

Pacific Journal of Mathematics

MOMENT SEQUENCES IN l^p

J. BOCKETT HUNTER

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Let $p > 0$. Conditions are derived, each necessary and sufficient, for a moment sequence to be in l^p . It is shown that the moment sequences in l^p are dense in l^p . For $p = 2$, these results were obtained by G. G. Johnson.

G. G. Johnson obtained a necessary and sufficient condition for a moment sequence to be in l^2 , and showed that the moment sequences in l^2 are dense in l^2 . This paper shows that the same conclusions hold in any l^p space. The proofs are similar to and improvements of those in G. G. Johnson, Pacific J. Math., 46(1973), 201-207.

LEMMA 1. Let $0 < p < 1$, $q > 0$. If $a_n = 1 - (n + 1)^{-p}$, then $\{a_n^n\} \in l^q$.

Proof. $a_n^{nq} = \exp(qn \log(1 - (n + 1)^{-p})) < \exp(qn(-(n + 1)^{-p})) = (\exp(qn(n + 1)^{-p}))^{-1} < [\sum_{k=0}^N (qn(n + 1)^{-p})^k / k!]^{-1}$, where N satisfies $N(1 - p) > 1$. Then

$$\sum_{n=1}^{\infty} a_n^{nq} < \sum_{n=1}^{\infty} [(qn(n + 1)^{-p})^N / N!]^{-1} = N! q^{-N} \sum_{n=1}^{\infty} [(n + 1)^p / n]^N,$$

which converges if and only if $\sum_{n=1}^{\infty} n^{-(1-p)N}$ converges, and the latter is a convergent p -series.

THEOREM 1. Let $p > 0$, $f \in BV[0, 1]$, $\mu_n = \int_0^1 t^n df$. For each $\{a_n\}$ such that $0 \leq a_n < 1$, and $\{a_n^n\} \in l^p$, the following are equivalent.

- (i) $\{\mu_n\} \in l^p$
- (ii) $\left\{ f(1) - (1 - a_n^n)^{-1} \int_{a_n}^1 f(t) dt^n \right\}_{n=1}^{\infty} \in l^p$.

Lemma 1 shows such $\{a_n^n\}$ exist.

Proof. Split the integral for μ_n at a_n and integrate by parts to obtain, as in [1], $\mu_n = a_n^n(\delta_n - \gamma_n) + (f(1) - \delta_n)$, where $\delta_n = (1 - a_n^n)^{-1} \int_{a_n}^1 f(t) dt^n$ and $\gamma_n = (a_n^n)^{-1} \int_0^{a_n} f(t) dt^n$. Since $|\delta_n - \gamma_n|$ is bounded, $\{a_n^n(\delta_n - \gamma_n)\} \in l^p$, so that $\{\mu_n\} \in l^p$ if and only if $\{f(1) - \delta_n\}_{n=1}^{\infty} \in l^p$.

LEMMA 2. If $g(t) = 1 - (1 - t)^\alpha$, $\alpha > 0$, and $\nu_n = \int_0^1 t^n dg$, then $\{\nu_n\} \in l^p$ if and only if $\alpha > 1/p$.

Proof. $\mu_n = \int_0^1 t^n dg = \Gamma(\alpha + 1)\Gamma(n + 1)/\Gamma(n + \alpha + 1)$. Using Stirling's formula or Gauss's test, $\sum_n \mu_n^p$ converges if and only if $\alpha > 1/p$. [3, pp. 92-93].

Consequently no l^p space contains all of the moment sequences.

COROLLARY. If there is δ , $0 < \delta < 1$, $B > 0$ and α such that

- (i) $\alpha > 1/p$ and $|f(1) - f(t)| \leq B|1 - t|^\alpha$ for t in $[\delta, 1]$, then $\{\mu_n\} \in l^p$
- (ii) $\alpha \leq 1/p$ and $f(1) - f(t) \geq B(1 - t)^\alpha$ for t in $[\delta, 1]$, or $f(t) - f(1) \geq B(1 - t)^\alpha$ for t in $[\delta, 1]$, then $\{\mu_n\} \notin l^p$.

