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JEAN-MARIE G. ROLIN

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Let X be a standard process and A be a continuous additive functional of X . The inverse of A is defined by $\tau_t = \inf\{s : A_s > t\}$. The aim of this paper is to prove that the process τ has conditionally independent increments with respect to the σ -algebra generated by the time changed process $\hat{X}_t = X_{\tau_t}$. However these increments are not necessarily stationary. Another interesting result is derived: the continuous part of the process τ is a continuous additive functional of the process \hat{X} .

The existence of regular conditional probabilities permits to consider the process τ as an additive process and under a necessary and sufficient condition, it is in fact a Levy process with increasing paths. The general theory of such processes is then used to obtain a Levy representation of the jumps of the process τ .

1. Introduction. Let us consider a standard process $X = (\Omega, \mathcal{M}; E, \mathcal{E}, \Delta; \mathcal{M}_t, X_t, \theta_t, P^x)$ and a continuous additive functional (C.A.F.) of X . We refer to [1] for all the notations and definitions of such concepts.

It is well known in the theory of the Lebesgue-Stieltjes integral that if we define

$$(1.1) \quad \tau_t = \inf\{s : A_s > t\}$$

then for all nonnegative Borel functions f on $[0, \infty]$ vanishing at infinity, the following formula holds

$$(1.2) \quad \int_0^\infty f(t) dA_t = \int_0^\infty f(\tau_t) dt.$$

The aim of this paper is to investigate some of the probabilistic properties of this "inverse" of the continuous additive functional A .

It is easy to see that for each s , τ_s is a stopping time for X and it is known that under some additional assumptions, the time changed process X_{τ_t} is a standard process (see [1]–V–2, 11, and [3]).

Some important results have been established by Blumenthal and Gettoor in the case where the fine support of A consists of a single point

x_0 . That is the theory of local times that shows in particular that the process (τ_t, P^{x_0}) is more or less equivalent to a subordinator. For a precise statement of this theorem, refer to [1]–V–3.

We are now going to show that in the general case, the process τ has conditionally independent increments with respect to the σ -algebra generated by the process \hat{X} .

II. The conditioning. Let $X = (\Omega, \mathcal{M}, E, \mathcal{E}, \Delta; X_t, \mathcal{M}_t, \theta_t, P^x)$ be a standard process with lifetime ξ and A be a continuous additive functional of X . We will suppose that for all ω in Ω , the functions $t \rightarrow A_t(\omega)$ are continuous on $[0, \infty]$ and the paths functions $t \rightarrow X_t(\omega)$ are right continuous on $[0, \infty]$ and have left-hand limits on $[0, \xi(\omega))$. Let us introduce some notations. We will write

$$\begin{aligned} \hat{X}_t &= X_{\tau_t} \\ \hat{\theta}_t &= \theta_{\tau_t} \\ \hat{\mathcal{F}}_t &= \mathcal{F}_{\tau_t} \end{aligned}$$

$\hat{\mathcal{F}}_t^0$ and $\hat{\mathcal{F}}^0$ will have their usual meanings relative to the process \hat{X} and $\hat{\mathcal{F}}_t$ and $\hat{\mathcal{F}}$ will be their respective completions by the family P^μ as sub- σ -algebras of \mathcal{F} . To make this precise, A will be in $\hat{\mathcal{F}}_t$ ($\hat{\mathcal{F}}$) if for each finite measure μ on $(E_\Delta, \mathcal{E}_\Delta)$ there exist sets B_μ in $\hat{\mathcal{F}}_t^0$ ($\hat{\mathcal{F}}^0$) and N_μ in \mathcal{F}^0 such that $P^\mu(N_\mu) = 0$ and $B_\mu - N_\mu \subset A \subset B_\mu \cup N_\mu$. Let us remark that Y in \mathcal{F} will be in $\hat{\mathcal{F}}_t$ ($\hat{\mathcal{F}}$) if for each finite measure μ on $(E_\Delta, \mathcal{E}_\Delta)$ there exists Z_μ in $\hat{\mathcal{F}}_t$ ($\hat{\mathcal{F}}$) such that $Y = Z_\mu$ almost surely P^μ .

It follows immediately from these definitions that $\hat{\mathcal{F}}_t$ is contained in $\hat{\mathcal{F}}_s$ and $\hat{\mathcal{F}}$ is contained in \mathcal{F} . Since by definition, \hat{X}_t is in $\hat{\mathcal{F}}^0 | \mathcal{E}_\Delta$, it is clear that \hat{X}_t is in $\hat{\mathcal{F}} | \mathcal{E}_\Delta^*$ for each t , where \mathcal{E}_Δ^* denotes the σ -algebra of universally measurable sets over $(E_\Delta, \mathcal{E}_\Delta)$. It is also easy to see that $\hat{\theta}_t$ is in $\hat{\mathcal{F}}_{t+s} | \hat{\mathcal{F}}_s$ for all t, s and in particular, $\hat{\theta}_t$ is in $\hat{\mathcal{F}} | \hat{\mathcal{F}}$ for each t .

Now if we consider the lifetime ξ of the process \hat{X} , i.e. $\xi = \inf\{t: \hat{X}_t = \Delta\}$ we note that

$$(2.1) \quad \xi = A_\xi = A_\infty \quad \text{a.s.}$$

since $\{\hat{X}_t = \Delta\} = \{\tau_t \geq \xi\} = \{A_t \leq t\}$.

We are now ready to state some lemmas. The simplicity of their proofs will permit us to omit them.

