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CAUCHY TRANSFORMS AND CHARACTERISTIC FUNCTIONS

KENNETH CLAYTON PIETZ

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The following problem arises in the study of rational approximation: classify all plane sets E such that $\hat{\mu}(z) \equiv \int d\mu(\zeta)/(\zeta - z) = \chi_E(z)$ area almost everywhere for some complex Borel measure μ . A partial solution to this problem for compact sets is given here. The main result is the following.

THEOREM. Let K be a compact plane set with connected dense interior. Then there is a measure μ such that $\hat{\mu} = \chi_K$ area a.e., if and only if K has finite Painlevé length.

1. Introduction. Throughout this paper, the word "measure" will mean a complex Borel measure supported on the complex plane \mathbb{C} . If μ is a compactly supported measure, we define the *Newtonian potential* of μ by the formula

$$U_{|\mu|}(z) = \int \frac{d|\mu|(\zeta)}{|\zeta - z|}.$$

It is well known that $U_{|\mu|}$ is finite $dxdy$ a.e. For each z such that $U_{|\mu|}(z) < \infty$ we define the *Cauchy transform* of μ by

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z}$$

The Cauchy transform is thus defined almost everywhere. We seek compact sets K such that $\chi_K = \hat{\mu}$ $dxdy$ a.e., for some μ .

It is easy to see that we may assume that K is connected. For, let $K = K_1 \cup K_2$ with K_1 and K_2 closed and disjoint. Let $\hat{\mu} = \chi_K$ a.e., write μ_i for $\mu|_{K_i}$, $i = 1, 2$ and define a function

$$f = \begin{cases} \hat{\mu}_1 & \text{on } \mathbb{C} - K_1 \\ -\hat{\mu}_2 & \text{on } \mathbb{C} - K_2 \end{cases}$$

By Liouville's theorem, $f \equiv 0$. It follows easily that $\hat{\mu}_1 = \chi_{K_1}$ and $\hat{\mu}_2 = \chi_{K_2}$.

For a compact $K \subseteq \mathbb{C}$ we denote by $R(K)$ the Banach algebra of continuous functions on K which are uniform limits of rational functions with poles off K . It is well known ([4]) that $\hat{\mu} = 0$ on $\mathbb{C} - K$ if and only if $\mu \in R(K)^\perp$.

2. Painlevé Length. By a *regular* neighborhood of a compact plane set K we mean an open set $V \supseteq K$ such that ∂V consists of finitely many rectifiable curves surrounding K in the usual sense of contour integration. We say that K has finite Painlevé length if there is a number l such that every open $U \supseteq K$ contains a regular neighborhood V of K such that ∂V has length at most l . The infimum of such numbers l is called the *Painlevé length* of K .

The following theorem is well known, but we include a proof for completeness.

2.1. THEOREM. *Let K be a compact connected plane set with Painlevé length $\kappa < \infty$. Then there is a measure μ such that $\hat{\mu} = \chi_\kappa dx dy$ a.e.*

Proof. Let $\{U_n\}$ be a decreasing sequence of open sets such that

- (i) $K = \bigcap_{n=1}^\infty U_n$
- (ii) ∂U_j is a rectifiable curve for each j
- (iii) $\text{Length } \partial U_j < \kappa + \frac{1}{j}$.

Define $\mu_j = 1/2\pi i dz$ on ∂U_j for each j . The sequence $\{\mu_n\}$ is bounded and hence a subsequence, again labeled $\{\mu_n\}$, converges weak-star to a limit μ .

For any $\phi \in C_0^\infty$, we have

$$\begin{aligned} & \frac{1}{\pi} \iint \frac{\partial \phi}{\partial \bar{z}} \hat{\mu}(z) dx dy \\ &= \int \left(-\frac{1}{\pi} \iint \frac{\partial \phi}{\partial \bar{z}} \frac{1}{z - \zeta} dx dy \right) d\mu(\zeta) \\ &= \int \phi(\zeta) d\mu(\zeta) = \lim_n \int \phi(\zeta) d\mu_n(\zeta) \\ &= \lim_n \frac{1}{\pi} \iint_{U_n} \frac{\partial \phi}{\partial \bar{z}} dx dy \\ &= \frac{1}{\pi} \iint_K \frac{\partial \phi}{\partial \bar{z}} dx dy \end{aligned}$$

using the theorems of Green and Fubini. It follows easily that $\hat{\mu} = \chi_\kappa$ a.e.

