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THE INDEX OF A TANGENT 2-FIELD

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Thomas, using an obstruction theory approach, evaluated the index of a tangent 2-field on M^m , $m \equiv 1(4)$ if M is a spin manifold. Atiyah using the Atiyah-Singer index theorem evaluated the index for all orientable manifolds. The purpose here is to give a proof of Atiyah's result in the spirit of Thomas' work.

Let M be a connected closed smooth orientable manifold of dimension m . Let k be any integer and suppose M admits k vector fields which are linearly independent everywhere except possibly at a finite number of points. The obstruction to making the k vector field linearly independent everywhere is called the index of the k -field and it is an element of

$$H^m(M, \pi_{m-1}(V_{m,k})) \simeq \pi_{m-1}(V_{m,k}).$$

Suppose $m = 2r + 1$ and let

$$\hat{\chi}_2(M) = \left(\dim \bigoplus_{i=0}^r H^i(M, Z_2) \right) \bmod 2. \quad \text{In [5],}$$

Thomas proved:

THEOREM. *Let M be a closed connected spin manifold, $m \equiv 1(4)$, $m > 1$ with $W_{m-1}(M) = 0$. Then the index of any 2-field with singularities is*

$$\hat{\chi}_2(M) \in Z_2 = \pi_{m-1}(V_{m,2}).$$

Thomas' method was to calculate the secondary obstruction to a cross section of the association $V_{m,2}$ bundle to the tangent bundle. Atiyah [1] showed that if

$$b = \left(\dim \bigoplus_{i=0}^r H^i(M, \text{Reals}) \right) \bmod 2$$

then the index of a 2-field for any orientable manifold with $W_{m-1}(M) = 0$ is b . Finally, Milnor, Lusztig, and Peterson [3] showed the relationship between these results by showing that

$$b + \hat{\chi}_2 = W_2 W_{m-2}.$$

It has always seemed that direct proof, in the spirit of Thomas, should be possible for the Atiyah result. In this paper we will provide such a proof, i.e., we will prove

THEOREM 1. *Let M be a closed connected orientable manifold $m \equiv 1(4)$, $m > 1$ with $W_{m-1}(M) = 0$. Then the index of any 2-field with finite singularities is*

$$(\hat{\chi}_2 + W_2 W_{m-2}) \in Z_2 = \pi_{m-1}(V_{m,2}).$$

2. Proof of the theorem. The proof has two key steps. The first is to show that a secondary operation on the Thom class involves the secondary obstruction; and the second step is to evaluate the cohomology operation.

Let $m \equiv 1 \pmod{4}$. Then

$$Sq^2 Sq^{m-1} + Sq^m Sq^1 = Sq^{m+1}$$

and, thus, on m -dimensional integral classes $Sq^2 Sq^{m-1} = 0$. Let E be the fiber of the map

$$K(Z, m) \xrightarrow{Sq^{m-1}} K(Z_2, 2m - 1).$$

Then the relation $Sq^2 Sq^{m-1} = 0$ defines a class $v \in H^{2m}(E, Z_2)$ which is defined up to a primary operation on the generator of $H^m(E, Z)$.

THEOREM 2.1. *Let $T(M)$ be the Thom complex of $\tau(M)$, where M is a manifold as in Theorem 1. There exists a map $f: T(M) \rightarrow E$ such that f^* in dimension m is an isomorphism and $f^*(v) = U \cup (O_2 + W_2 W_{m-2})$ where O_2 is the index of the 2-field.*

This is proved in [4].

THEOREM 2.2. *For the data as in Theorem 2.1, $f^*(v) = \hat{\chi}_2(U \cup \mu)$ where μ generates $H^m(M, Z_2)$.*

This is the new result which we prove in §3. The main theorem is a direct consequence of these two results.

3. Proof of Theorem 2.2. Recall that the tangent bundle embeds in a natural way as a neighborhood of the diagonal in $M \times M =$

M^2 . Let $j: M^2 \rightarrow T(\tau(M))$ be the obvious map. Let $\{\alpha_i; i = 1, \dots, q\}$ a basis for

$$\bigoplus_{j=0}^{(m-1)/2} H^j(M, Z_2) \quad \text{and} \quad \{\beta\}$$

be the dual basis, i.e.,

$$\alpha_i \cup \beta_j = \delta_{ij}\mu.$$

PROPOSITION 3.1 [Theorem 2.6 [5]].

