ON THE GROUP OF PERMUTATIONS WITH COUNTABLE SUPPORT

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Let $S_X$ denote the group of permutations of the set $X$. If $\aleph_\alpha$ is an infinite cardinal, the set of permutations having support with cardinality less than or equal to $\aleph_\alpha$ is a normal subgroup of $S_X$. The principal result of this paper is a constructive proof that $S_X$ is generated by its cycles, if $X$ is countably infinite. Of particular interest is the corollary that for any set $X$, the cycles of $S_X$ generate the subgroup of permutations with countable support.

If $f \in S_X$ and $x \in X$, then let $O_f(x)$ denote the orbit of $x$ under $f$. The set $X$ is the disjoint union of the distinct orbits of $f$ [1]. In case $f(x) \neq x$, $O_f(x)$ is called a nontrivial orbit of $f$. Let $S(f)$ denote the support of the permutation $f$. If $S(f)$ consists of exactly one nontrivial orbit, then $f$ is called a cycle. Let $C_X$ be the subgroup of $S_X$ consisting of all finite products of cycles. If $X$ is finite, then $C_X = S_X$. For an uncountable set $X$, $C_X$ is a proper subgroup of $S_X$. We now show that $C_X = S_X$ in the remaining case.

**THEOREM.** If $X$ is countably infinite, then $S_X$ is generated by its cycles.

**Proof.** Clearly, the subgroup $C_X$ of $S_X$ generated by its cycles is a normal subgroup. But the only normal subgroups of $S_X$ are $\{1\}$, the set of even permutations of finite support, the set of all permutations of finite support, or $S_X$ (see, e.g., [2]). Hence, $C_X = S_X$.

**COROLLARY.** For any set $X$, the cycles of $S_X$ generate the subgroup of permutations with countable support.

**Proof.** Clear.

However, one can give a more constructive proof by means of the following lemma.

**LEMMA.** Let $f \in S_X$ such that $S(f)$ is a countably infinite union of finite orbits, or a countably infinite union of countably infinite orbits. Then $f$ is the product of two cycles in $S_X$.

**Proof.** Suppose that $S(f) = \bigcup \{O_f(x_i) | i \in \mathbb{Z}\}$, where $O_f(x_i)$ is finite for each integer $i$, and $O_f(x_i) \cap O_f(x_j) = \emptyset$ if $i \neq j$. Let $O_f(x_{-i}) = O_f(x_i)^{-1}$.
\[ \{a_1, a_2, \ldots, a_p\}, \quad O_f(x_0) = \{b_1, b_2, \ldots, b_q\}, \quad \text{and} \quad O_f(x_1) = \{c_1, c_2, \ldots, c_r\}. \]

It follows that
\[
\begin{align*}
    f &= \cdots (a_1a_2 \cdots a_p)(b_1b_2 \cdots b_q)(c_1c_2 \cdots c_r) \cdots \\
    &= (\cdots a_1 a_2 a_p b_1 b_2 \cdots b_q c_1 c_2 \cdots c_r) (\cdots c_1 b_1 a_1 \cdots).
\end{align*}
\]

Now, suppose that \( S(f) \) consists of orbits which are countably infinite. Chose a partition \( A \cup B \) of \( S(f) \) such that \( A = \{x_i | i \in \mathbb{Z}\} \) and \( B = \{y_i | i \in \mathbb{Z} \text{ and } i \geq 0\} \). Let \( g \) denote the infinite cycle \( (\cdots x_{-3}x_{-2}x_{-1}x_0 y_0 x_1 y_1 x_2 y_2 \cdots) \), and let
\[
h = (\cdots x_{-3}x_{-2}x_{-1}y_0 y_1 x_0 y_2 y_3 x_1 y_4 y_5 x_2 y_6 y_7 x_3 \cdots).
\]

Then
\[
gh = (\cdots x_{-3}x_{-2}x_{-1}x_0 y_0 x_1 y_1 \cdots) (\cdots x_{-3}x_{-2}x_{-1}y_0 y_1 x_0 y_2 y_3 x_1 \cdots) \\
    = (\cdots x_{-3}x_{-2}x_{-1}y_2 y_8 y_{20} \cdots)(\cdots x_{-3}x_{-2}x_{-1}y_0 y_4 y_{12} \cdots) \\
    (\cdots x_{3}x_{1}x_{0}y_{1}y_{6}y_{16} \cdots)(\cdots x_{2(2i+1)}x_{2(i+1)}y_{2(i+1)}y_{2(i+1)+4} \cdots) \cdots.
\]

It is easy to see that \( gh \) fixes none of the elements in the set \( A \cup B \). Hence \( S(gh) = S(f) \). Since each cycle of \( gh \) contains at most one \( y \) with an odd subscript, \( gh \) has infinitely many cycles. Clearly, each of these cycles is infinite. Using the fact [2] that \( f \) and \( gh \) are conjugate in \( S_X \) if and only if \( f \) and \( gh \) have the same support structure, there exists a permutation \( t \) such that \( f = t^{-1}(gh)t = (t^{-1}gt)(t^{-1}ht) \), where \( t^{-1}gt \) and \( t^{-1}ht \) are necessarily cycles in \( S_X \). This completes the proof of the lemma.

The theorem follows from this, since if \( f \) is a permutation on \( X \), then \( f = f_1 f_2 \), where \( f_1 \) agrees with \( f \) on its finite orbits and \( f_2 \) agrees with \( f \) on its infinite orbits.

**Remark.** It is known [3] that if \( G \) is an abelian group, then \( G \) is isomorphic to a group of permutations on some set \( X \), where each permutation has countable support. It follows that each abelian group is isomorphic to a subgroup of \( C_X \), for some set \( X \).

**References**


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