Proof. of (i)

$$\begin{aligned} \left| f(1) - (1 - a_n^n)^{-1} \int_{a_n}^1 f(t) dt^n \right| &= \left| (1 - a_n^n)^{-1} \int_{a_n}^1 (f(1) - f(t)) dt^n \right| \\ &\leq (1 - a_n^n)^{-1} \int_{a_n}^1 |f(1) - f(t)| dt^n \\ &\leq (1 - a_n^n)^{-1} \int_{a_n}^1 B(1 - t)^\alpha dt^n \\ &= B \left[g(1) - (1 - a_n^n)^{-1} \int_{a_n}^1 g(t) dt^n \right] \end{aligned}$$

which we shall call ψ_n . By Lemma 2 and Theorem 1, $\{\psi_n\} \in l^p$.

The proof of (ii) is analogous to that of (i).

For each integer $k > 0$, $m > 0$, define the moment sequence $c_{km} =$

$\{c_{nkm}\}_{n=0}^\infty$ by $c_{nkm} = (-1)^m k^m m!^{-1} \sum_{r=0}^m \binom{m}{r} (-1)^r (r/k)^n = m!^{-1} \Delta_\omega^m x^n$,
 where $\Delta_\omega f(x) = [f(x + \omega) - f(x)]/\omega$, $\omega = k^{-1}$, and $x = 0$.

THEOREM 2. Let $p > 0$. The moment sequences c_{km} belong to and have dense linear span in l^p .

Proof. For $m > n$, $\Delta_\omega^m x^n = 0$. From [2, p. 13], with $f(x) = x^{n+m}$, $\Delta_\omega^m f(x) = f^{(m)}(\xi)$ for some ξ between 0 and $m\omega$, so that $|\Delta_\omega^m x^{n+m}| \leq \max_{0 \leq \xi \leq m\omega} |f^{(m)}(\xi)| = (n + m)! (m\omega)^n / n!$.

Using these facts we have, for $0 < p \leq 1$,

$$\begin{aligned} \sum_{\substack{n=0 \\ n \neq m}}^{\infty} |c_{nkm}|^p &= \sum_{n=m+1}^{\infty} |c_{nkm}|^p = \sum_{n=1}^{\infty} |c_{n+m,k,m}|^p = \sum_{n=1}^{\infty} |m^{-1} \Delta_{\omega}^m x^{n+m}|^p \\ &\leq m^{1-p} \sum_{n=1}^{\infty} (n+m)! (mk^{-1})^{np} / n! \\ &= m^{1-p} [(1 - (m/k)^p)^{-m-1} - 1]. \end{aligned}$$

Therefore the sum is finite and tends to 0 as $k \rightarrow \infty$.

Since $\Delta_{\omega}^m x^m = m!$, $c_{mkm} = 1$. For

$$\begin{aligned} 0 < p \leq 1, e^m &= \{\delta_{jm}\}_{j=0}^{\infty}, \|c_{km} - e^m\|_p \\ &= |c_{mkm} - 1|^p + \sum_{\substack{n=0 \\ n \neq m}}^{\infty} |c_{nkm}|^p \rightarrow 0, \text{ as } k \rightarrow \infty. \end{aligned}$$

But the $\{e^m\}_{m=0}^{\infty}$ form a basis for l^p so that the c_{km} have dense linear span in l^p for $0 < p \leq 1$.

For any $p' > p$, $l^{p'} \supset l^p$ and the $l^{p'}$ topology is weaker than that of l^p ([4, p. 203]). Therefore the $c_{km} \in l^{p'}$ and $c_{km} \rightarrow e^m$ in $l^{p'}$, so the c_{km} have dense linear span in each l^p space.

I wish to thank B. E. Rhoades for bringing [1] to my attention, and for his suggestions in the preparation of this paper.

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