- (a) $j^*U = A + tA$ where $A = \sum_{i=0}^q (\alpha_i \otimes \beta_i)$.
- (b) $A \cup tA = \hat{\chi}_2(M)\mu \otimes \mu$.

Let $\tilde{\Omega}_m$ be the secondary operation defined over $K(Z_2, m)$ based on $Sq^2Sq^{m-1} + Sq^1(Sq^{m-1}Sq^1) = 0$.

PROPOSITION 3.2. [Thomas 2.6 [5]]. If $Sq^{m-1}U = 0$ then $Sq^{m-1}A = 0$.

Proof. An easy application of the Cartan formula shows that $Sq^{m-1}A \in H^{m-1}(M, Z_2) \otimes H^m(M, Z_2)$. Thus, $Sq^{m-1}A$ and $Sq^{m-1}(tA)$ are in different graded subgroups of $H^*(M^2)$ and so could add to zero only if each were zero separately.

PROPOSITION 3.3. $Sq^{m-1}Sq^1A = \langle V, \cup Sq^1V_r, [M] \rangle \mu \times \mu$ for any choice of basis α_i where V_r is the r dimensional Wu class.

Proof. Since $Sq^{m-1}Sq^1A = Sq^{m-1}Sq^1(\sum \alpha_i \otimes \beta_i)$ ($\dim \alpha_i = r$) it suffices to verify that if $H^r(M, Z_2)$ is a vector space of rank t and if N is a linear transformation taking the α_i to the new basis $\bar{\alpha}_i$ then

$$Sq^{m-1}Sq^1(\sum N\alpha_i \otimes N^*\beta_j) = Sq^{m-1}Sq^1(\sum \alpha_i \otimes \beta_j).$$

Moreover, N can be written as a composite of permutations (which obviously leave it invariant) and transformations of the form

$$N_{ij}\alpha_k = \begin{cases} \alpha_k & k \neq j \\ \alpha_i + \alpha_j & k = j \end{cases}$$

So the lemma is true if it is true for N_{ij} . Now $N_{ij}^*\beta_k = \begin{cases} \beta_k & k \neq i \\ \beta_i + \beta_j & k = i \end{cases}$. Thus the difference between the two sums is easily seen to be 0.

Now notice that $Sq^{m-1}Sq^1(\alpha_i \otimes \beta_i) = V_r Sq^1 \alpha_i \otimes V_r \beta_i$ and since $V_r Sq^1 \alpha_i = (Sq^1 V_r) \alpha_i$, if $Sq^1 V_r = 0$ then the lemma is true. Assume then $Sq^1 V_r \neq 0$, and give a basis for $H^r(M, Z_2)$ by choosing $\alpha_1, \dots, \alpha_{r-1}$ to span $\langle Sq^1 V_r \rangle^\perp$ and filling out to a basis by requiring α_i be dual to $Sq^1 V_r$. Then

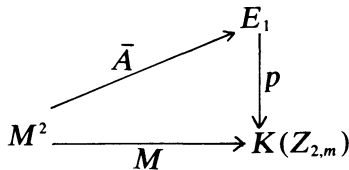
$$Sq^{m-1}Sq^1(\sum \alpha_i \otimes \beta_i) = Sq^1 V_r \alpha_i \otimes V_r Sq^1 v,$$

and the lemma follows.

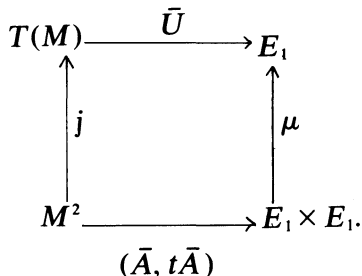
PROPOSITION 3.4. *Theorem 2.2 is true if $Sq^{m-1}Sq^1 A = 0$, i.e., if $V_r \cup Sq^1 V_r = 0$.*

With the additional hypothesis that $W_2 = 0$ this is exactly what Thomas proved in [5]. The proof which follows is the same as Thomas' up to the point where it is shown that the indeterminacy does not kill the argument.

