CAUCHY TRANSFORMS AND CHARACTERISTIC FUNCTIONS

KENNETH CLAYTON PIETZ
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The following problem arises in the study of rational approximation: classify all plane sets \( E \) such that 
\[
\mu(z) = \frac{1}{2\pi i} \int \frac{d\mu(\zeta)}{(\zeta - z)} = \chi_E(z)
\]
area almost everywhere for some complex Borel measure \( \mu \). A partial solution to this problem for compact sets is given here. The main result is the following.

**Theorem.** Let \( K \) be a compact plane set with connected dense interior. Then there is a measure \( \mu \) such that 
\[
\mu = \chi_K \text{ area a.e., if and only if } K \text{ has finite Painlevé length.}
\]

1. **Introduction.** Throughout this paper, the word “measure” will mean a complex Borel measure supported on the complex plane \( \mathbb{C} \). If \( \mu \) is a compactly supported measure, we define the *Newtonian potential* of \( \mu \) by the formula

\[
U_{|\mu|}(z) = \frac{1}{2\pi} \int \frac{|\mu|(|\zeta|)}{|\zeta - z|}.
\]

It is well known that \( U_{|\mu|} \) is finite \( dxdy \) a.e. For each \( z \) such that \( U_{|\mu|}(z) < \infty \) we define the *Cauchy transform* of \( \mu \) by

\[
\hat{\mu}(z) = \frac{1}{2\pi i} \int \frac{\mu(|\zeta|)}{|\zeta - z|}.
\]

The Cauchy transform is thus defined almost everywhere. We seek compact sets \( K \) such that \( \chi_K = \hat{\mu} \ dxdy \) a.e., for some \( \mu \).

It is easy to see that we may assume that \( K \) is connected. For, let \( K = K_1 \cup K_2 \) with \( K_1 \) and \( K_2 \) closed and disjoint. Let \( \hat{\mu} = \chi_K \) a.e., write \( \mu_i \) for \( \mu|_{K_i}, i = 1,2 \) and define a function

\[
f = \begin{cases} 
\hat{\mu}_1 & \text{on } \mathbb{C} - K_1 \\
-\hat{\mu}_2 & \text{on } \mathbb{C} - K_2
\end{cases}
\]

By Liouville's theorem, \( f \equiv 0 \). It follows easily that \( \hat{\mu}_1 = \chi_{K_1} \) and \( \hat{\mu}_2 = \chi_{K_2} \).
For a compact $K \subseteq \mathbb{C}$ we denote by $R(K)$ the Banach algebra of continuous functions on $K$ which are uniform limits of rational functions with poles off $K$. It is well known ([4]) that $\hat{\mu} = 0$ on $\mathbb{C} - K$ if and only if $\mu \in R(K)^\perp$. 

2. Painlevé Length. By a regular neighborhood of a compact plane set $K$ we mean an open set $V \supseteq K$ such that $\partial V$ consists of finitely many rectifiable curves surrounding $K$ in the usual sense of contour integration. We say that $K$ has finite Painlevé length if there is a number $l$ such that every open $U \supseteq K$ contains a regular neighborhood $V$ of $K$ such that $\partial V$ has length at most $l$. The infimum of such numbers $l$ is called the Painlevé length of $K$.

The following theorem is well known, but we include a proof for completeness.

2.1. Theorem. Let $K$ be a compact connected plane set with Painlevé length $\kappa < \infty$. Then there is a measure $\mu$ such that $\hat{\mu} = \chi_\kappa$ dx dy a.e.

Proof. Let $\{U_n\}$ be a decreasing sequence of open sets such that

(i) $K = \bigcap_{n=1}^{\infty} U_n$

(ii) $\partial U_j$ is a rectifiable curve for each $j$

(iii) $\text{Length } \partial U_j < \kappa + \frac{1}{j}$

Define $\mu_j = \frac{1}{2\pi i} dz$ on $\partial U_j$ for each $j$. The sequence $\{\mu_n\}$ is bounded and hence a subsequence, again labeled $\{\mu_n\}$, converges weak-star to a limit $\mu$.

