Pacific Journal of Mathematics

CAUCHY TRANSFORMS AND CHARACTERISTIC FUNCTIONS

Kenneth Clayton Pietz

Vol. 58, No. 2 April 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

CAUCHY TRANSFORMS AND CHARACTERISTIC FUNCTIONS

KENNETH C. PIETZ

The following problem arises in the study of rational approximation: classify all plane sets E such that $\hat{\mu}(z) \equiv \int d\mu(\zeta)/(\zeta - z) = \chi_E(z)$ area almost everywhere for some complex Borel measure μ. A partial solution to this problem for compact sets is given here. The main result is the following.

THEOREM. **Let K be a compact plane set with connected dense interior. Then there is a measure μ such that $\hat{\mu} = \chi_K$ area a.e., if and only if K has finite Painlevé length.**

1. Introduction. Throughout this paper, the word "measure" will mean a complex Borel measure supported on the complex plane **C**.. If μ is a compactly supported measure, we define the *Newtonian potential* of μ by the formula

$$U_{|\mu|}(z) = \int \frac{d|\mu|(\zeta)}{|\zeta - z|}.$$

It is well known that $U_{|\mu|}$ is finite $dxdy$ a.e. For each z such that $U_{|\mu|}(z) < \infty$ we define the *Cauchy transform* of μ by

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z}$$

The Cauchy transform is thus defined almost everywhere. We seek compact sets K such that $\chi_K = \hat{\mu}\ dxdy$ a.e., for some μ.

It is easy to see that we may assume that K is connected. For, let $K = K_1 \cup K_2$ with K_1 and K_2 closed and disjoint. Let $\hat{\mu} = \chi_K$ a.e., write μ_i for $\mu|_{K_i}$, $i = 1, 2$ and define a function

$$f = \begin{cases} \hat{\mu}_1 & \text{on} \quad \mathbf{C} - K_1 \\ -\hat{\mu}_2 & \text{on} \quad \mathbf{C} - K_2 \end{cases}.$$

By Liouville's theorem, $f \equiv 0$. It follows easily that $\hat{\mu}_1 = \chi_{K_1}$ and $\hat{\mu}_2 = \chi_{K_2}$.

For a compact $K \subseteq \mathbf{C}$ we denote by $R(K)$ the Banach algebra of continuous functions on K which are uniform limits of rational functions with poles off K. It is well known ([4]) that $\hat{\mu} = 0$ on $\mathbf{C} - K$ if and only if $\mu \in R(K)^{\perp}$.

2. Painlevé Length.

By a *regular* neighborhood of a compact plane set K we mean an open set $V \supseteq K$ such that ∂V consists of finitely many rectifiable curves surrounding K in the usual sense of contour integration. We say that K has finite Painlevé length if there is a number l such that every open $U \supseteq K$ contains a regular neighborhood V of K such that ∂V has length at most l. The infimum of such numbers l is called the *Painlevé length* of K.

The following theorem is well known, but we include a proof for completeness.

2.1. THEOREM. *Let K be a compact connected plane set with Painlevé length $\kappa < \infty$. Then there is a measure μ such that $\hat{\mu} = \chi_K\, dxdy$ a.e.*

Proof. Let $\{U_n\}$ be a decreasing sequence of open sets such that
(i) $K = \bigcap_{n=1}^{\infty} U_n$
(ii) ∂U_j is a rectifiable curve for each j
(iii) Length $\partial U_j < \kappa + \frac{1}{j}$.

Define $\mu_j = 1/2\pi i\, dz$ on ∂U_j for each j. The sequence $\{\mu_n\}$ is bounded and hence a subsequence, again labeled $\{\mu_n\}$, converges weak-star to a limit μ.

For any $\phi \in C_0^{\infty}$, we have

$$\begin{aligned}
&\frac{1}{\pi}\int\int \frac{\partial \phi}{\partial \bar{z}} \hat{\mu}(z)\, dxdy \\
&= \int\left(-\frac{1}{\pi}\int\int \frac{\partial \phi}{\partial \bar{z}} \frac{1}{z-\zeta}\, dxdy\right) d\mu(\zeta) \\
&= \int \phi(\zeta)\, d\mu(\zeta) = \lim_n \int \phi(\zeta)\, d\mu_n(\zeta) \\
&= \lim_n \frac{1}{\pi}\int\int_{U_n} \frac{\partial \phi}{\partial \bar{z}}\, dxdy \\
&= \frac{1}{\pi}\int\int_K \frac{\partial \phi}{\partial \bar{z}}\, dxdy
\end{aligned}$$

using the theorems of Green and Fubini. It follows easily that $\hat{\mu} = \chi_K$ a.e.