Proof of 3.4. Let (E_1, u, v) be the universal example for $\tilde{\Omega}_m$, i.e., E_1 is a two stage Postnikov system with k -invariants Sq^{m-1} and $Sq^{m-1}Sq^1$ over a $K(Z_2, m)$. The class u is the image of the fundamental class of $H^m(K(Z_2, m))$ in $H^m(E_1)$. The class $v \in H^{2m}(E_1)$ is defined by the relation $Sq^2 Sq^m + Sq^1(Sq^{m-1}Sq^1) = 0$. The hypotheses imply that there is a commutative diagram



where $A^*(\kappa) = A$ and κ_m is the fundamental class of $K(Z_{2,m})$. Let $t\bar{A}$ be the composite $M^2 \xrightarrow{t} M^2 \xrightarrow{A} E$. Consider the diagram (not necessarily commutative)



The argument which Thomas used goes as follows: First

$$\mu^*(v) = v \otimes 1 + p^*(\kappa \otimes \kappa) + 1 \otimes v,$$

since v is not primitive, (see [5] or [2]). Then, since $(\bar{A}, t\bar{A})^*(v \otimes 1) = (\bar{A}, t\bar{A})^*(1 \otimes v)$, we see that

$$(\mu(\bar{A}, t\bar{A}))^*v = A \cup tA = \hat{\chi}_2(M)(\mu \otimes \mu).$$

Now $d(\mu(\bar{A}, t\bar{A}), \bar{U}j)$, the difference class, is a map into

$$K(\mathbb{Z}_2, 2m - 2) \times K(\mathbb{Z}_2, 2m - 1)$$

and thus is a pair of cohomology classes, (a, b) . It follows from the definition of the secondary operation that

$$(\bar{U}j)^*v = (\mu(\bar{A}, t\bar{A}))^*v + Sq^2a + Sq^1b.$$

Since M is orientable $Sq^1b = 0$ and since in Thomas' case $Sq^2W_2(M) = 0$, $Sq^2a = 0$. What we need to show is that in the case of our diagram the same conclusion holds.

LEMMA 3.5. *Let $(a, b) \in H^{2m-2}(M^2, \mathbb{Z}_2) \otimes H^{2m-1}(M^2, \mathbb{Z}_2)$ be the pair of cohomology classes $(a, b) = d(v(\bar{A}, t\bar{A}), \bar{U}j)$. The class a is invariant under t^* .*

The proof is given in §4. We continue the proof of 3.4.

Thus if $(a, b) = d(v(\bar{A}, t\bar{A}), j\bar{U})$ then a is a symmetric class, i.e.,

$$a = a_1 \otimes \mu + a_2 \otimes a_2 + \mu \otimes a_1.$$

Now $Sq^2a = 0$ if a is symmetric; and, therefore, if we use the diagram * with the maps as given we see that

$$(\bar{U}j)^*v = \hat{\chi}_2(M)(\mu \otimes \mu).$$

This is 2.2 under the hypothesis of 3.4.

We now consider the case where $V_r \cup Sq^1V_r \neq 0$. Let $A' = A - V_r \otimes Sq^1V_r$. Then $j^*U = A' + tA' + Sq^1(V_r \otimes V_r)$. Let (E, u, v) be the universal example for the operation Ω based on the relation $Sq^2Sq^{m-1} = 0$ which holds on integral classes. The class $u \in H^m(E, \mathbb{Z})$ is the fundamental class and $v \in H^{2m}(E, \mathbb{Z}_2)$ is based on the relation. Let $f: M^2 \rightarrow E$ be such that $f^*u = A' + tA'$ and suppose $f = -tf$. Then $\Omega(A' + tA') = (\hat{\chi}(M) - 1)(\mu \otimes \mu)$. Note that Ω is also defined on $Sq^1(V_r \otimes V_r)$. Let E_2 be the fiber of the map $K(\mathbb{Z}_2, m - 1) \xrightarrow{\delta Sq^{m-3}} K(\mathbb{Z}, 2m - 3)$. Let u_2 be the fundamental class. Suppose a map

defining Ω on $Sq^1(V_r \otimes V_r)$ factors $M^2 \xrightarrow{k} E_2 \xrightarrow{g} E$ where $g^*u = Sq^1u_2$ and $k^*u_2 = V_r \otimes V_r$. The indeterminacy of the value of Ω via such factorization is $k^*(Sq^2H^{2m-2}(E_2))$ but it is easy to see that $H^{2m-2}(E_2)$ is generated by primary operations on u_2 and primary operations on a

symmetric class are symmetric and thus $k^*(Sq^2H^{2m-2}(E_2)) = 0$. Thus to complete the proof of 2.2 we need to show that k exists, (Lemma 3.6), and we need to evaluate Ω on such a factorization (Lemma 3.7).

