

Pacific Journal of Mathematics

**LOGARITHMIC CONVEXITY RESULTS FOR HOLOMORPHIC
SEMIGROUPS**

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The classical logarithmic convexity inequality, for solutions of $u' = -Au$ with A a self adjoint operator on Hilbert space, yield that u is small at intermediate times, $0 < t \leq T$, provided that u is small at T and bounded at 0. Use of the Carleman inequality for analytic functions allows one to easily generalize this result to the case of operators A which are generators of holomorphic semigroups on Banach space.

The basic logarithmic convexity result states that $\log\|u(t)\|$ is a convex function of t for solutions of the ordinary differential equation on Hilbert space, $u' = -Au$, where A is a self adjoint operator. The simplest and earliest proof known to the author appears in [1]; it involves merely differentiating $\log\|u(t)\|$ twice and use of symmetry and the Cauchy-Schwartz inequality.

Log convexity is equivalent to the following inequality: if $0 \leq t \leq T$,

$$(1) \quad \|u(T)\| \leq \epsilon, \quad \|u(0)\| \leq E, \quad \text{then} \quad \|u(t)\| \leq \epsilon^{t/T} E^{1-t/T}.$$

It thus provides a stability estimate for the problem of backward solution of $u' = -Au$ with a prescribed bound, for if u and v are two solutions to this equation, both closely fitting measured data g at time T and satisfying prescribed bounds at time 0; i.e.,

$$(2) \quad \begin{aligned} \|u(T) - g\| &\leq \epsilon, & \|v(T) - g\| &\leq \epsilon \\ \|u(0)\| &\leq E, & \|v(0)\| &\leq E, \end{aligned}$$

then at intermediate times we have

$$(3) \quad \|u(t) - v(t)\| \leq 2\epsilon^{t/T} E^{1-t/T}, \quad 0 \leq t \leq T.$$

We wish to show that the basic log convexity inequality (1) generalizes quite easily, by use of the Carleman inequality, to the class of operators A on Banach space which are generators of holomorphic semigroups. Such operators are usually defined in terms of existence and certain bounds for the resolvent operator $(A - zI)^{-1}$ in certain sectors of the complex plane, see Kato [4], or see Friedman [2] for a

more concise and introductory treatment. From these bounds it follows that there exist constants $M \geq 1$, k real, and $0 < \psi \leq \pi/2$ such that:

(4) (i) A generates a semigroup $e^{-\tau A}$ which is strongly continuous at $\tau = 0$ and satisfies the semigroup property, not only for real τ , but also for all complex $\tau = t + is$ in the closure of the sector $\Gamma_\psi = \{\tau \neq 0: \arg \tau < \psi\}$,

(4) (ii) $e^{-\tau A}$ is analytic with respect to τ in Γ_ψ ,

(4) (iii) $\|e^{-\tau A}\| \leq M e^{kt}$ on $\bar{\Gamma}_\psi$.

This is a particularly large and interesting class of operators. It includes, for example (see [2]), essentially all elliptic operators on $L^2(\Omega)$ corresponding to zero Dirichlet data on $\partial\Omega$ for which the Gårding inequality applies, and all elliptic operators on $L^p(\Omega)$ corresponding to regular elliptic boundary value problems.

In the Hilbert space case it suffices that A be a "sectorial operator," i.e.,

(5) (i) the numerical range of A lies in the sector $\{z: \arg(z + k) \leq \pi/2 - \psi\}$,

(5) (ii) A is closed,

(5) (iii) the resolvent $(A - zI)^{-1}$ exists at at least one point z outside this sector. Under these hypotheses (4) holds with $M = 1$.

THEOREM. *Let A be the generator of a holomorphic semigroup on Banach space, with corresponding $M \geq 1$, $0 < \psi \leq \pi/2$, $0 < \psi \leq \pi/2$, and real k , in (4). Let $u(t)$ be a solution of the ordinary differential equation $u' = -Au$ (that is, $u(t) = e^{tA}u(0)$, $t \geq 0$) satisfying*

$$(6) \quad \|u(T)\| \leq \epsilon, \quad \|u(0)\| \leq E.$$

Then

$$(7) \quad \|u(t)\| \leq M e^{k(t-Tw(t))} \epsilon^{w(t)} E^{1-w(t)}, \quad 0 \leq t \leq T,$$

where $w(\tau)$ is the harmonic function on the "bent strip"

$$S = \{\tau = t + is: |\arg \tau| < \psi, |\arg(\tau - T)| > \psi\}$$

which is bounded and continuous on \bar{S} , and which assumes the values 0 and 1 respectively on the left and right hand boundary arcs of S .

Proof. It suffices to assume that $k = 0$, for the general case then follows by considering $e^{-k\tau}u(\tau)$ instead of $u(\tau)$ itself.

The vector valued function $u(\tau) = e^{-\tau A}u(0)$ is analytic on S , continuous and bounded on \bar{S} , and bounded in norm by ME and Me respectively on the left and right hand boundary arcs of S . The same conditions then hold for the complex valued function $f(\tau) = v^*(u(\tau))$, where v^* is any element of unit norm in the dual Banach space. The Carleman inequality (whose proof after all merely involves dominating the subharmonic function $\log|f(z)|$ by the harmonic function $(\log \epsilon)w(z) + (\log E)(1 - w(z))$, see [3]) then yields that

$$(8) \quad |f(\tau)| \leq \epsilon^{w(\tau)} E^{1-w(\tau)} \quad \text{on } \bar{S}.$$

Since the norm of a vector u is the supremum of its values $|v^*(u)|$ over all v^* of unit norm, we obtain (7) as desired.

REMARK. Notice that when A is self-adjoint and semi-bounded from below, then the numerical range of A lies on the segment $[-k, \infty)$ of the real axis, $\psi = \pi/2$, S is the vertical strip $\{\tau = t + is : 0 < t < T\}$, $w(\tau) \equiv t/T$, and we hence obtain (1) as a special case of (7).

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