

Pacific Journal of Mathematics

**ON THE GROUP OF PERMUTATIONS WITH COUNTABLE
SUPPORT**

JUSTIN THOMAS LLOYD AND WILLIAM SMILEY

ON THE GROUP OF PERMUTATIONS WITH COUNTABLE SUPPORT

JUSTIN T. LLOYD AND W. G. SMILEY, III

Let S_X denote the group of permutations of the set X . If \aleph_α is an infinite cardinal, the set of permutations having support with cardinality less than or equal to \aleph_α is a normal subgroup of S_X . The principal result of this paper is a constructive proof that S_X is generated by its cycles, if X is countably infinite. Of particular interest is the corollary that for any set X , the cycles of S_X generate the subgroup of permutations with countable support.

If $f \in S_X$ and $x \in X$, then let $O_f(x)$ denote the orbit of x under f . The set X is the disjoint union of the distinct orbits of f [1]. In case $f(x) \neq x$, $O_f(x)$ is called a *nontrivial orbit* of f . Let $S(f)$ denote the support of the permutation f . If $S(f)$ consists of exactly one nontrivial orbit, then f is called a *cycle*. Let C_X be the subgroup of S_X consisting of all finite products of cycles. If X is finite, then $C_X = S_X$. For an uncountable set X , C_X is a proper subgroup of S_X . We now show that $C_X = S_X$ in the remaining case.

THEOREM. *If X is countably infinite, then S_X is generated by its cycles.*

Proof. Clearly, the subgroup C_X of S_X generated by its cycles is a normal subgroup. But the only normal subgroups of S_X are $\{1\}$, the set of even permutations of finite support, the set of all permutations of finite support, or S_X (see, e.g., [2]). Hence, $C_X = S_X$.

COROLLARY. *For any set X , the cycles of S_X generate the subgroup of permutations with countable support.*

Proof. Clear.

However, one can give a more constructive proof by means of the following lemma.

LEMMA. *Let $f \in S_X$ such that $S(f)$ is a countably infinite union of finite orbits, or a countably infinite union of countably infinite orbits. Then f is the product of two cycles in S_X .*

Proof. Suppose that $S(f) = \cup \{O_f(x_i) \mid i \in Z\}$, where $O_f(x_i)$ is finite for each integer i , and $O_f(x_i) \cap O_f(x_j) = \phi$ if $i \neq j$. Let $O_f(x_{-1}) =$

$\{a_1, a_2, \dots, a_p\}$, $O_f(x_0) = \{b_1, b_2, \dots, b_q\}$, and $O_f(x_1) = \{c_1, c_2, \dots, c_r\}$. It follows that

$$f = \dots (a_1 a_2 \dots a_p) (b_1 b_2 \dots b_q) (c_1 c_2 \dots c_r) \dots$$

$$= (\dots a_1 a_2 \dots a_p b_1 b_2 \dots b_q c_1 c_2 \dots c_r \dots) (\dots c_1 b_1 a_1 \dots).$$

Now, suppose that $S(f)$ consists of orbits which are countably infinite. Chose a partition $A \cup B$ of $S(f)$ such that $A = \{x_i \mid i \in Z\}$ and $B = \{y_i \mid i \in Z \text{ and } i \geq 0\}$. Let g denote the infinite cycle $(\dots x_{-3} x_{-2} x_{-1} x_0 y_0 x_1 y_1 x_2 y_2 \dots)$, and let

$$h = (\dots x_{-3} x_{-2} x_{-1} y_0 y_1 x_0 y_2 y_3 x_1 y_4 y_5 x_2 y_6 y_7 x_3 \dots).$$

Then

$$gh = (\dots x_{-3} x_{-2} x_{-1} x_0 y_0 x_1 y_1 \dots) (\dots x_{-3} x_{-2} x_{-1} y_0 y_1 x_0 y_2 y_3 x_1 \dots)$$

$$= (\dots x_{-5} x_{-3} x_{-1} y_2 y_8 y_{20} \dots) (\dots x_{-6} x_{-4} x_{-2} y_0 y_4 y_{12} \dots) \cdot$$

$$(\dots x_3 x_1 x_0 y_1 y_6 y_{16} \dots) \dots (\dots x_{2(2j)+1} x_{2j} y_{2j+1} y_{2(2j+1)+4} \dots) \dots.$$

It is easy to see that gh fixes none of the elements in the set $A \cup B$. Hence $S(gh) = S(f)$. Since each cycle of gh contains at most one y with an odd subscript, gh has infinitely many cycles. Clearly, each of these cycles is infinite. Using the fact [2] that f and gh are conjugate in S_X if and only if f and gh have the same support structure, there exists a permutation t such that $f = t^{-1}(gh)t = (t^{-1}gt)(t^{-1}ht)$, where $t^{-1}gt$ and $t^{-1}ht$ are necessarily cycles in S_X . This completes the proof of the lemma.