LEMMA 3.6. $\delta Sq^{m-3}(V_r \otimes V_r) = 0$

Proof. Since $W_{m-1}(M) = 0$ and $W_{m-2}(M)$ is the reduction of an integer class $\delta_* W_{m-3}(M)$ we see that $W_{2m-4}(M \otimes M) = W_{m-2}(M) \otimes W_{m-2}(M)$ is the restriction of an integer class and so $\delta(W_{2m-4}(M \otimes M)) = 0$ but $\delta(W_{2m-4}(M \otimes M)) = \delta Sq^{m-3}(V_r \otimes V_r)$.

LEMMA 3.7. *Let c be a class of dimension $m - 1$ with $\delta Sq^{m-3}c = 0$, where δ is the Bockstein $H^*(\ , Z_2) \rightarrow H^{*+1}(\ , Z)$. Then (E, u, v) is defined on Sq^1c and equals $Sq^{m-1}Sq^2c$ modulo a primary operation on Sq^1c .*

This is proved in §5.

This finishes the proof since $Sq^{m-1}Sq^2(V_r \otimes V_r) = Sq^r Sq^1 V_r \otimes Sq^r Sq^1 V_r$ and $Sq^r Sq^1 V_r = Sq^2 Sq^{r-1} V_r$. Now $Sq^r Sq^1 V_r \neq 0$ iff $V_r \cup Sq^1 V_r \neq 0$ and iff $Sq^2 Sq^{r-1} V_r = Sq^2 W_{m-2} \neq 0$ but $V_2 = W_2$ and if $V_r \cup Sq^1 V_r \neq 0$, $Sq^{m-1}Sq^2(V_r \otimes V_r) \neq 0$ and $W_2 W_{m-2} \neq 0$. This completes the proof.

4. *Proof of 3.5.* Let \bar{E} be the fiber of the map $K(Z_2, m) \xrightarrow{Sq^{m-1}} K(Z_2, 2m - 1)$. Let $[X]^k$ be a Z_2 homology skeleton of the space X , i.e., $i^*: H^j(X, Z_2) \rightarrow H^j([X]^k, Z_2)$ is an isomorphism for $j \leq k$ and $H^j([X]^k, Z_2) = 0$ for $j > k$. Then

$$[[M^2/[M^2]^{m-1}]^{2m-1}, \bar{E}] \cong [\Sigma^{-2}([M^2/[M^2]^{m-1}]^{2m-2}), \Omega^2 \bar{E}]$$

and

$$[M^2/[M^2]^{m-1}, \bar{E}] \cong [[M^2/[M^2]^{m-1}]^{2m-2}, \bar{E}].$$

Therefore

$$\Sigma^{-2}[M^2/[M^2]^{m-1}]\Omega^2 \bar{E} \cong [M^2/[M^2]^{m-1}, \bar{E}] = A.$$

This isomorphism is not canonical since it depends on the particular desuspension used. Suppose we choose one so that j desuspends to

$$j': \Sigma^{-2}([M^2/[M^2]^{m-1}]^{2m-2}) \rightarrow \Sigma^{-2}([T(M)]^{2m-2}).$$

Since $\Omega^2 \bar{E} = K(Z_2, m - 2) \times K(Z_2, 2m - 4)$ we see that A is isomorphic to some extension of $H^m(M^2, Z_2)$ by $H^{m-2}(M^2, Z_2)$. The extension is determined by the loop multiplication in $\Omega^2 \bar{E}$.

The following lemma is an easy calculation.

LEMMA 4.1. For any class $a \in A$ represented by (a_1, a_2) with $a_1 \in H^m(M^2, Z_2)$, $2a$ is represented by $(0, Sq^{m-2}a_1)$.

