 + d$. Then $f(x) \geqq f(x_0)$ and if $x \perp y$ we have

$$\begin{aligned} f(x+y) &= c\,\|x - x_0\|^2 - 2c\langle y, x_0\rangle + c\,\|y\|^2 + d \\ &= c\,\|x - x_0\|^2 + c\,\|y - x_0\|^2 - c\,\|x_0\|^2 + d \\ &= f(x) + f(y) - d - c\,\|x_0\|^2 = f(x) + f(y) - f(0). \end{aligned}$$

We now show that the converse holds.

COROLLARY 3.6. *Let $f\colon X \to R$ satisfy*: (a) *there is an $x_0 \in X$ such that $f(x) \geqq f(x_0)$ for all $x \in X$,* (b) *if $x \perp y$ then*

$$f(x+y) = f(x) + f(y) - f(0).$$

Then is a $c \geqq 0$ such that $f(x) = c\,\|x - x_0\|^2 + f(x_0)$ and if $c \neq 0$, x_0 is unique.

Proof. Let $g(x) = f(x + x_0) - f(x_0)$. Then $g\colon X \to R^+$. Let $x \perp y$ and write $x = x_1 + x_2 + x_3$ where x_1 is a multiple of x, x_2 is a multiple of y and x_3 is orthogonal to x and y. Then $g(x+y) = g(x) + g(y)$. Hence $g(x) = c\,\|x\|^2$ for some $c \geqq 0$ and $f(x + x_0) = c\,\|x\|^2 + f(x_0)$. Hence $f(x) = c\,\|x - x_0\|^2 + f(x_0)$. If $c \neq 0$ and $f(x) \geqq f(y_0)$ for all $x \in X$ then $f(y_0) = f(x_0)$ and $f(y_0) = c\,\|y_0 - x_0\|^2 + f(x_0)$. Thus $\|y_0 - x_0\| = 0$ so $y_0 = x_0$.

COROLLARY 3.7. *Let $f\colon X \to R$ be orthogonally additive. If there is an $x_0 \in X$ such that $f(x_0) = \|x_0\|^2$ and $|f(x)| \leqq \|x_0\|\,\|x\|$ for all $x \in X$, then $f(x) = \langle x, x_0\rangle$ for all $x \in X$.*

Proof. Let $g(x) = \|x\|^2 - 2f(x) + \|x_0\|^2$. Then

$$g(x) \geqq \|x\|^2 - 2\|x\|\,\|x_0\| + \|x_0\|^2 = (\|x\| + \|x_0\|)^2 \geqq 0 = g(x_0).$$

Also $x \perp y$ implies $g(x+y) = g(x) + g(y) - g(0)$. Hence by Corollary 3.6 there is a $c \geqq 0$ such that $g(x) = c\,\|x - x_0\|^2$. Therefore

$$\begin{aligned} 2f(x) = \|x\|^2 + \|x_0\|^2 - c\,\|x - x_0\|^2 &= (1-c)\,\|x\|^2 \\ &+ (1-c)\,\|x_0\|^2 + 2c\langle x, x_0\rangle. \end{aligned}$$

Since $f(0) = 0$ we have $(1-c)\,\|x_0\|^2 = 0$. Thus either $c = 1$ or $x_0 = 0$. If $x_0 = 0$ then $|1 - c|\,\|x\|^2 = 2|f(x)| \leqq 0$ for all $x \in X$ so again $c = 1$. Hence $f(x) = \langle x, x_0\rangle$.

ACKNOWLEDGMENT. The authors would like to thank the referee whose comments helped to improve this paper and generalize some of the results.

REFERENCES

1. R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc., **61** (1947), 265–292.

2. K. Sundaresan, *Orthogonality and nonlinear functionals on Banach spaces*, Proc. Amer. Math. Soc., **34** (1972), 187–190.

Received March 28, 1974 and in revised form December 26, 1974.

UNIVERSITY OF DENVER

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics

Vol. 58, No. 2 April, 1975