

Pacific Journal of Mathematics

MOMENT SEQUENCES IN l^p

J. BOCKETT HUNTER

MOMENT SEQUENCES IN l^p

J. BOCKETT HUNTER

Let $p > 0$. Conditions are derived, each necessary and sufficient, for a moment sequence to be in l^p . It is shown that the moment sequences in l^p are dense in l^p . For $p = 2$, these results were obtained by G. G. Johnson.

G. G. Johnson obtained a necessary and sufficient condition for a moment sequence to be in l^2 , and showed that the moment sequences in l^2 are dense in l^2 . This paper shows that the same conclusions hold in any l^p space. The proofs are similar to and improvements of those in G. G. Johnson, Pacific J. Math., 46(1973), 201–207.

LEMMA 1. Let $0 < p < 1$, $q > 0$. If $a_n = 1 - (n + 1)^{-p}$, then $\{a_n^n\} \in l^q$.

Proof. $a_n^{nq} = \exp(qn \log(1 - (n + 1)^{-p})) < \exp(qn(-(n + 1)^{-p})) = (\exp(qn(n + 1)^{-p}))^{-1} < [\sum_{k=0}^N (qn(n + 1)^{-p})^k/k!]^{-1}$, where N satisfies $N(1 - p) > 1$. Then

$$\sum_{n=1}^{\infty} a_n^{nq} < \sum_{n=1}^{\infty} [(qn(n + 1)^{-p})^N/N!]^{-1} = N! q^{-N} \sum_{n=1}^{\infty} [(n + 1)^p/n]^N,$$

which converges if and only if $\sum_{n=1}^{\infty} n^{-(1-p)N}$ converges, and the latter is a convergent p -series.

THEOREM 1. Let $p > 0$, $f \in BV[0, 1]$, $\mu_n = \int_0^1 t^n df$. For each $\{a_n\}$ such that $0 \leq a_n < 1$, and $\{a_n^n\} \in l^p$, the following are equivalent.

- (i) $\{\mu_n\} \in l^p$
- (ii) $\left\{ f(1) - (1 - a_n^n)^{-1} \int_{a_n}^1 f(t) dt^n \right\}_{n=1}^{\infty} \in l^p$.

Lemma 1 shows such $\{a_n^n\}$ exist.

Proof. Split the integral for μ_n at a_n and integrate by parts to obtain, as in [1], $\mu_n = a_n^n(\delta_n - \gamma_n) + (f(1) - \delta_n)$, where $\delta_n = (1 - a_n^n)^{-1} \int_{a_n}^1 f(t) dt^n$ and $\gamma_n = (a_n^n)^{-1} \int_0^{a_n} f(t) dt^n$. Since $|\delta_n - \gamma_n|$ is bounded, $\{a_n^n(\delta_n - \gamma_n)\} \in l^p$, so that $\{\mu_n\} \in l^p$ if and only if $\{f(1) - \delta_n\}_{n=1}^{\infty} \in l^p$.

LEMMA 2. If $g(t) = 1 - (1-t)^\alpha$, $\alpha > 0$, and $\nu_n = \int_0^1 t^n dg$, then $\{\nu_n\} \in l^p$ if and only if $\alpha > 1/p$.

Proof. $\mu_n = \int_0^1 t^n dg = \Gamma(\alpha + 1)\Gamma(n + 1)/\Gamma(n + \alpha + 1)$. Using Stirling's formula or Gauss's test, $\sum_n \mu_n^p$ converges if and only if $\alpha > 1/p$. [3, pp. 92-93].

Consequently no l^p space contains all of the moment sequences.

COROLLARY. If there is δ , $0 < \delta < 1$, $B > 0$ and α such that

(i) $\alpha > 1/p$ and $|f(1) - f(t)| \leq B|1-t|^\alpha$ for t in $[\delta, 1]$, then $\{\mu_n\} \in l^p$

(ii) $\alpha \leq 1/p$ and $f(1) - f(t) \geq B(1-t)^\alpha$ for t in $[\delta, 1]$, or $f(t) - f(1) \geq B(1-t)^\alpha$ for t in $[\delta, 1]$, then $\{\mu_n\} \notin l^p$.