LEMMA 2.1. *Let T be a $\{\hat{\mathcal{F}}_t\}$ stopping time. Then τ_T is a (\mathcal{F}_t) stopping time and $\hat{\mathcal{F}}_T = \mathcal{F}_{\tau_T}$. Moreover for all t ,*

$$(2.2) \quad \tau_{T+t} = \tau_T + \tau_t \circ \hat{\theta}_T \quad \text{a.s.}$$

and

$$(2.3) \quad A_{\tau} = T \text{ on } \{T < \hat{\xi}\}.$$

LEMMA 2.2. *Let T be a $\{\mathcal{F}_t\}$ stopping time. Then A_T is a $\{\hat{\mathcal{F}}_t\}$ stopping time and \mathcal{F}_T is contained in $\hat{\mathcal{F}}_{A_T}$. Moreover for all t ,*

$$(2.4) \quad \tau_{A_T+t} = T + \tau_t \circ \theta_T \text{ a.s. on } \{A_T < \infty\}.$$

LEMMA 2.3. *Let Y be in $\hat{\mathcal{F}}$ and T be a $\{\mathcal{F}_t\}$ stopping time. Then*

$$(2.5) \quad Y \circ \hat{\theta}_{A_T} = Y \circ \theta_T \text{ a.s. on } \{A_T < \infty\}.$$

In particular, if we take $T \equiv 0$, then

$$(2.6) \quad Y = Y \circ \hat{\theta}_0 \text{ a.s.}$$

Let us turn now to some considerations related to the support of the continuous additive functional A . We will denote it by F . By definition

$$F = \{x \in E : P^x(\tau_0 = 0) = 1\}$$

It is known (see [1]-V-3) that F is a nearly Borel set which is finely perfect, i.e. the set of regular points for F is precisely F , and that is a consequence of the fact that

$$(2.7) \quad T_F = \tau_0 \text{ a.s.}$$

where T_F is the hitting time of the set F . Moreover for all x in E_Δ ,

$$P^x[\hat{X}_t \notin F \text{ for some } t < \hat{\xi}] = 0.$$

Using this result, we can and we will from now on, suppose that the process \hat{X} lives on $F \cup \{\Delta\}$. It is also easy to prove that for all $\{\mathcal{F}_t\}$ stopping times T ,

$$(2.8) \quad \{X_T \in F\} = \{\tau_0 \circ \theta_T = 0\} \text{ a.s.}$$

In the sequel, we will have to deal with expressions of the form $E^x(Z|\hat{\mathcal{F}}_t)(\omega)$ where Z is in $b\hat{\mathcal{F}}$. It is not difficult to see, using the fact that $\hat{\mathcal{F}}_t^0$ is countably generated and the martingale convergence theorem, that we can choose a version which is jointly measurable in x and

ω . More precisely: if Z is in $b\mathcal{F}$ and $t \leq \infty$, then there exists $Z_t^x(\omega)$ in $b\mathcal{E}_\Delta^* \otimes \hat{\mathcal{F}}_t$ such that for all x in E_Δ , $E^x(Z|\hat{\mathcal{F}}_t) = Z_t^x$ a.s. P^x . Since $E^x(Z|\hat{\mathcal{F}}_t)$ is only defined a.s. P^x , we will always suppose when writing expressions such as $E^x(Z|\hat{\mathcal{F}}_t)(\omega)$ that it is jointly measurable in x and ω .

We now come to an important lemma.

LEMMA 2.4. *Let $Z_1^x(\omega), Z_2^x(\omega)$ be in $b\mathcal{E}_\Delta^* \otimes \hat{\mathcal{F}}$ and such that, for each x in $F \cup \{\Delta\}$, $Z_1^x = Z_2^x$ a.s. P^x . Then*

$$Z_1^{x_0} = Z_2^{x_0} \quad \text{a.s.}$$

Proof. Clearly $Z_1^{x_0}$ is in $b\hat{\mathcal{F}}$ and by the preceding lemma, for all finite measures μ on $(E_\Delta, \mathcal{E}_\Delta)$ and for all A in $\hat{\mathcal{F}}$

$$E^\mu(1_A Z_1^{x_0}) = E^\mu E^{x_0}(1_A Z_1^{x_0}) = \int_{F \cup \{\Delta\}} E^x(1_A Z_1^{x_0}) P^\mu(\hat{X}_0 \in dx).$$

Now if x is in F , $\tau_0 = 0$ a.s. P^x and $\hat{X}_0 = x$ a.s. P^x . If $x = \Delta$, $\tau_0 = \infty$ a.s. P^Δ and $\hat{X}_0 = \Delta$ a.s. P^Δ . Hence for all x in $F \cup \{\Delta\}$,

$$E^x(1_A Z_1^{x_0}) = E^x(1_A Z_1^x) = E^x(1_A Z_2^x) = E^x(1_A Z_2^{x_0})$$

and

$$E^\mu(1_A Z_1^{x_0}) = E^\mu(1_A Z_2^{x_0}).$$

That implies that

$$Z_1^{x_0} = Z_2^{x_0} \quad \text{a.s.} \quad P^\mu,$$

and the conclusion holds since μ is arbitrary.

In the sequel, we will usually omit the ω 's when writing expressions such as $E^{x_t(\hat{\theta}_s, \omega)}[Z|\hat{\mathcal{F}}_u](\hat{\theta}_s, \omega)$. We will write

$$E^{x_t}[Z|\hat{\mathcal{F}}_v] \circ \hat{\theta}_s(\omega) = E^{x_t(\hat{\theta}_s, \omega)}[Z|\hat{\mathcal{F}}_v](\hat{\theta}_s, \omega)$$

$$E^{x_t}[Z|\hat{\mathcal{F}}_v](\hat{\theta}_s)(\omega) = E^{x_t(\omega)}[Z|\hat{\mathcal{F}}_v](\hat{\theta}_s, \omega).$$

For instance, we have almost surely

$$E^{x_t}(Z|\hat{\mathcal{F}}_v) \circ \hat{\theta}_s = E^{x_{t+s}}[Z|\hat{\mathcal{F}}_v](\hat{\theta}_s).$$

We are now ready to state the main theorem of this section.