Since t^* on (Imj^*) is fixed and since $t^*Sq^{m-2}a = Sq^{m-2}t^*a$, the subset in $(H^m(M^2, Z_2), H^{2m-2}(M, Z_2))$ consisting of classes which are invariant under t^* is subgroup. Let $E_1 \xrightarrow{p} \bar{E}$ be the natural projection. Clearly $j\bar{U}P$ and $(\bar{A} + t\bar{A})P$ are maps in this subgroup and their difference is a where $(a, b) = d(j\bar{U}, \bar{A} + t\bar{A})$. Hence, $t^*a = a$.

5. Proof of 3.7. We will need to study several two stage Postnikov systems simultaneously and so some additional notation is needed. Let β be a vector of primary operations and $K(G)$ a generalized Eilenberg-MacLane space

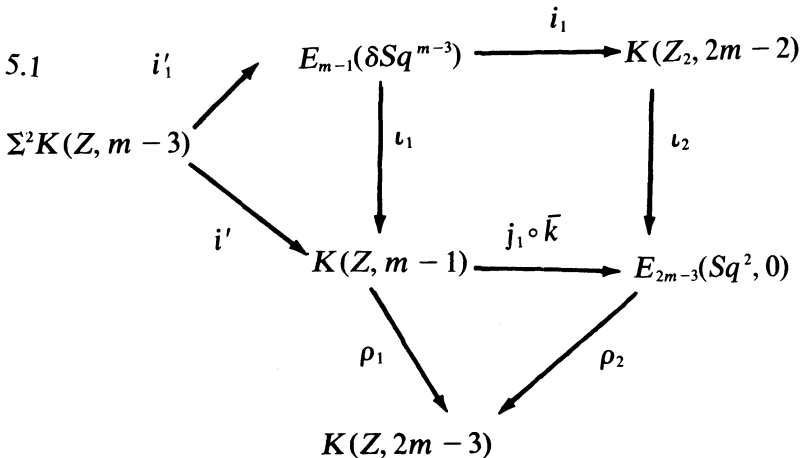
$$K(G) = \Pi K(G_i, i).$$

Let $E_m(\beta, g)$ be the fiber of the map

$$K(Z_q, m) \xrightarrow{\beta} K(G).$$

For our purposes q is either 0 or 2. We will use u_m to represent the characteristic class in $H^m(E_m)$. If $\alpha\beta = 0$ is a relation on m -dimensional class then there is a class $v(\alpha) \in H^*(E_m)$ based on this relation. The triple $(E_m(\beta, q), u, v(\alpha))$, thus, represents the universal example for a secondary operation defined on a class $a \in H^m(X, Z_q)$ with $\beta a = 0$. Note also that $v(\alpha)$ could belong to different $E(\beta, q)$. For example $Sq^2Sq^{m-1} = 0$ and $Sq^2Sq^{m-2} = 0$ on $m-1$ dim integer classes so $v(Sq^2) \in H^*(E(Sq^{m-1}, 0))$ and a different $v(Sq^2) \in H^*(E(Sq^m, 0))$. It is usually clear from the context.

The proof of 3.7 uses the following diagram



The maps are defined as follows:

$$j^*u_m = \delta u_{m-1}; \quad j^*u_{2m-3} = \delta Sq^{m-3}; \quad k^*u_{m-1} = u_{m-1}.$$

First we need to prove the existence of the diagram. The map j is the one induced from the diagram

$$\begin{array}{ccccc}
 & & E_m(Sq^{m-1}, 0) & \rightarrow & K(Z_{2,m}) & \xrightarrow{Sq^{m-1}} & K(Z_2, 2m-1) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 5.2 & & j & & \delta & & \\
 & & E_{m-1}(Sq^{m-1}Sq^1, 2) & \rightarrow & K(Z_2, m-1) & &
 \end{array}$$

The map j_1 is induced from the diagram

$$\begin{array}{ccccc}
 & & E_{m-1}(Sq^{m-1}Sq^1, 2) & \rightarrow & K(Z_{2,m-1}) & \xrightarrow{Sq^{m-1}Sq^1} & K(Z_2, 2m-1) \\
 & & \downarrow & & \downarrow & & \nearrow \\
 5.3 & & j_1 & & \delta Sq^{m-3} & & Sq^2 \\
 & & E_{2m-3}(Sq^2, 0) & \rightarrow & K(Z, 2m-3) & &
 \end{array}$$

together with the observation that $Sq^2\delta Sq^{m-3} = Sq^{m-1}Sq^1$ on $m-1$ dimensional classes, $m \equiv 1(4)$.