The converse of this theorem is not true. This is easily seen by taking a closed disc, for example, and attaching a set with zero area but infinite Painlevé length. The converse can also fail when $K = \overline{K^0}$, as the following example shows.

2.2. EXAMPLE. Let $\{x_i\}_{i=1}^{\infty}$ be an enumeration of the rationals in (0,1), let $\{r_i\}_{i=1}^{\infty}$ be any monotone decreasing sequence of positive numbers such that $\Sigma_{i=0}^{\infty} r_i < \infty$, and let $K_0 = \{(x, y): x \in (0, 1), y = x \sin 1/x\} \cup (0, 0)$. We note that K_0 has infinite length.

Let $K = K_0 \cup \bigcup_{n=1}^{\infty} \bar{\Delta}(P_n; r_n)$, where $P_n = (x_n, x_n \sin 1/x_n)$ and the x_n, r_n are chosen inductively so that

(i) $\bar{\Delta}(P_i; r_i \cap \bar{\Delta}(P_j; r_i) = \phi$ for $i \neq j$

(ii) $K_0 \cap \bar{\Delta}(P_j; r_j)$ is connected for each j

(iii) $\left\{x \in R: \left(x, x \sin \frac{1}{x}\right) \in K_0\right\} - \bigcup_{n=1}^{\infty} \bar{\Delta}(P_n; r_n)$ contains no interval.

Evidently $K = \overline{K^0}$ and K has infinite Painlevé length. But if we let $\mu = 1/2\pi i\, dz$ on the boundaries of the $\Delta(P_n; r_n)$, we have $\hat{\mu} = \chi_K$ a.e.

The interior of the compact set in this example is dense, but not connected. In the next section we show that if K^0 is connected and dense in K, and if there is a measure μ such that $\hat{\mu} = \chi_K$ a.e., then K must have finite Painlevé length.

3. Wermer's theorem and some extensions.

The following theorem of John Wermer appears as a solution to a problem in [7].

THEOREM. *Let U be the region bounded by a Jordan curve Γ and assume there is a measure μ on Γ such that $\hat{\mu}(z) = 1$ for $z \in U$, $\hat{\mu}(z) = 0$ for $z \notin \Gamma \cup U$. Then Γ is rectifiable.*

We obtain some more general results, using ideas from Ahern and Sarason ([1]), Davie ([2]), and Gamelin and Garnett ([5]). However, many of the points in Wermer's original proof are retained.

The algebra $R(K)$ is called a *Dirichlet algebra* if it has no nonzero real annihilating measures.

Two points p_1 and p_2 of K are said to be in the same *Gleason part*, or simply *part*, of K if whenever $\{f_n\}$ is a sequence in $R(K)$ such that $\|f_n\|_K \leqq 1$ and $|f_n(p_1)| \to 1$, then also $|f_n(p_2)| \to 1$. This is an equivalence relation on K.

A discussion of the properties of Dirichlet algebras and parts may be found in [4].

3.1 THEOREM. *Let K be a compact plane set such that $R(K)$ is a Dirichlet algebra. Assume μ is a measure such that $\hat{\mu} = 1$ on K^0, $\hat{\mu} = 0$ off K. Then the components $\{U_i\}_{i\in I}$ of K^0 are simply connected, ∂U_i is a rectifiable curve for each i, and $\Sigma_{i\in I}$* length *$\partial U_i < \infty$. Furthermore $\mu = 1/2\pi i\, d\zeta$ on $\cup_{i\in I}\partial U_i$ with appropriate orientation.*

Proof. Theorem 5.1 of [5] implies that the components $\{U_i\}_{i\in I}$ of K^0 are simply connected, and Theorem 11.1 of [5] shows that the nontrivial parts of K are precisely the U_i. Glicksberg's decomposition theorem (VI 3.4 of [4]) then gives $\mu = \sum_{i\in I} \mu_i$ where μ_i is supported on $\bar{U}_i$ for each i. Theorem VI 3.3 of [4] implies that $\mu_i \in R(\bar{U}_i)^\perp$ for each i and it follows that $\hat{\mu}_i = 1$ on U_i, $\mu_i = 0$ off $\bar{U}_i$. It is easy to see that $R(\bar{U}_i)$ is Dirichlet for each i.

