Pacific Journal of Mathematics

ON THE GROUP OF PERMUTATIONS WITH COUNTABLE SUPPORT

JUSTIN THOMAS LLOYD AND WILLIAM SMILEY

Vol. 58, No. 2 April 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

ON THE GROUP OF PERMUTATIONS WITH COUNTABLE SUPPORT

JUSTIN T. LLOYD AND W. G. SMILEY, III

Let S_X denote the group of permutations of the set X. If $\aleph_\alpha$ is an infinite cardinal, the set of permutations having support with cardinality less than or equal to $\aleph_\alpha$ is a normal subgroup of S_X. The principal result of this paper is a constructive proof that S_X is generated by its cycles, if X is countably infinite. Of particular interest is the corollary that for any set X, the cycles of S_X generate the subgroup of permutations with countable support.

If $f \in S_X$ and $x \in X$, then let $O_f(x)$ denote the orbit of x under f. The set X is the disjoint union of the distinct orbits of f [1]. In case $f(x) \neq x$, $O_f(x)$ is called a *nontrivial orbit* of f. Let $S(f)$ denote the support of the permutation f. If $S(f)$ consists of exactly one nontrivial orbit, then f is called a *cycle*. Let C_X be the subgroup of S_X consisting of all finite products of cycles. If X is finite, then $C_X = S_X$. For an uncountable set X, C_X is a proper subgroup of S_X. We now show that $C_X = S_X$ in the remaining case.

THEOREM. *If X is countably infinite, then S_X is generated by its cycles.*

Proof. Clearly, the subgroup C_X of S_X generated by its cycles is a normal subgroup. But the only normal subgroups of S_X are $\{1\}$, the set of even permutations of finite support, the set of all permutations of finite support, or S_X (see, e.g., [2]). Hence, $C_X = S_X$.

COROLLARY. *For any set X, the cycles of S_X generate the subgroup of permutations with countable support.*

Proof. Clear.

However, one can give a more constructive proof by means of the following lemma.

LEMMA. *Let $f \in S_X$ such that $S(f)$ is a countably infinite union of finite orbits, or a countably infinite union of countably infinite orbits. Then f is the product of two cycles in S_X.*

Proof. Suppose that $S(f) = \cup \{O_f(x_i) \mid i \in Z\}$, where $O_f(x_i)$ is finite for each integer i, and $O_f(x_i) \cap O_f(x_j) = \phi$ if $i \neq j$. Let $O_f(x_{-1}) =$

$\{a_1, a_2, \cdots, a_p\}$, $O_f(x_0) = \{b_1, b_2, \cdots, b_q\}$, and $O_f(x_1) = \{c_1, c_2, \cdots, c_r\}$. It follows that

$$f = \cdots (a_1 a_2 \cdots a_p)(b_1 b_2 \cdots b_q)(c_1 c_2 \cdots c_r) \cdots$$
$$= (\cdots a_1 a_2 \cdots a_p b_1 b_2 \cdots b_q c_1 c_2 \cdots c_r \cdots)(\cdots c_1 b_1 a_1 \cdots).$$

Now, suppose that $S(f)$ consists of orbits which are countably infinite. Chose a partition $A \cup B$ of $S(f)$ such that $A = \{x_i \mid i \in Z\}$ and $B = \{y_i \mid i \in Z \text{ and } i \geqq 0\}$. Let g denote the infinite cycle $(\cdots x_{-3} x_{-2} x_{-1} x_0 y_0 x_1 y_1 x_2 y_2 \cdots)$, and let

$$h = (\cdots x_{-3} x_{-2} x_{-1} y_0 y_1 x_0 y_2 y_3 x_1 y_4 y_5 x_2 y_6 y_7 x_3 \cdots).$$

Then

$$gh = (\cdots x_{-3} x_{-2} x_{-1} x_0 y_0 x_1 y_1 \cdots)(\cdots x_{-3} x_{-2} x_{-1} y_0 y_1 x_0 y_2 y_3 x_1 \cdots)$$
$$= (\cdots x_{-5} x_{-3} x_{-1} y_2 y_8 y_{20} \cdots)(\cdots x_{-6} x_{-4} x_{-2} y_0 y_4 y_{12} \cdots)\cdot$$
$$(\cdots x_3 x_1 x_0 y_1 y_6 y_{16} \cdots) \cdots (\cdots x_{2(2j)+1} x_{2j} y_{2j+1} y_{2(2j+1)+4} \cdots) \cdots.$$

It is easy to see that gh fixes none of the elements in the set $A \cup B$. Hence $S(gh) = S(f)$. Since each cycle of gh contains at most one y with an odd subscript, gh has infinitely many cycles. Clearly, each of these cycles is infinite. Using the fact [2] that f and gh are conjugate in S_X if and only if f and gh have the same support structure, there exists a permutation t such that $f = t^{-1}(gh)t = (t^{-1}gt)(t^{-1}ht)$, where $t^{-1}gt$ and $t^{-1}ht$ are necessarily cycles in S_X. This completes the proof of the lemma.

The theorem follows from this, since if f is a permutation on X, then $f = f_1 f_2$, where f_1 agrees with f on its finite orbits and f_2 agrees with f on its infinite orbits.

REMARK. It is known [3] that if G is an abelian group, then G is isomorphic to a group of permutations on some set X, where each permutation has countable support. It follows that each abelian group is isomorphic to a subgroup of C_X, for some set X.

REFERENCES

1. M. Hall, Jr., *The theory of groups*, the Macmillan Company, New York, 1959.

2. A. Karrass and K. Solitar, *Some remarks on the infinite symmetric groups*, Math. Zeit., **66** (1956), 64–69.

3. M. Kneser and S. Swierczkowski, *Embeddings in groups of countable permutations*, Coll. Math., 7 (1960), 177–179.

Received January 2, 1974. This work was supported in part by a grant from the Office of Research at the University of Houston.

UNIVERSITY OF HOUSTON

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $ 72.00 a year (6 Vols., 12 issues). Special rate: $ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics

Vol. 58, No. 2 April, 1975