The map k exists because of the same relation. The map i is the double adjoint and since $i^*\delta Sq^{m-3}u = 0$ the lifting \bar{i} exists.

Lemma 3.7 can be rephrased in this notation by the following.

PROPOSITION 5.4. *The class $v(Sq^2)$ can be chosen so that $k^*j^*v(Sq^2) = Sq^{m-1}Sq^2u_{m-1}$.*

The first formula we need is

$$j^*v(Sq^2) = j^*(v(Sq^2) + p^*(\gamma)).$$

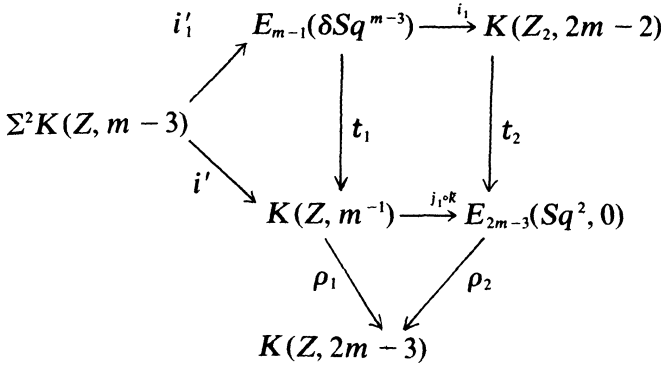
This follows directly from diagram 5.2 and 5.3. Indeed, either diagram allows one to define an operation in $E_{m-1}(Sq^{m-1}Sq^1, 2)$ based on the relation $Sq^2Sq^{m-1}Sq^1 = 0$. These two differ by some class in the base.

The second formula we need is $k^*j^*(v(Sq^2)) = 0$ modulo the indeterminacy, i.e., there is a choice of k such that the formula is true. This implies that $k^*j^*v(Sq^2) = k^*p^*\gamma$. We shall be finished when we evaluate

PROPOSITION 5.5. $\gamma = Sq^{m-1}Sq^2u_{m-1}$.

Proof. The map $K(Z, m - 1) \rightarrow K(Z_2, m - 1)$ lifts to a map $\bar{k}: K(Z, m - 1) \rightarrow E_{m-1}(Sq^{m-1}Sq^1, 2)$. Clearly, $\bar{k}^*j^*v(Sq^2) = 0$. Thus, $\bar{k}^*j^*v(Sq^2) = \gamma u_{m-1}$. Note that anything which is lost in γ by evaluating it on an interger class is part of the ambiguity in defining $v \in H^{2m}(E_m(Sq^{m-1}, 0))$.

We have the following diagram



A direct check of the appropriate exact sequence shows that

$$i_1^* \kappa_{2m-2} = v(Sq^2).$$

It follows from [4] that $i_1^*(v(Sq^2)) = \sigma^2(\kappa \cup Sq^2 \kappa)$. Since $\iota_2^* v(Sq^2) = Sq^2 \kappa_{2m-2}$, we see that

$$\iota_2^*(j_1 \circ \bar{k})^* v(Sq^2) = Sq^2[v(Sq^2)].$$

Since $\ker i_1^* = \ker \iota_2^*$ in this dimension we have

$$\begin{aligned}
 i_1^*(j_1 \circ \bar{k})^* v(Sq^2) &= Sq^2(\sigma^2(\kappa \cup Sq^2 \kappa)) \\
 &= Sq^{m-1} Sq^2(\sigma^2 \kappa).
 \end{aligned}$$

Thus, $(j_1 \circ \bar{k})^* v(Sq^2) = Sq^{m-1} Sq^2 \kappa_{m-1}$. This proves the proposition and completes the proof of the theorem.

It is interesting to note that the above argument proves the following theorem.

THEOREM 5.6. In $H^*(K(Z, m - 1))$, $m \equiv 1(4)$, $\varphi_{1,1}(\delta Sq^{m-4}) = Sq^{m-1} Sq^2 \text{ mod the indeterminacy where } \varphi_{1,1} \text{ is the secondary operation defined on integer classes based on } Sq^2 Sq^2 = 0$.

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