We may therefore restrict our attention to one pair (μ_i, U_i), which we relabel (μ, U). It is well known that μ is absolutely continuous with respect to harmonic measure for points in U, since $R(\bar{U})$ is Dirichlet.

By expanding $\hat{\mu}$ in a Laurent series, we obtain $\int_{\partial U} z^k d\mu(z) = \delta_{-1,k}$. We can assume $0 \in U$. Let ϕ be the Riemann map of $\Delta = \{|z| < 1\}$ onto U such that $\phi(0) = 0$. Write ρ_0 for harmonic measure at 0 on $\partial\Delta$, and λ_0 the same on ∂U.

LEMMA (Ahern-Sarason [1]; Davie [2]). *The function ϕ has a measurable extension ϕ^* to a subset E of $\partial\Delta$ of full measure such that ϕ^* is one-to-one on E with a measurable inverse. The operator $T: L^1\{\lambda_0\} \to L^1\{\rho_0\}$ defined by $Tf = f \circ \phi^*$ is an isometric isomorphism which maps $L^\infty\{\lambda_0\}$ isometrically onto $L^\infty\{\rho_0\}$.*

Claim I. The function $1/\phi^*$ is not in the $L^\infty\{\rho_0\}$ closure of the linear span of $\{\phi^{*k} : k \neq -1\}$. To see this, note that $\mu \ll \lambda_0$ implies $d\mu = g d\lambda_0$ for some $g \in L^1\{\lambda_0\}$ so that $Tg \in L^1\{\rho_0\}$. Now suppose there is a sequence $\{Q_j\}_{j=1}^\infty$ of linear combinations of $\{\phi^{*k} : k \neq -1\}$ which converges to $1/\phi^*$ in $L^\infty\{\rho_0\}$. Then also $Q_j Tg \to 1/\phi^*\, Tg$ in $L^1\{\rho_0\}$ and $T^{-1}\{Q_j\}g \to z^{-1}g$ in $L^1\{\lambda_0\}$. But $\int T^{-1}\{Q_j\} g d\lambda_0 = 0$ for all j and $\int z^{-1} g d\lambda_0 = 1$, a contradiction. This establishes the claim and shows that there is an $h \in L^1\{\rho_0\}$ such that $\int \phi^{*k} \bar{h} d\rho_0 = \delta_{-1,k}$.

LEMMA (Ahern-Sarason [1]). *Let* $f \in H^\infty(U)$. *Then there is a sequence* $\{h_n\}_{n=1}^\infty$ *in* $R(\bar{U})$, *with* $\|h_n\|_\infty \leqq \|f\|_\infty$ *for all* n, *such that* $\{h_n(z)\} \to f(z)$ *for all* $z \in U_0$.

Claim II. The equality $\int \zeta \bar{h}(\zeta) d\rho_0(\zeta) = 0$ holds. To prove this, apply the above lemma to ϕ^{-1}. By Mergelyan's theorem ([4]), $R(\bar{U})$ is equal to $P(\bar{U})$, the uniform closure in $C(\bar{U})$ of the polynomials in z. Hence, there is a bounded sequence $P_n(z)$ of polynomials converging pointwise to ϕ^{-1} in U. So $\{P_n(\phi(\zeta))\} \to \zeta$ for all $\zeta \in \Delta$. By Alaoglu's theorem, there is a subsequence, again labeled $\{P_n(\phi^*)\}$ which converges weak-star on $\partial\Delta$ to some Ψ, i.e., converges over L^1. We need only show $\Psi = \zeta$. For fixed k,

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi\left(e^{i\theta}\right) e^{ik\theta} d\theta = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} P_n\left(\phi^*\left(e^{i\theta}\right)\right) e^{ik\theta} d\theta = \delta_{-1,k}.$$

So Ψ and ζ have the same Fourier coefficients, and $\Psi = \zeta$. But now

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\partial\Delta} P_n(\phi^*(\zeta))\bar{h}(\zeta) d\rho_0(\zeta) \\ &= \int_{\partial\Delta} \zeta\bar{h}(\zeta) d\rho_0(\zeta). \\ &= \int_{\partial\Delta} \zeta\bar{h}(\zeta) d\rho_0(\zeta) \end{aligned}$$

which establishes the claim.