The converse of this theorem is not true. This is easily seen by taking a closed disc, for example, and attaching a set with zero area but infinite Painlevé length. The converse can also fail when $K = \overline{K^0}$, as the following example shows.

2.2. EXAMPLE. Let $\{x_i\}_{i=1}^\infty$ be an enumeration of the rationals in $(0,1)$, let $\{r_i\}_{i=1}^\infty$ be any monotone decreasing sequence of positive numbers such that $\sum_{i=0}^\infty r_i < \infty$, and let $K_0 = \{(x, y) : x \in (0, 1), y = x \sin 1/x\} \cup (0, 0)$. We note that K_0 has infinite length.

Let $K = K_0 \cup \bigcup_{n=1}^\infty \bar{\Delta}(P_n; r_n)$, where $P_n = (x_n, x_n \sin 1/x_n)$ and the x_n, r_n are chosen inductively so that

- (i) $\bar{\Delta}(P_i; r_i) \cap \bar{\Delta}(P_j; r_j) = \phi$ for $i \neq j$
- (ii) $K_0 \cap \bar{\Delta}(P_j; r_j)$ is connected for each j
- (iii) $\left\{x \in \mathbb{R} : \left(x, x \sin \frac{1}{x}\right) \in K_0\right\} - \bigcup_{n=1}^\infty \bar{\Delta}(P_n; r_n)$ contains no interval.

Evidently $K = \overline{K^0}$ and K has infinite Painlevé length. But if we let $\mu = 1/2\pi i dz$ on the boundaries of the $\Delta(P_n; r_n)$, we have $\hat{\mu} = \chi_K$ a.e.

The interior of the compact set in this example is dense, but not connected. In the next section we show that if K^0 is connected and dense in K , and if there is a measure μ such that $\hat{\mu} = \chi_K$ a.e., then K must have finite Painlevé length.

3. Wermer's theorem and some extensions. The following theorem of John Wermer appears as a solution to a problem in [7].

THEOREM. *Let U be the region bounded by a Jordan curve Γ and assume there is a measure μ on Γ such that $\hat{\mu}(z) = 1$ for $z \in U$, $\hat{\mu}(z) = 0$ for $z \notin \Gamma \cup U$. Then Γ is rectifiable.*

We obtain some more general results, using ideas from Ahern and Sarason ([1]), Davie ([2]), and Gamelin and Garnett ([5]). However, many of the points in Wermer's original proof are retained.

The algebra $R(K)$ is called a *Dirichlet algebra* if it has no nonzero real annihilating measures.

Two points p_1 and p_2 of K are said to be in the same *Gleason part*, or simply *part*, of K if whenever $\{f_n\}$ is a sequence in $R(K)$ such that $\|f_n\|_K \leq 1$ and $|f_n(p_1)| \rightarrow 1$, then also $|f_n(p_2)| \rightarrow 1$. This is an equivalence relation on K .

A discussion of the properties of Dirichlet algebras and parts may be found in [4].

3.1 THEOREM. *Let K be a compact plane set such that $R(K)$ is a Dirichlet algebra. Assume μ is a measure such that $\hat{\mu} = 1$ on K^0 , $\hat{\mu} = 0$ off K . Then the components $\{U_i\}_{i \in I}$ of K^0 are simply connected, ∂U_i is a rectifiable curve for each i , and $\sum_{i \in I} \text{length } \partial U_i < \infty$. Furthermore $\mu = 1/2\pi i d\zeta$ on $\cup_{i \in I} \partial U_i$ with appropriate orientation.*

Proof. Theorem 5.1 of [5] implies that the components $\{U_i\}_{i \in I}$ of K^0 are simply connected, and Theorem 11.1 of [5] shows that the nontrivial parts of K are precisely the U_i . Glicksberg's decomposition theorem (VI 3.4 of [4]) then gives $\mu = \sum_{i \in I} \mu_i$ where μ_i is supported on \bar{U}_i for each i . Theorem VI 3.3 of [4] implies that $\mu_i \in R(\bar{U}_i)^\perp$ for each i and it follows that $\hat{\mu}_i = 1$ on U_i , $\mu_i = 0$ off \bar{U}_i . It is easy to see that $R(\bar{U}_i)$ is Dirichlet for each i .