The theorem follows from this, since if f is a permutation on X , then $f = f_1 f_2$, where f_1 agrees with f on its finite orbits and f_2 agrees with f on its infinite orbits.

REMARK. It is known [3] that if G is an abelian group, then G is isomorphic to a group of permutations on some set X , where each permutation has countable support. It follows that each abelian group is isomorphic to a subgroup of C_X , for some set X .

REFERENCES

1. M. Hall, Jr., *The theory of groups*, the Macmillan Company, New York, 1959.
2. A. Karrass and K. Solitar, *Some remarks on the infinite symmetric groups*, *Math. Zeit.*, **66** (1956), 64-69.
3. M. Kneser and S. Swierczkowski, *Embeddings in groups of countable permutations*, *Coll. Math.*, **7** (1960), 177-179.

Received January 2, 1974. This work was supported in part by a grant from the Office of Research at the University of Houston.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1975 Pacific Journal of Mathematics
All Rights Reserved

Zvi Artstein and John Allen Burns, <i>Integration of compact set-valued functions</i>	297
Mark Benard, <i>Characters and Schur indices of the unitary reflection group [321]³</i>	309
Simeon M. Berman, <i>A new characterization of characteristic functions of absolutely continuous distributions</i>	323
Monte Boisen and Philip B. Sheldon, <i>Pre-Prüfer rings</i>	331
Hans-Heinrich Brungs, <i>Three questions on duo rings</i>	345
Iracema M. Bund, <i>Birbaum-Orlicz spaces of functions on groups</i>	351
John D. Elwin and Donald R. Short, <i>Branched immersions between 2-manifolds of higher topological type</i>	361
Eric Friedlander, <i>Extension functions for rank 2, torsion free abelian groups</i>	371
Jon Froemke and Robert Willis Quackenbush, <i>The spectrum of an equational class of groupoids</i>	381
Barry J. Gardner, <i>Radicals of supplementary semilattice sums of associative rings</i>	387
Shmuel Glasner, <i>Relatively invariant measures</i>	393
George Rudolph Gordh, Jr. and Sibe Mardesic, <i>Characterizing local connectedness in inverse limits</i>	411
Siegfried Graf, <i>On the existence of strong liftings in second countable topological spaces</i>	419
Stanley P. Gudder and D. Strawther, <i>Orthogonally additive and orthogonally increasing functions on vector spaces</i>	427
Darald Joe Hartfiel and Carlton James Maxson, <i>A characterization of the maximal monoids and maximal groups in β_X</i>	437
Robert E. Hartwig and S. Brent Morris, <i>The universal flip matrix and the generalized faro-shuffle</i>	445
William Emery Haver, <i>Mappings between ANRs that are fine homotopy equivalences</i>	457
J. Bockett Hunter, <i>Moment sequences in l^p</i>	463
Barbara Jeffcott and William Thomas Spears, <i>Semimodularity in the completion of a poset</i>	467
Jerry Alan Johnson, <i>A note on Banach spaces of Lipschitz functions</i>	475
David W. Jonah and Bertram Manuel Schreiber, <i>Transitive affine transformations on groups</i>	483
Karsten Juul, <i>Some three-point subset properties connected with Menger's characterization of boundaries of plane convex sets</i>	511
Ronald Brian Kirk, <i>The Haar integral via non-standard analysis</i>	517
Justin Thomas Lloyd and William Smiley, <i>On the group of permutations with countable support</i>	529
Erwin Lutwak, <i>Dual mixed volumes</i>	531
Mark Mahowald, <i>The index of a tangent 2-field</i>	539
Keith Miller, <i>Logarithmic convexity results for holomorphic semigroups</i>	549
Paul Milnes, <i>Extension of continuous functions on topological semigroups</i>	553
Kenneth Clayton Pietz, <i>Cauchy transforms and characteristic functions</i>	563
James Ted Rogers Jr., <i>Whitney continua in the hyperspace $C(X)$</i>	569
Jean-Marie G. Rolin, <i>The inverse of a continuous additive functional</i>	585
William Henry Ruckle, <i>Absolutely divergent series and isomorphism of subspaces</i>	605
Rolf Schneider, <i>A measure of convexity for compact sets</i>	617
Alan Henry Schoenfeld, <i>Continuous measure-preserving maps onto Peano spaces</i>	627
V. Merriline Smith, <i>Strongly superficial elements</i>	643
Roger P. Ware, <i>A note on quadratic forms over Pythagorean fields</i>	651
Roger Allen Wiegand and Sylvia Wiegand, <i>Finitely generated modules over Bezout rings</i>	655
Martin Ziegler, <i>A counterexample in the theory of definable automorphisms</i>	665