Proof. of (i)

$$\begin{aligned} \left| f(1) - (1 - a_n^n)^{-1} \int_{a_n}^1 f(t) dt^n \right| &= \left| (1 - a_n^n)^{-1} \int_{a_n}^1 (f(1) - f(t)) dt^n \right| \\ &\leq (1 - a_n^n)^{-1} \int_{a_n}^1 |f(1) - f(t)| dt^n \\ &\leq (1 - a_n^n)^{-1} \int_{a_n}^1 B(1-t)^\alpha dt^n \\ &= B \left[g(1) - (1 - a_n^n)^{-1} \int_{a_n}^1 g(t) dt^n \right] \end{aligned}$$

which we shall call ψ_n . By Lemma 2 and Theorem 1, $\{\psi_n\} \in l^p$.

The proof of (ii) is analogous to that of (i).

For each integer $k > 0$, $m > 0$, define the moment sequence $c_{km} =$

$$\{c_{nkm}\}_{n=0}^\infty \text{ by } c_{nkm} = (-1)^m k^m m^{-1} \sum_{r=0}^m \binom{m}{r} (-1)^r (r/k)^n = m^{-1} \Delta_\omega^m x^n,$$

where $\Delta_\omega f(x) = [f(x + \omega) - f(x)]/\omega$, $\omega = k^{-1}$, and $x = 0$.

THEOREM 2. Let $p > 0$. The moment sequences c_{km} belong to and have dense linear span in l^p .

Proof. For $m > n$, $\Delta_\omega^m x^n = 0$. From [2, p. 13], with $f(x) = x^{n+m}$, $\Delta_\omega^m f(x) = f^{(m)}(\xi)$ for some ξ between 0 and $m\omega$, so that $|\Delta_\omega^m x^{n+m}| \leq \max_{0 \leq \xi \leq m\omega} |f^{(m)}(\xi)| = (n+m)!(m\omega)^n/n!$.

Using these facts we have, for $0 < p \leq 1$,

$$\begin{aligned} \sum_{\substack{n=0 \\ n \neq m}}^{\infty} |c_{nkm}|^p &= \sum_{n=m+1}^{\infty} |c_{nkm}|^p = \sum_{n=1}^{\infty} |c_{n+m,k,m}|^p = \sum_{n=1}^{\infty} |m!^{-1} \Delta_{\omega}^m x^{n+m}|^p \\ &\cong m!^{-p} \sum_{n=1}^{\infty} (n+m)! (mk^{-1})^{np} / n! \\ &= m!^{1-p} [(1 - (m/k)^p)^{-m-1} - 1]. \end{aligned}$$

Therefore the sum is finite and tends to 0 as $k \rightarrow \infty$.

Since $\Delta_{\omega}^m x^m = m!$, $c_{mkm} = 1$. For

$$\begin{aligned} 0 < p \leq 1, e^m &= \{\delta_{jm}\}_{j=0}^{\infty}, \|c_{km} - e^m\|_p \\ &= |c_{mkm} - 1|^p + \sum_{\substack{n=0 \\ n \neq m}}^{\infty} |c_{nkm}|^p \rightarrow 0, \text{ as } k \rightarrow \infty. \end{aligned}$$

But the $\{e^m\}_{m=0}^{\infty}$ form a basis for l^p so that the c_{km} have dense linear span in l^p for $0 < p \leq 1$.

For any $p' > p$, $l^{p'} \supset l^p$ and the $l^{p'}$ topology is weaker than that of l^p ([4, p. 203]). Therefore the $c_{km} \in l^{p'}$ and $c_{km} \rightarrow e^m$ in $l^{p'}$, so the c_{km} have dense linear span in each l^p space.

I wish to thank B. E. Rhoades for bringing [1] to my attention, and for his suggestions in the preparation of this paper.

REFERENCES

1. G. G. Johnson, *Moment Sequences in Hilbert Space*, Pacific J. Math., **46** (1973), 201-207.
2. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer Verlag 1924.
3. B. E. Rhoades, *Spectra of some Hausdorff Operators*, Acta Sci. Math., **32** (1971), 91-100.
4. A. Wilansky, *Functional Analysis*, Blaisdell Pub. Co., 1964.