We may therefore restrict our attention to one pair (μ_i, U_i) , which we relabel (μ, U) . It is well known that μ is absolutely continuous with respect to harmonic measure for points in U , since $R(\bar{U})$ is Dirichlet.

By expanding $\hat{\mu}$ in a Laurent series, we obtain $\int_{\partial U} z^k d\mu(z) = \delta_{-1,k}$. We can assume $0 \in U$. Let ϕ be the Riemann map of $\Delta = \{|z| < 1\}$ onto U such that $\phi(0) = 0$. Write ρ_0 for harmonic measure at 0 on $\partial \Delta$, and λ_0 the same on ∂U .

LEMMA (Ahern-Sarason [1]; Davie [2]). *The function ϕ has a measurable extension ϕ^* to a subset E of $\partial \Delta$ of full measure such that ϕ^* is one-to-one on E with a measurable inverse. The operator $T: L^1\{\lambda_0\} \rightarrow L^1\{\rho_0\}$ defined by $Tf = f \circ \phi^*$ is an isometric isomorphism which maps $L^\infty\{\lambda_0\}$ isometrically onto $L^\infty\{\rho_0\}$.*

Claim I. The function $1/\phi^*$ is not in the $L^\infty\{\rho_0\}$ closure of the linear span of $\{\phi^{**k}: k \neq -1\}$. To see this, note that $\mu \ll \lambda_0$ implies $d\mu = gd\lambda_0$ for some $g \in L^1\{\lambda_0\}$ so that $Tg \in L^1\{\rho_0\}$. Now suppose there is a sequence $\{Q_j\}_{j=1}^\infty$ of linear combinations of $\{\phi^{**k}: k \neq -1\}$ which converges to $1/\phi^*$ in $L^\infty\{\rho_0\}$. Then also $Q_j Tg \rightarrow 1/\phi^* Tg$ in $L^1\{\rho_0\}$ and $T^{-1}\{Q_j\}g \rightarrow z^{-1}g$ in $L^1\{\lambda_0\}$. But $\int T^{-1}\{Q_j\}gd\lambda_0 = 0$ for all j and $\int z^{-1}gd\lambda_0 = 1$, a contradiction. This establishes the claim and shows that there is an $h \in L^1\{\rho_0\}$ such that $\int \phi^{**k} h d\rho_0 = \delta_{-1,k}$.

LEMMA (Ahern-Sarason [1]). *Let $f \in H^\infty(U)$. Then there is a sequence $\{h_n\}_{n=1}^\infty$ in $R(\bar{U})$, with $\|h_n\|_\infty \leq \|f\|_\infty$ for all n , such that $\{h_n(z)\} \rightarrow f(z)$ for all $z \in U_0$.*

Claim II. The equality $\int \zeta \bar{h}(\zeta) d\rho_0(\zeta) = 0$ holds. To prove this, apply the above lemma to ϕ^{-1} . By Mergelyan's theorem ([4]), $R(\bar{U})$ is equal to $P(\bar{U})$, the uniform closure in $C(\bar{U})$ of the polynomials in z . Hence, there is a bounded sequence $P_n(z)$ of polynomials converging pointwise to ϕ^{-1} in U . So $\{P_n(\phi(\zeta))\} \rightarrow \zeta$ for all $\zeta \in \Delta$. By Alaoglu's theorem, there is a subsequence, again labeled $\{P_n(\phi^*)\}$ which converges weak-star on $\partial\Delta$ to some Ψ , i.e., converges over L^1 . We need only show $\Psi = \zeta$. For fixed k ,

$$\frac{1}{2\pi} \int_0^{2\pi} \Psi(e^{i\theta}) e^{ik\theta} d\theta = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} P_n(\phi^*(e^{i\theta})) e^{ik\theta} d\theta = \delta_{-1,k}.$$

So Ψ and ζ have the same Fourier coefficients, and $\Psi = \zeta$. But now

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\partial\Delta} P_n(\phi^*(\zeta)) \bar{h}(\zeta) d\rho_0(\zeta) \\ &= \int_{\partial\Delta} \zeta \bar{h}(\zeta) d\rho_0(\zeta). \\ &= \int_{\partial\Delta} \zeta \bar{h}(\zeta) d\rho_0(\zeta) \end{aligned}$$

which establishes the claim.