For any $\phi \in C_0$, we have

$$\frac{1}{\pi} \int \int \frac{\partial \phi}{\partial \bar{z}} \hat{\mu}(z) \, dx \, dy$$

$$= \int \left( -\frac{1}{\pi} \int \int \frac{\partial \phi}{\partial \bar{z}} \frac{1}{z - \bar{\zeta}} \, dx \, dy \right) d\mu(\zeta)$$

$$= \int \phi(\zeta) \, d\mu(\zeta) = \lim_n \int \phi(\zeta) \, d\mu_n(\zeta)$$

$$= \lim_n \frac{1}{\pi} \int \int_{U_n} \frac{\partial \phi}{\partial \bar{z}} \, dx \, dy$$

$$= \frac{1}{\pi} \int \int_K \frac{\partial \phi}{\partial \bar{z}} \, dx \, dy$$

using the theorems of Green and Fubini. It follows easily that $\hat{\mu} = \chi_\kappa$ a.e.
The converse of this theorem is not true. This is easily seen by taking a closed disc, for example, and attaching a set with zero area but infinite Painlevé length. The converse can also fail when \( K = K^0 \), as the following example shows.

2.2. **Example.** Let \( \{x_i\}^n_{i=1} \) be an enumeration of the rationals in \((0,1)\), let \( \{r_i\}_{i=1}^{n} \) be any monotone decreasing sequence of positive numbers such that \( \sum_{i=0}^{n} r_i < \infty \), and let \( K_0 = \{(x,y): x \in (0,1), y = x \sin \frac{1}{x}\} \cup (0,0) \). We note that \( K_0 \) has infinite length.

Let \( K = K_0 \cup \bigcup_{n=1}^{\infty} \Delta(P_n; r_n) \), where \( P_n = (x_n, x_n \sin \frac{1}{x_n}) \) and the \( x_n, r_n \) are chosen inductively so that

(i) \( \Delta(P_i; r_i) \cap \Delta(P_j; r_j) = \emptyset \) for \( i \neq j \)

(ii) \( K_0 \cap \Delta(P_j; r_j) \) is connected for each \( j \)

(iii) \( \left\{ x \in R: \left(x, x \sin \frac{1}{x}\right) \in K_0 \right\} \cup \bigcup_{n=1}^{\infty} \Delta(P_n; r_n) \) contains no interval.

Evidently \( K = K^0 \) and \( K \) has infinite Painlevé length. But if we let \( \mu = \frac{1}{2\pi i} \, dz \) on the boundaries of the \( \Delta(P_n; r_n) \), we have \( \hat{\mu} = \chi_K \) a.e.

The interior of the compact set in this example is dense, but not connected. In the next section we show that if \( K^0 \) is connected and dense in \( K \), and if there is a measure \( \mu \) such that \( \hat{\mu} = \chi_K \) a.e., then \( K \) must have finite Painlevé length.

3. **Wermer’s theorem and some extensions.** The following theorem of John Wermer appears as a solution to a problem in [7].

**Theorem.** Let \( U \) be the region bounded by a Jordan curve \( \Gamma \) and assume there is a measure \( \mu \) on \( \Gamma \) such that \( \hat{\mu}(z) = 1 \) for \( z \in U, \hat{\mu}(z) = 0 \) for \( z \not\in \Gamma \cup U \). Then \( \Gamma \) is rectifiable.

We obtain some more general results, using ideas from Ahern and Sarason ([1]), Davie ([2]), and Gamelin and Garnett ([5]). However, many of the points in Wermer’s original proof are retained.

The algebra \( R(K) \) is called a **Dirichlet algebra** if it has no nonzero real annihilating measures.

Two points \( p_1 \) and \( p_2 \) of \( K \) are said to be in the same **Gleason part**, or simply **part**, of \( K \) if whenever \( \{f_n\} \) is a sequence in \( R(K) \) such that \( \|f_n\|_K \leq 1 \) and \( |f_n(p_1)| \to 1 \), then also \( |f_n(p_2)| \to 1 \). This is an equivalence relation on \( K \).
A discussion of the properties of Dirichlet algebras and parts may be found in [4].

3.1 THEOREM. Let $K$ be a compact plane set such that $R(K)$ is a Dirichlet algebra. Assume $\mu$ is a measure such that $\hat{\mu} = 1$ on $K^0$, $\hat{\mu} = 0$ off $K$. Then the components $\{U_i\}_{i \in I}$ of $K^0$ are simply connected, $\partial U_i$ is a rectifiable curve for each $i$, and $\Sigma_{i \in I}$ length $\partial U_i < \infty$. Furthermore $\mu = 1/2\pi i d\zeta$ on $\cup_{i \in I} \partial U_i$ with appropriate orientation.