THEOREM 2.5. *Let μ be a finite measure on $(E_\Delta, \mathcal{E}_\Delta)$ and let Y be in $b\tilde{\mathcal{F}}_t$ and Z be in $b\mathcal{F}$. Then*

$$(2.9) \quad E^\mu(YZ \circ \hat{\theta}_t | \hat{\mathcal{F}}) = E^\mu(Y | \hat{\mathcal{F}})E^{\hat{x}_0}(Z | \hat{\mathcal{F}}) \circ \hat{\theta}_t$$

a.s. P^μ for all t .

This theorem has several immediate corollaries.

COROLLARY 2.6. *Let μ and Z be as in 2.5. Then*

$$(2.10) \quad E^\mu(Z \circ \hat{\theta}_t | \hat{\mathcal{F}}) = E^{\hat{x}_0}(Z | \hat{\mathcal{F}}) \circ \hat{\theta}_t \quad \text{a.s. } P^\mu \text{ for all } t.$$

If we take $\mu = \epsilon_x$ and if we apply Lemma 2.4, we get the following results.

COROLLARY 2.7. *Let Y be in $b\tilde{\mathcal{F}}_t$ and Z be in $b\mathcal{F}$. Then, for all t , almost surely*

$$(2.11) \quad E^{\hat{x}_0}(YZ \circ \hat{\theta}_t | \hat{\mathcal{F}}) = E^{\hat{x}_0}(Y | \hat{\mathcal{F}})E^{\hat{x}_0}(Z | \hat{\mathcal{F}}) \circ \hat{\theta}_t.$$

In particular, if we set $Y \equiv 1$,

$$(2.12) \quad E^{\hat{x}_0}(Z \circ \hat{\theta}_t | \hat{\mathcal{F}}) = E^{\hat{x}_0}(Z | \hat{\mathcal{F}}) \circ \hat{\theta}_t \quad \text{a.s. for all } t.$$

Proof. Let us consider the following random variable

$$W = \prod_1^n f_i(\hat{X}_i),$$

where f_i are in $b\mathcal{E}_\Delta$ for $1 \leq i \leq n$ and $0 \leq t_1 < t_2 < \dots < t_n$. Clearly we can write $W = W_1 W_2 \circ \hat{\theta}_t$ a.s. where W_1 is in $\hat{\mathcal{F}}_t^0$ and W_2 is in $\hat{\mathcal{F}}^0$. Now

$$E^\mu(WYZ \circ \hat{\theta}_t) = E^\mu[W_1 Y E^{\hat{x}_t}(W_2 Z)].$$

We know that \hat{X}_i is in F almost surely on $\{t < \hat{\xi}\}$. On $\{t \geq \hat{\xi}\}$ $\tau_t = \infty$ and consequently $\hat{X}_t = \Delta$. On the other hand, we already saw that $\hat{X}_0 = x$ a.s. P^x for all x in $F \cup \{\Delta\}$. Therefore for all x in $F \cup \{\Delta\}$,

$$E^x([W_2 E^{\hat{x}_0}(Z | \hat{\mathcal{F}})]) = E^x[W_2 E^x(Z | \hat{\mathcal{F}})] = E^x(W_2 Z).$$

So, we have

$$\begin{aligned} E^\mu(WYZ \circ \hat{\theta}_t) &= E^\mu[W_1 Y E^{\hat{x}_t}(W_2 E^{\hat{x}_0}(Z | \hat{\mathcal{F}}))] \\ &= E^\mu[W_1 Y E^{\hat{x}_0}(Z | \hat{\mathcal{F}}) \circ \hat{\theta}_t] \\ &= E^\mu[W E^\mu(Y | \hat{\mathcal{F}}) E^{\hat{x}_0}(Z | \hat{\mathcal{F}}) \circ \hat{\theta}_t], \end{aligned}$$

if we recall that by convention $E^{X_0}(Z|\hat{\mathcal{F}})$ is in $\hat{\mathcal{F}}$, and $\hat{\theta}_t$ is in $\hat{\mathcal{F}}|\hat{\mathcal{F}}$. Using the monotone class theorem, we see that the last equality is true for all W in $b\hat{\mathcal{F}}^0$. If W is in $b\hat{\mathcal{F}}$ there exists W_μ in $b\hat{\mathcal{F}}^0$ such that $W = W_\mu$ a.s. P^μ . Hence the equality holds for every W in $b\hat{\mathcal{F}}$ and the theorem is proven. Using the corollary 2.6, we see that the formula of the Theorem 2.5 may be written

$$(2.13) \quad E^\mu(YZ \circ \hat{\theta}_t|\hat{\mathcal{F}}) = E^\mu(Y|\hat{\mathcal{F}}) E^\mu(Z \circ \hat{\theta}_t|\hat{\mathcal{F}}) \text{ a.s. } P^\mu \text{ for all } t.$$

The intuitive meaning of the Theorem 2.5, is now clear. What happened before and after the time τ_t , are conditionally independent given the process in the support of the continuous additive functional.

We will end this section by a proposition which is closely related to Theorem 2.5.