Similarly $\int \zeta^k \bar{h}(\zeta) d\rho_0(\zeta) = 0$ for all $k \geq 0$, and by the F. and M. Riesz theorem, $\bar{h}d\rho_0 = wdz$, $w \in H^1$. Then for any k , $0 < r < 1$,

$$\int_{|z|=r} \phi^k(z) w(z) dz = \int_{|z|=1} \phi^{*k}(z) w(z) dz = \delta_{-1,k}$$

But also $1/2\pi i \int_{|z|=r} \phi^k(z) \phi'(z) dz = \delta_{-1,k}$, so $(w(z) - \phi'(z)/2\pi i) dz$ annihilates all integral powers of ϕ^* , hence all integral powers of z , so that $w(z) = \phi'(z)/2\pi i$, and $\phi' \in H^1$. This implies that ∂U is a rectifiable Jordan curve (see e.g., [3], p. 44). The theorem is now clear.

By similar methods we can prove:

3.2 THEOREM. *Let K be a compact plane set such that $\text{Re}(R(K))$ has finite defect in $C_R(\partial K)$. Then the components $\{U_i\}_{i \in I}$ of K° are finitely connected and there is a measure μ on ∂K such that $\hat{\mu} = 1$ on K° , $\hat{\mu} = 0$ off K if and only if the following three conditions hold.*

- (i) For each i , ∂U_i is a cycle composed of rectifiable curves.
- (ii) $\sum_{i \in I} \text{length } \partial U_i < \infty$
- (iii) $\mu = \frac{1}{2\pi i} d\zeta$ on $\cup_{i \in I} \partial U$ with appropriate orientation.

3.4 THEOREM. *Let K be a compact plane set with connected dense interior. Then there is a measure μ with $\hat{\mu} = 1$ on K^0 , $\hat{\mu} = 0$ off K if and only if*

- (i) *The components of $C - K$ are bounded by rectifiable curves $\{\gamma_i\}_{i \in I}$ with finite total length and*
- (ii) *$\mu = 1/2\pi i d\zeta$ on $\cup_{i \in I} \gamma_i$ with appropriate orientation.*

Proof. As before, the sufficiency of the two conditions is obvious. To prove the necessity, let Δ be a large disk containing K , and let $\lambda = 1/2\pi i d\zeta|_{\partial\Delta} - \mu$. Then $\hat{\lambda} = 1$ on $(\bar{\Delta} - K^0)^0 = \Delta - K$, and $\hat{\lambda} = 0$ off $\bar{\Delta} = K^0$.

The hypotheses imply that $\bar{\Delta} - K^0$ is finitely connected. In fact, the complement of $\bar{\Delta} - K^0$ has two components, $C - \bar{\Delta}$ and K^0 . Also, the components of $(\bar{\Delta} - K^0)^0 = \Delta - K$ are simply connected. As before, $R(\bar{\Delta} - K^0)$ is a Dirichlet algebra so we can apply Theorem 3.1 to $\bar{\Delta} - K^0$. The conclusions (i) and (ii) follow easily.

3.4 COROLLARY. *Let K be a compact plane set with connected dense interior. Then there is a measure μ with $\hat{\mu} = \chi_K dx dy$ a.e. if and only if K has finite Painlevé length.*

REFERENCES

1. P. R. Ahern and Donald Sarason, *On Some Hypo-Dirichlet Algebras of Analytic Functions*, Amer. J. Math., **89**, 4 (1967), 932-941.
2. A. M. Davie, *Dirichlet Algebras of Analytic Functions*, J. Functional Analysis, **6** (1970), 348-356.
3. P. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
4. T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.
5. T. W. Gamelin and John Garnett, *Pointwise Bounded Approximation and Dirichlet Algebras*, J. Functional Analysis, **8** (1971), 360-404.
6. John Garnett, *Analytic capacity and Measure*, Lecture Notes in Math. No. 297, Springer, 1972.
7. John Wermer, *Solution to a Problem in Advanced Problems and Solutions*, Amer. Math. Monthly, **64** (1957), 372.

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