Proof. Theorem 5.1 of [5] implies that the components $\{U_i\}_{i \in I}$ of $K^0$ are simply connected, and Theorem 11.1 of [5] shows that the nontrivial parts of $K$ are precisely the $U_i$. Glicksberg’s decomposition theorem (VI 3.4 of [4]) then gives $\mu = \Sigma_{i \in I} \mu_i$ where $\mu_i$ is supported on $\bar{U}_i$ for each $i$. Theorem VI 3.3 of [4] implies that $\mu_i \in R(\bar{U}_i)$ for each $i$ and it follows that $\hat{\mu}_i = 1$ on $U_i$, $\mu_i = 0$ off $\bar{U}_i$. It is easy to see that $R(\bar{U}_i)$ is Dirichlet for each $i$.

We may therefore restrict our attention to one pair $(\mu_i, U_i)$, which we relabel $(\mu, U)$. It is well known that $\mu$ is absolutely continuous with respect to harmonic measure for points in $U$, since $R(U)$ is Dirichlet.

By expanding $\hat{\mu}$ in a Laurent series, we obtain $\int_{\partial U} z^k d\mu(z) = \delta_{-1,k}$. We can assume $0 \in U$. Let $\phi$ be the Riemann map of $\Delta = \{ |z| < 1 \}$ onto $U$ such that $\phi(0) = 0$. Write $\rho_0$ for harmonic measure at 0 on $\partial \Delta$, and $\lambda_0$ the same on $\partial U$.

LEMMA (Ahern-Sarason [1]; Davie [2]). The function $\phi$ has a measurable extension $\phi^*$ to a subset $E$ of $\partial \Delta$ of full measure such that $\phi^*$ is one-to-one on $E$ with a measurable inverse. The operator $T: L^1(\lambda_0) \to L^1(\rho_0)$ defined by $Tf = f \circ \phi^*$ is an isometric isomorphism which maps $L^\infty(\lambda_0)$ isometrically onto $L^\infty(\rho_0)$.

Claim I. The function $1/\phi^*$ is not in the $L^\infty(\rho_0)$ closure of the linear span of $\{\phi^{*k} : k \neq -1 \}$. To see this, note that $\mu \prec \lambda_0$ implies $d\mu = gd\lambda_0$ for some $g \in L^1(\lambda_0)$ so that $Tg \in L^1(\rho_0)$. Now suppose there is a sequence $\{Q_i\}_{i=1}^\infty$ of linear combinations of $\{\phi^{*k} : k \neq -1 \}$ which converges to $1/\phi^*$ in $L^\infty(\rho_0)$. Then also $Q_i Tg \to 1/\phi^* Tg$ in $L^1(\rho_0)$ and $T^{-1}(Q_i)g \to z^{-i}g$ in $L^1(\lambda_0)$. But $\int T^{-1}(Q_i)g d\lambda_0 = 0$ for all $j$ and $\int z^{-i}gd\lambda_0 = 1$, a contradiction. This establishes the claim and shows that there is an $h \in L^1(\rho_0)$ such that $\int \phi^{*i} \tilde{h}d\rho_0 = \delta_{-1,k}$.
Lemma (Ahern-Sarasjon [1]). Let \( f \in H^\alpha(U) \). Then there is a sequence \( \{h_n\}_{n=1}^\infty \) in \( R(\bar{U}) \), with \( \|h_n\|_\infty \leq \|f\|_\infty \) for all \( n \), such that \( \{h_n(z)\} \to f(z) \) for all \( z \in U_0 \).

Claim II. The equality \( \int \zeta \bar{h}(\zeta)\,d\rho_0(\zeta) = 0 \) holds. To prove this, apply the above lemma to \( \phi^{-1} \). By Mergelyan’s theorem ([4]), \( R(\bar{U}) \) is equal to \( P(U) \), the uniform closure in \( C(\bar{U}) \) of the polynomials in \( z \). Hence, there is a bounded sequence \( P_n(z) \) of polynomials converging pointwise to \( \phi^{-1} \) in \( U \). So \( \{P_n(\phi(\zeta))\} \to \zeta \) for all \( \zeta \in \Delta \). By Alaoglu’s theorem, there is a subsequence, again labeled \( \{P_n(\phi^*)\} \) which converges weak-star on \( \partial \Delta \) to some \( \Psi \), i.e., converges over \( L^1 \). We need only show \( \Psi = \zeta \). For fixed \( k \),

\[
\frac{1}{2\pi} \int_0^{2\pi} \Psi(e^{i\theta}) \, e^{\bar{w} e^{i\theta}} \, d\theta = \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} P_n(\phi^*(e^{i\theta})) \, e^{\bar{w} e^{i\theta}} \, d\theta = \delta_{-1,k}.
\]