Similarly $\int \zeta^k \bar{h}(\zeta) d\rho_0(\zeta) = 0$ for all $k \geqq 0$,and by the F. and M. Riesz theorem, $\bar{h} d\rho_0 = w dz$, $w \in H^1$. Then for any k, $0 < r < 1$,

$$\int_{|z|=r} \phi^k(z) w(z) dz = \int_{|z|=1} \phi^{*k}(z) w(z) dz = \delta_{-1,k}$$

But also $1/2\pi i \int_{|z|=r} \phi^k(z)\,\phi'(z)\,dz = \delta_{-1,k}$, so $(w(z) - \phi'(z)/2\pi i)\,dz$ annihilates all integral powers of ϕ^*, hence all integral powers of z, so that $w(z) = \phi'(z)/2\pi i$, and $\phi' \in H^1$. This implies that ∂U is a rectifiable Jordan curve (see e.g., [3], p. 44). The theorem is now clear.

By similar methods we can prove:

3.2 THEOREM. *Let K be a compact plane set such that* $\mathrm{Re}(R(K))$ *has finite defect in* $C_R(\partial K)$. *Then the components* $\{U_i\}_{i\in I}$ *of* K° *are finitely connected and there is a measure* μ *on* ∂K *such that* $\hat{\mu} = 1$ *on* K°, $\hat{\mu} = 0$ *off* K *if and only if the following three conditions hold.*

(i) *For each i, ∂U_i is a cycle composed of rectifiable curves.*

(ii) $\Sigma_{i \in I}$ length $\partial U_i < \infty$

(iii) $\mu = \frac{1}{2\pi i} d\zeta$ *on* $\cup_{i \in I} \partial U$ *with appropriate orientation.*

3.4 THEOREM. *Let K be a compact plane set with connected dense interior. Then there is a measure μ with $\hat{\mu} = 1$ on K^0, $\hat{\mu} = 0$ off K if and only if*

(i) *The components of $\mathbf{C} - K$ are bounded by rectifiable* curves $\{\gamma_i\}_{i \in I}$ *with finite total length and*

(ii) $\mu = 1/2\pi i\, d\zeta$ *on* $\cup_{i \in I} \gamma_i$ *with appropriate orientation.*

Proof. As before, the sufficiency of the two conditions is obvious. To prove the necessity, let Δ be a large disk containing K, and let $\lambda = 1/2\pi i\, d\zeta|_{\partial\Delta} - \mu$. Then $\hat{\lambda} = 1$ on $(\bar{\Delta} - K^0)^0 = \Delta - K$, and $\hat{\lambda} = 0$ off $\bar{\Delta} = K^0$.

The hypotheses imply that $\bar{\Delta} - K^0$ is finitely connected. In fact, the complement of $\bar{\Delta} - K^0$ has two components, $\mathbf{C} - \bar{\Delta}$ and $\bar{K}^0$. Also, the components of $(\bar{\Delta} - K^0)^0 = \Delta - K$ are simply connected. As before, $R(\bar{\Delta} - K^0)$ is a Dirichlet algebra so we can apply Theorem 3.1 to $\bar{\Delta} - K^0$. The conclusions (i) and (ii) follow easily.

3.4 COROLLARY. *Let K be a compact plane set with connected dense interior. Then there is a measure μ with $\hat{\mu} = \chi_K dxdy$* a.e. *if and only if K has finite Painlevé length.*

REFERENCES

1. P. R. Ahern and Donald Sarason, *On Some Hypo-Dirichlet Algebras of Analytic Functions*, Amer. J. Math., **89**, 4 (1967), 932–941.

2. A. M. Davie, *Dirichlet Algebras of Analytic Functions*, J. Functional Analysis, **6** (1970), 348–356.

3. P. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.

4. T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

5. T. W. Gamelin and John Garnett, *Pointwise Bounded Approximation and Dirichlet Algebras*, J. Functional Analysis, **8** (1971), 360–404.

6. John Garnett, *Analytic capacity and Measure*, Lecture Notes in Math. No. 297, Springer, 1972.

7. John Wermer, Solution to a Problem in *Advanced Problems and Solutions*, Amer. Math. Monthly, **64** (1957), 372.

Received June 10, 1974. This paper contains material from the author's thesis, submitted to the University of California, Los Angeles. The author would like to thank his advisor, John Garnett, for his help and encouragement.

LOCKHEED ELECTRONICS COMPANY, HOUSTON, TEXAS
AND
UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $ 72.00 a year (6 Vols., 12 issues). Special rate: $ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics

Vol. 58, No. 2 April, 1975