PROPOSITION 2.8. *Let μ be a finite measure on $(E_\Delta, \mathcal{E}_\Delta)$ and let Y be in $b\hat{\mathcal{F}}_t$. Then*

$$(2.14) \quad E^\mu(Y|\hat{\mathcal{F}}) = E^\mu(Y|\hat{\mathcal{F}}_t) \text{ a.s. } P^\mu.$$

Proof. Let us prove first that for all Z in $b\hat{\mathcal{F}}$,

$$(2.15) \quad E^\mu(Z|\hat{\mathcal{F}}_t) = E^\mu(Z|\hat{\mathcal{F}}_t) \text{ a.s. } P^\mu.$$

If we consider $Z = \prod_1^n f_i(\hat{X}_{t_i})$ where f_i are in $b\mathcal{E}_\Delta$ and $0 \leq t_1 < t_2 < \dots < t_n$, then as before, we can write $Z = Z_1 Z_2 \circ \hat{\theta}_t$ a.s. where Z_1 is in $b\hat{\mathcal{F}}_t^0$ and Z_2 is in $b\hat{\mathcal{F}}^0$. Hence

$$E^\mu(Z|\hat{\mathcal{F}}_t) = Z_1 E^{X_t} Z_2 \text{ a.s. } P^\mu,$$

and since the right hand side is in $\hat{\mathcal{F}}_t$, (2.8) holds. By the monotone class theorem and the properties of the completion, (2.8) is clearly true for all Z in $b\hat{\mathcal{F}}$.

Now, for Z in $b\hat{\mathcal{F}}$ and Y in $b\hat{\mathcal{F}}_t$,

$$\begin{aligned} E^\mu(YZ) &= E^\mu[YE^\mu(Z|\hat{\mathcal{F}}_t)] \\ &= E^\mu[YE^\mu(Z|\hat{\mathcal{F}}_t)] \\ &= E^\mu[ZE^\mu(Y|\hat{\mathcal{F}}_t)]. \end{aligned}$$

Hence $E^\mu(Y|\hat{\mathcal{F}}) = E^\mu(Y|\hat{\mathcal{F}}_t)$ a.s. P^μ .

If we use Lemma 2.4, this proposition has a straightforward corollary.

COROLLARY 2.9. *Let Y be in $b\hat{\mathcal{F}}_1$. Then $E^{x_0}(Y|\hat{\mathcal{F}})$ is in $\hat{\mathcal{F}}$.*

In the following chapter we will be mainly concerned with the operator $T: b\mathcal{F} \rightarrow b\hat{\mathcal{F}}$ defined by

$$(2.16) \quad TZ = E^{x_0}(Z|\hat{\mathcal{F}}),$$

Even if this operator is not a conditional expectation, it has all its important properties. For instance

$$(2.17) \quad T(\alpha Z) = \alpha TZ.$$

$$(2.18) \quad T(Z_1 + Z_2) = TZ_1 + TZ_2.$$

$$(2.19) \quad TZ \geq 0 \quad \text{if} \quad Z \geq 0.$$

$$(2.20) \quad TZ_n \uparrow TZ \quad \text{if} \quad Z_n \uparrow Z.$$

(2.21) If Y is in $b\hat{\mathcal{F}}$ and Z is in $b\mathcal{F}$ then $T(YZ) = YTZ$.
In particular, Y is in $b\hat{\mathcal{F}}$ if and only if $TY = Y$.

$$(2.22) \quad T1 = 1,$$

all these statements being true almost surely. They are easy to verify by applying Lemma 2.4 to the corresponding properties of the conditional expectations with respect to the measures P^x . For instance, if $Z \geq 0$, let $B = \{(x, \omega): E^x[Z|\hat{\mathcal{F}}](\omega) \geq 0\}$. Then 1_B is in $b\mathcal{E}_\Delta^* \otimes \hat{\mathcal{F}}$ and $1_B(x, \cdot) = 1$ a.s. P^x for all x in E_Δ . So $1_B[\hat{X}_0(\omega), \omega] = 1$ a.s. Hence $TZ \geq 0$ a.s. Also if Z_n increases to Z ,

$$\sup_n E^x(Z_n|\hat{\mathcal{F}}) = E^x(Z|\hat{\mathcal{F}}) \quad \text{a.s. } P^x$$

for all x in E_Δ . Hence $\sup_n TZ_n = TZ$ a.s.

It is also useful to remark that if $Z = 0$ a.s. P^x for all x in $F \cup \{\Delta\}$, then $TZ = 0$ a.s.

Moreover the main theorem of this section and its corollaries may be expressed in terms of T by the following statement:

Let Y be in $b\mathcal{F}_t$ and Z be in $b\mathcal{F}$ then TY is in $b\mathcal{F}_t$,

$$(2.23) \quad T(YZ \circ \hat{\theta}_t) = (TY)(TZ) \circ \hat{\theta}_t \text{ a.s.}$$

and in particular

$$(2.24) \quad T(Z \circ \hat{\theta}_t) = (TZ) \circ \hat{\theta}_t \text{ a.s.}$$

The aim of the next chapter will be to prove that under certain conditions T may be considered as an integral operator.

III. The regularization. In this section, we will suppose that X is a standard process with the property that the measurable space (Ω, \mathcal{F}^0) is a standard Borel space. This is the case if the process X is of function space type, i.e., if Ω consists of all the functions from $[0, \infty)$ into E_Δ which are right continuous on $[0, \infty)$ and have left-hand limits on $[0, \infty)$ and if the random variables X_t are the coordinate functions $[X_t(\omega) = \omega(t)]$. In this situation (Ω, \mathcal{F}^0) is a Polish space and then a standard Borel space. See for instance [6] and [7].