So \( \Psi \) and \( \zeta \) have the same Fourier coefficients, and \( \Psi = \zeta \). But now

\[
0 = \lim_{n \to \infty} \int_{\partial \Delta} P_n(\phi^*(\zeta)) \bar{h}(\zeta)\,d\rho_0(\zeta)
= \int_{\partial \Delta} \bar{h}(\zeta)\,d\rho_0(\zeta).
= \int_{\partial \Delta} \zeta \bar{h}(\zeta)\,d\rho_0(\zeta)
\]

which establishes the claim.

Similarly \( \int \zeta^k \bar{h}(\zeta)\,d\rho_0(\zeta) = 0 \) for all \( k \geq 0 \), and by the F. and M. Riesz theorem, \( \bar{h}d\rho_0 = wdz \), \( w \in H^1 \). Then for any \( k \), \( 0 < r < 1 \),

\[
\int_{|z|=r} \phi^k(z)w(z)\,dz = \int_{|z|=1} \phi^*k(z)w(z)\,dz = \delta_{-1,k}
\]

But also \( 1/2\pi i \int_{|z|=r} \phi^k(z) \phi'(z)\,dz = \delta_{-1,k} \), so \( (w(z) - \phi'(z)/2\pi i)\,dz \) annihilates all integral powers of \( \phi^* \), hence all integral powers of \( z \), so that \( w(z) = \phi'(z)/2\pi i \), and \( \phi^* \in H^1 \). This implies that \( \partial U \) is a rectifiable Jordan curve (see e.g., [3], p. 44). The theorem is now clear.

By similar methods we can prove:

3.2 Theorem. Let \( K \) be a compact plane set such that \( \Re (R(K)) \) has finite defect in \( C_\alpha(\partial K) \). Then the components \( \{U_i\}_{i \in I} \) of \( K^c \) are finitely connected and there is a measure \( \mu \) on \( \partial K \) such that \( \mu = 1 \) on \( K^c \), \( \mu = 0 \) off \( K \) if and only if the following three conditions hold.
(i) For each \( i \), \( \partial U_i \) is a cycle composed of rectifiable curves.

(ii) \( \sum_{i \in I} \text{length } \partial U_i < \infty \)

(iii) \( \mu = \frac{1}{2\pi i} d\zeta \text{ on } \bigcup_{i \in I} \partial U \) with appropriate orientation.

3.4 Theorem. Let \( K \) be a compact plane set with connected dense interior. Then there is a measure \( \mu \) with \( \hat{\mu} = 1 \) on \( K^0 \), \( \hat{\mu} = 0 \) off \( K \) if and only if

(i) The components of \( C - K \) are bounded by rectifiable curves \( \{\gamma_i\}_{i \in I} \) with finite total length and

(ii) \( \mu = 1/2\pi i d\zeta \text{ on } \bigcup_{i \in I} \gamma_i \) with appropriate orientation.

Proof. As before, the sufficiency of the two conditions is obvious. To prove the necessity, let \( \Delta \) be a large disk containing \( K \), and let \( \lambda = 1/2\pi i d\zeta \big|_{\Delta} - \mu \). Then \( \lambda = 1 \) on \( (\Delta - K^0)^0 = \Delta - K \), and \( \lambda = 0 \) off \( \Delta = K^0 \).

The hypotheses imply that \( \Delta - K^0 \) is finitely connected. In fact, the complement of \( \Delta - K^0 \) has two components, \( C - \Delta \) and \( K^0 \). Also, the components of \( (\Delta - K^0)^0 = \Delta - K \) are simply connected. As before, \( R(\Delta - K^0) \) is a Dirichlet algebra so we can apply Theorem 3.1 to \( \Delta - K^0 \). The conclusions (i) and (ii) follow easily.

3.4 Corollary. Let \( K \) be a compact plane set with connected dense interior. Then there is a measure \( \mu \) with \( \hat{\mu} = \chi_K dx dy \) a.e. if and only if \( K \) has finite Painlevé length.

References


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