Received December 6, 1973.

INDIANA UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1975 Pacific Journal of Mathematics
All Rights Reserved

Pacific Journal of Mathematics

Vol. 58, No. 2

April, 1975

Zvi Artstein and John Allen Burns, <i>Integration of compact set-valued functions</i>	297
Mark Benard, <i>Characters and Schur indices of the unitary reflection group [321]³</i>	309
Simeon M. Berman, <i>A new characterization of characteristic functions of absolutely continuous distributions</i>	323
Monte Boisen and Philip B. Sheldon, <i>Pre-Prüfer rings</i>	331
Hans-Heinrich Brungs, <i>Three questions on duo rings</i>	345
Iracema M. Bund, <i>Birnbaum-Orlicz spaces of functions on groups</i>	351
John D. Elwin and Donald R. Short, <i>Branched immersions between 2-manifolds of higher topological type</i>	361
Eric Friedlander, <i>Extension functions for rank 2, torsion free abelian groups</i>	371
Jon Froemke and Robert Willis Quackenbush, <i>The spectrum of an equational class of groupoids</i>	381
Barry J. Gardner, <i>Radicals of supplementary semilattice sums of associative rings</i>	387
Shmuel Glasner, <i>Relatively invariant measures</i>	393
George Rudolph Gordh, Jr. and Sibe Mardesic, <i>Characterizing local connectedness in inverse limits</i>	411
Siegfried Graf, <i>On the existence of strong liftings in second countable topological spaces</i>	419
Stanley P. Gudder and D. Strawther, <i>Orthogonally additive and orthogonally increasing functions on vector spaces</i>	427
Darald Joe Hartfiel and Carlton James Maxson, <i>A characterization of the maximal monoids and maximal groups in β_X</i>	437
Robert E. Hartwig and S. Brent Morris, <i>The universal flip matrix and the generalized faro-shuffle</i>	445
William Emery Haver, <i>Mappings between ANRs that are fine homotopy equivalences</i>	457
J. Bockett Hunter, <i>Moment sequences in l^p</i>	463
Barbara Jeffcott and William Thomas Spears, <i>Semimodularity in the completion of a poset</i>	467
Jerry Alan Johnson, <i>A note on Banach spaces of Lipschitz functions</i>	475
David W. Jonah and Bertram Manuel Schreiber, <i>Transitive affine transformations on groups</i>	483
Karsten Juul, <i>Some three-point subset properties connected with Menger's characterization of boundaries of plane convex sets</i>	511
Ronald Brian Kirk, <i>The Haar integral via non-standard analysis</i>	517
Justin Thomas Lloyd and William Smiley, <i>On the group of permutations with countable support</i>	529
Erwin Lutwak, <i>Dual mixed volumes</i>	531
Mark Mahowald, <i>The index of a tangent 2-field</i>	539
Keith Miller, <i>Logarithmic convexity results for holomorphic semigroups</i>	549
Paul Milnes, <i>Extension of continuous functions on topological semigroups</i>	553
Kenneth Clayton Pietz, <i>Cauchy transforms and characteristic functions</i>	563
James Ted Rogers Jr., <i>Whitney continua in the hyperspace $C(X)$</i>	569
Jean-Marie G. Rolin, <i>The inverse of a continuous additive functional</i>	585
William Henry Ruckle, <i>Absolutely divergent series and isomorphism of subspaces</i>	605
Rolf Schneider, <i>A measure of convexity for compact sets</i>	617
Alan Henry Schoenfeld, <i>Continuous measure-preserving maps onto Peano spaces</i>	627
V. Merriline Smith, <i>Strongly superficial elements</i>	643
Roger P. Ware, <i>A note on quadratic forms over Pythagorean fields</i>	651
Roger Allen Wiegand and Sylvia Wiegand, <i>Finitely generated modules over Bezout rings</i>	655
Martin Ziegler, <i>A counterexample in the theory of definable automorphisms</i>	665