Under this hypothesis, we can prove the following theorem.

THEOREM 3.1. *There exists a function*

$$P^\omega(A): \Omega X\mathcal{F}^0 \rightarrow R$$

such that, for each ω in Ω , $P^\omega(\cdot)$ is a probability measure on \mathcal{F}^0 , for each set A in \mathcal{F}^0 , $P \cdot (A)$ is in $b\mathcal{F}$ and for all Z in $b\mathcal{F}^0$, the following relation holds almost surely in ω .

$$(3.1) \quad \begin{aligned} TZ(\omega) &= E^{X_0(\omega)}[Z | \hat{\mathcal{F}}](\omega) \\ &= E^\omega Z = \int_\Omega Z(\omega') P^\omega(d\omega'). \end{aligned}$$

Proof. Let Q be the rational numbers. Since (Ω, \mathcal{F}^0) is a standard Borel space, there exists an increasing right continuous sequence of sets in \mathcal{F}^0 , A_r , r in Q , such that

$$\bigcap_{r \in Q} A_r = \phi \quad \text{and} \quad \bigcup_{r \in Q} A_r = \Omega.$$

\mathcal{F}^0 is the σ -algebra generated by this collection. Moreover, if F is a probability distribution function on this sequence, i.e., an increasing right continuous set function on this sequence with the property that

$\inf_{r \in Q} F(A_r) = 0$ and $\sup_{r \in Q} F(A_r) = 1$, then F can be extended in a unique way to a probability measure on \mathcal{F}^0 . Indeed, this statement becomes evident if we take $A_r = \varphi^{-1}[(-\infty, r]]$ where φ is a bijective measurable function from Ω into a Borel subset of the real line such that φ^{-1} is measurable, and such a function exists by the definition of a standard Borel space.

Now let $Q^\omega(A_r)$ be versions of $T1_{A_r}(\omega)$. Let us define

$$N_1 = \bigcup_{r \in Q} \left\{ \omega : \inf_{\substack{s \in Q \\ s > r}} Q^\omega(A_s) \neq Q^\omega(A_r) \right\}$$

$$N_2 = \left\{ \omega : \sup_{r \in Q} Q^\omega(A_r) \neq 1 \right\}$$

$$N_3 = \left\{ \omega : \inf_{r \in Q} Q^\omega(A_r) \neq 0 \right\}$$

$$N = N_1 \cup N_2 \cup N_3.$$

Clearly, N is in $\hat{\mathcal{F}}$. Moreover for all finite measures μ on $(E_\Delta, \mathcal{E}_\Delta)$, $\mu(N) = 0$. Indeed, $T1_{A_r} \leq \inf_{s > r} T1_{A_s}$ a.s. and if s_n decreases to r , $1_{A_{s_n}}$ decreases to 1_{A_r} and consequently $T1_{A_{s_n}}$ decreases to $T1_{A_r}$ a.s. This implies that $T1_{A_r} = \inf_{s > r} T1_{A_s}$ a.s. Similarly $\sup_{r \in Q} T1_{A_r} = 1$ a.s. and $\inf_{r \in Q} T1_{A_r} = 0$ a.s.

Now let F be any probability distribution function on the sequence $\{A_r : r \text{ in } Q\}$ and let us define

$$P^\omega(A_r) = Q^\omega(A_r) 1_{N^c}(\omega) + F(A_r) 1_N(\omega).$$

clearly $P \cdot (A_r)$ is in $b\hat{\mathcal{F}}$ for all r in Q and for all ω in Ω P^ω is a probability distribution function on the sequence A_r . Let us also denote by P^ω , the unique extension of P^ω to a probability measure on \mathcal{F}^0 . If we define $\mathcal{C} = \{A \in \mathcal{F}^0 : P \cdot (A) \in \hat{\mathcal{F}}\}$ then \mathcal{C} is a σ -algebra containing A_r for all r in Q . Hence for all A in \mathcal{F}^0 , $P \cdot (A)$ is in $\hat{\mathcal{F}}$. Now let $H = \{Z \in b\mathcal{F}^0 : TZ = E \cdot Z \text{ a.s.}\}$. H is a linear space containing 1_{A_r} for all r in Q . Moreover if Z_n in H^+ increases to Z bounded, then Z is in H . By the monotone class theorem, the proof is complete if we remark that the collection $\{A_r - A_s : s < r, s, r \in Q\}$ is a π -system generating \mathcal{F}^0 .

From now on, we will restrict our attention to the stochastic process $\tau = \{\tau_t : 0 \leq t < \infty\}$. Unhappily, the measures P^ω we have just constructed can only be defined on \mathcal{F}^0 and in the general case, τ is not \mathcal{F}^0 measurable. However, if we suppose that there exists a reference

measure for X (see [1] V-1), then the C.A.F. A is equivalent to a perfect C.A.F. B such that each B_t is in \mathcal{F}^0 (see [1] V-2.1 and 2.10). So without loss of generality, we may and we will assume that A is a perfect C.A.F. and each A_t is \mathcal{F}^0 measurable.

Since $\{\tau_t < s\} = \{A_s > t\}$, that implies that the process τ is \mathcal{F}^0 measurable. Moreover in this situation the support of the C.A.F. is a Borel set for $P^x(\tau_0 = 0)$ is in $b\mathcal{E}_\Delta$.

One more remark: Later we will have to consider the increments of the process τ , i.e., $\tau_{t+s} - \tau_t$, and this is not defined on the set $\{\tau_t = \infty\}$. However if we set $\tau_{t+s} - \tau_t = \infty$ on $\{\tau_t = \infty\}$, this random variable is in \mathcal{F}^0 and $\tau_{t+s} - \tau_t = \tau_s \circ \hat{\theta}_t$ a.s. Indeed

$$\begin{aligned} P^x[\tau_s \circ \hat{\theta}_t < \infty, \tau_t = \infty] &= E^x[P^{X_t}(\tau_s < \infty); \tau_t = \infty] \\ &= P^\Delta(\tau_s < \infty)P^x(\tau_t = \infty) = 0 \end{aligned}$$

for $\tau_0 = \infty$ a.s. P^Δ .

We are now ready to state the main theorem of this section.

THEOREM 3.2. *There exists a set N in \mathcal{F} with $P^\mu(N) = 0$ for all finite measures μ on $(E_\Delta, \mathcal{E}_\Delta)$, such that for all ω in $\{\hat{\xi} > 0\} - N$, the process*

$$\{\tau_t : 0 \leq t < \hat{\xi}(\omega)\}$$

is an additive process on $(\Omega, \mathcal{F}^0, P^\omega)$ (i.e., a process with independent increments such that $\tau_0 = 0$ a.s. P^ω).

Proof. Let us prove first that for all $t \geq 0$ the σ -algebras

$$\mathcal{K}_t = \sigma\{\tau_s : 0 \leq s \leq t\}$$

and

$$\mathcal{L}_t = \sigma\{\tau_{t+s} - \tau_t : 0 < s < \infty\}$$

are independent with respect to P^ω for almost all ω . Indeed the right continuity of τ , implies that \mathcal{K}_t and \mathcal{L}_t are generated by countable π -systems containing Ω , let us say \mathcal{K}_t^0 and \mathcal{L}_t^0 respectively. On the other hand, \mathcal{K}_t is contained in \mathcal{F}_t and \mathcal{L}_t is contained in $\hat{\theta}_t^{-1}(\mathcal{F})$. Using Corollary 2.7, this implies that

$$P^\omega(A \cap B) = P^\omega(A)P^\omega(B) \text{ a.s.}$$

for A in \mathcal{H}_t and B in \mathcal{L}_t .

Let us define

$$N_t = \bigcup_{A \in \mathcal{H}_t^c} \bigcup_{B \in \mathcal{L}_t^c} \{\omega : P^\omega(A \cap B) \neq P^\omega(A)P^\omega(B)\}.$$

Clearly N_t is in $\hat{\mathcal{F}}$ and $P^\mu(N_t) = 0$ for all μ . Using twice the monotone class theorem, it is easy to see that for all ω in N_t^c ,

$$P^\omega(A \cap B) = P^\omega(A)P^\omega(B)$$

for all A in \mathcal{H}_t , B in \mathcal{L}_t .

Now let $N_1 = \bigcup_{t \in Q^+} N_t$. Then for all ω in N_1^c , $t \geq 0$, $s > 0$, $\tau_{t+s} - \tau_t$ is independent of \mathcal{H}_t with respect to P^ω . For if A is in \mathcal{H}_t and $u \geq 0$, let us choose r_n in Q^+ such that $r_n \downarrow t$ and $r_n < t + s$. Then, by the right continuity of τ , we have

$$\begin{aligned} P^\omega[A \cap \{\tau_{t+s} - \tau_t \leq v\}] &= \lim_{r_n \downarrow t} P^\omega[A \cap \{\tau_{t+s} - \tau_{r_n} \leq v\}] \\ &= \lim_{r_n \downarrow t} P^\omega(A) P^\omega(\tau_{t+s} - \tau_{r_n} \leq v) \\ &= P^\omega(A) P^\omega(\tau_{t+s} - \tau_t \leq v). \end{aligned}$$

Now since $\{\hat{\xi} \leq t\} = \{\hat{X}_t = \Delta\}$, $\hat{\xi}$ is in $\hat{\mathcal{F}}$ and so

$$(3.2) \quad P^\omega[\hat{\xi} \neq \hat{\xi}(\omega)] = 0$$

for almost all ω . Also $e^{-\tau_0} = 1_F(\hat{X}_0)$ a.s. P^ω for all x in $F \cup \{\Delta\}$. Hence by Lemma 2.4

$$(3.3) \quad E^\omega(e^{-\tau_0}) = 1_F[\hat{X}_0(\omega)]$$

for almost all ω . Let N_2 be the set of ω 's for which either (3.2) or (3.3) is not satisfied and let $N = N_1 \cup N_2$. Clearly N is in $\hat{\mathcal{F}}$ and $P^\mu(N) = 0$ for all μ . If ω is in $\{\hat{\xi} > 0\} - N$, $\tau_0 = 0$ a.s. P^ω and the process τ is finite a.s. P^ω on $[0, \hat{\xi}(\omega))$ since $\hat{\xi} = \hat{\xi}(\omega)$ a.s. P^ω . Also for all $t < \hat{\xi}(\omega)$ and s in $(0, \hat{\xi}(\omega) - t)$ $\tau_{t+s} - \tau_t$ is independent of $\sigma\{\tau_\cup : 0 \leq \cup \leq t\}$ with respect to P^ω . That concludes the proof of the theorem.

Let us remark that there is no interest in considering the process τ with respect to P^ω for ω in $\{\hat{\xi} = 0\}$ because (3.3) implies that $\tau_0 = \infty$ a.s. P^ω for almost all ω in $\{\hat{\xi} = 0\}$.

Let us also note that the process τ is not homogeneous. Indeed

$$(3.4) \quad P^\omega(\tau_{t+s} - \tau_t \in B) = P^{\hat{\xi}(\omega)}(\tau_s \in B)$$

for almost all ω in Ω .

If we consider the particular case where the support of the C.A.F. is a single point x_0 , then

$$\hat{\mathcal{F}}^0 = \sigma(A_\infty)$$

for $\{\hat{X}_t = x_0\} = \{A_\infty > t\}$. Moreover there exists $\gamma \geq 0$, such that $P^{x_0}(A_\infty > t) = e^{-\gamma t}$. If $\gamma = 0$, $A_\infty = \infty$ a.s. P^{x_0} and it follows easily that $P^\omega = P^{x_0}$ for almost all ω in Ω . In this situation, we have as a corollary of Theorem 3.2, that the process $\tau = \{\tau_t : 0 \leq t < \infty\}$ is a homogeneous additive process with increasing paths on $(\Omega, \mathcal{F}^0, P^{x_0})$.

It is now clear that Theorem 3.2 generalizes the theorem which appears in [1]-V-3.21.

In order to obtain a Levy's decomposition of the process τ , Theorem 3.2 is not sufficient. We also need the fact that the process τ is continuous in probability with respect to P^ω for almost all ω in $\{\hat{\xi} > 0\}$ or equivalently the functions $t \rightarrow E^\omega(e^{-\tau_t})$ are continuous on $[0, \hat{\xi}(\omega))$ for almost all ω in $\{\hat{\xi} > 0\}$. Indeed, since $E^\omega(e^{-\tau_t}) = E^\omega(e^{-\tau_{t-s}}) E^\omega(e^{-\tau_{t-s}})$ if we let s decrease to zero, we have $E^\omega(e^{-\tau_t}) = E^\omega(e^{-\tau_{t-}}) E^\omega[e^{-\tau_{t-t-}}]$ and then if $t < \hat{\xi}(\omega)$ $\tau_{t-} = \tau_t$ a.s. P^ω if and only if $E^\omega(e^{-\tau_t}) = E^\omega(e^{-\tau_{t-}})$.

It is easy to see that in the general case, this condition will not be satisfied. However we have the following theorem.

THEOREM 3.3. *There exists N in $\hat{\mathcal{F}}$ with $P^\mu(N) = 0$ for all finite measures μ on $(E_\Delta, \mathcal{E}_\Delta)$, such that for all ω in $\{\hat{\xi} > 0\} - N$, the function $t \rightarrow E^\omega(e^{-\tau_t})$ is continuous on $[0, \hat{\xi}(\omega))$ if and only if the following assumption holds.*

Assumption 3.4. For all $\{\hat{\mathcal{F}}_t\}$ stopping times T ,

$$\tau_{T-} = \tau_T \text{ a.s. on } \{0 < T < \hat{\xi}\}$$

Proof. Let $\Omega_0 = \Omega - N$ where N is the set of measure zero appearing in the statement of Theorem 3.2. Let us set $C_t(\omega) = e^{-\tau_t(\omega)}$ and $\hat{C}_t(\omega) = E^\omega(e^{-\tau_t})$ if ω is in Ω_0 and $\hat{C}_t(\omega) = 0$ otherwise. Then we define for $\epsilon > 0$,

$$T_\epsilon = \inf \{t > 0 : \hat{C}_{t-} - \hat{C}_t \geq \epsilon\}$$

and $T_\epsilon = \hat{\xi}$ if the set in braces is empty. Now if we prove that for all finite measures μ on $(E_\Delta, \mathcal{E}_\Delta)$, $P^\mu(T_\epsilon < \hat{\xi}) = 0$, the sufficiency is established. Indeed if $N' = \cup_n \{T_n^1 < \hat{\xi}\}$, $P^\mu(N') = 0$ and for all ω in $\{\hat{\xi} > 0\} - N'$, $\hat{C}_{t-}(\omega) = \hat{C}_t(\omega)$ for all $t < \hat{\xi}(\omega)$. This implies the continuity of $E^\omega(e^{-\tau_t})$ on $[0, \hat{\xi}(\omega)]$ for all ω in $\{\hat{\xi} > 0\} - (N \cup N')$.

Let us write $T_\epsilon = T$. Since $\hat{C}_t = 0$ on $\{\hat{\xi} \leq t\}$, $T \leq \hat{\xi}$. Moreover T is a $\{\mathcal{F}_t\}$ stopping time for $\{T \leq t\} = \{T \leq t, \hat{\xi} > t\} \cup \{\hat{\xi} \leq t\}$ and if Q_t denotes the rationals in $(0, t)$

$$\begin{aligned} \{T \leq t < \hat{\xi}\} &= \bigcap_m \bigcup_{\substack{r \in Q_t, \cup(t) \\ s \in Q_t}} \{\hat{C}_s - \hat{C}_r \geq \epsilon, \hat{\xi} > t\} \\ 0 < r - s &< \frac{1}{m}. \end{aligned}$$

Hence $\{T \leq t\}$ is in \mathcal{F}_t since \hat{C}_t is clearly in \mathcal{F}_t . Also if $T(\omega) < \hat{\xi}(\omega)$, $\hat{C}_{T(\omega)-}(\omega) - \hat{C}_{T(\omega)}(\omega) \geq \epsilon$.

Now if $G(\omega, \omega')$ is in $b\mathcal{F} \otimes \mathcal{F}^0$ and if $\bar{G}(\omega) = G(\omega, \omega)$ it is easy to see that for almost all ω ,

$$E^{X_0(\omega)}[\bar{G} | \mathcal{F}](\omega) = \int_{\Omega} G(\omega, \omega') P^\omega(d\omega')$$

Since for all $\{\mathcal{F}_t\}$ stopping times T , $\tau_{T(\omega)}(\omega')$ and $\tau_{T(\omega)-}(\omega')$ are clearly in $\mathcal{F} \otimes \mathcal{F}^0$, we have

$$\hat{C}_T = E^{X_0}(C_T | \mathcal{F}) \quad \text{a.s. and}$$

$$\hat{C}_{T-} = E^{X_0}(C_{T-} | \mathcal{F}) \quad \text{a.s.}$$

Hence

$$\begin{aligned} P^\mu(T < \hat{\xi}) &\leq \frac{1}{\epsilon} E^\mu[\hat{C}_{T-} - \hat{C}_T; T < \hat{\xi}] \\ &\leq \frac{1}{\epsilon} E^\mu\{E^{X_0}[(C_{T-} - C_T)1_{(T < \hat{\xi})}] | \mathcal{F}; \hat{\xi} > 0\} \\ &\leq \frac{1}{\epsilon} E^\mu\{E^{X_0}[(C_{T-} - C_T)1_{(T < \hat{\xi})}] \circ \theta_0; \hat{\xi} > 0\} \\ &\leq \frac{1}{\epsilon} E^\mu\{E^{X_0}[C_{T-} - C_T; T < \hat{\xi}]; \hat{\xi} > 0\} \\ &\leq 0 \end{aligned}$$

for $C_{T-} = C_T$ a.s. on $\{0 < T < \hat{\xi}\}$.

For the necessity of Assumption 3.4, note that for all $\{\mathcal{F}_t\}$ stopping times T , $\hat{C}_{T-} = \hat{C}_T$ a.s. on $\{0 < T < \hat{\xi}\}$. Then for all x in F ,

$$\begin{aligned} 0 &= E^x[\hat{C}_{T-} - \hat{C}_T; 0 < T < \hat{\xi}] \\ &= E^x[E^{X_0}(C_{T-} - C_T | \mathcal{F}); 0 < T < \hat{\xi}] \\ &= E^x[C_{T-} - C_T; 0 < T < \hat{\xi}] \end{aligned}$$

and so $\tau_{T-} = \tau_T$ a.s. P^x on $\{0 < T < \hat{\xi}\}$ for all x in F . Now since $\tau_t = \tau_0 + \tau_t \circ \theta_0$ for all t almost surely, it is clear that $\tau_{t-} = \tau_0 + \tau_{t-} \circ \theta_0$ for all t almost surely and then $\tau_T - \tau_{T-} = (\tau_T - \tau_{T-}) \circ \hat{\theta}_0$ a.s. on $\{0 < T < \hat{\xi}\}$. Hence for all finite measures μ on $(E_\Delta, \mathcal{E}_\Delta)$, $P^\mu(\tau_T - \tau_{T-} = 0, 0 < T < \hat{\xi}) = P^\mu[(\tau_T - \tau_{T-}) \circ \hat{\theta}_0 = 0, 0 < T \circ \hat{\theta}_0 < \hat{\xi} \circ \hat{\theta}_0, \hat{\xi} > 0] = E^\mu[P^{X_0}(\tau_T - \tau_{T-} = 0, 0 < T < \hat{\xi}); \hat{\xi} > 0] = 0$ since X_0 is in F on $\{\hat{\xi} > 0\}$.

This finishes the proof of Theorem 3.3 and we have this straightforward corollary.

COROLLARY 3.5. *Under the Assumption 3.4, there exists N in \mathcal{F} with $P^\mu(N) = 0$ for all finite measures μ on $(E_\Delta, \mathcal{E}_\Delta)$, such that for all ω in $\{\hat{\xi} > 0\} - N$, the process $\{\tau_t : 0 \leq t < \hat{\xi}(\omega)\}$ is a Levy process with increasing paths on $(\Omega, \mathcal{F}^0, P^\omega)$.*

IV. The decomposition. Since τ is an increasing right continuous process, we can decompose it into its continuous and purely discontinuous parts. Let

$$(4.1) \quad \tau_t = \tau_0 + \tau_t^c + \tau_t^i$$

be this decomposition. If we denote by $K(\omega)$ the set of discontinuity points of the function $t \rightarrow \tau_t(\omega)$ for $t > 0$, then

$$(4.2) \quad \tau_t^c = \int_{(0,t]} 1_{K^c}(s) d\tau_s.$$

$$(4.3) \quad \tau_t^i = \sum_{0 < s \leq t} (\tau_s - \tau_{s-}).$$

Let us first restrict our attention to the continuous part of τ .

LEMMA 4.1. *Let τ_t^c be the continuous part of τ_t . Then, almost surely,*

$$(4.4) \quad \tau_t^c = \int_0^{\tau_t} 1_F(X_s) ds \text{ for all } t.$$

Proof. Let us recall the following change of variables formula. If $a(t)$ is a nonnegative increasing right continuous function on $[0, \infty]$ and

$$\bar{a}(t) = \inf \{s : a(s) > t\},$$

then for all nonnegative Borel functions g on $[0, \infty)$

$$\int_{(0, \infty)} g(t) da(t) = \int_{a(0)}^{a(\infty)} g[\bar{a}(t)] dt,$$

where $a(\infty) = \lim_{t \uparrow \infty} a(t)$. Applying this formula to (4.2), we have

$$\tau_t^c = \int_{\tau_0}^{\tau_\infty} 1_{(0, t]}(A_s) 1_{K^c}(A_s) ds.$$

It is easy to see that

$$1_{(0, t]}(A_s) = 1_{(\tau_0, \tau_t]}(s).$$

Moreover A_s is in K if and only if s is in $R^c \cup L^c$, where $R(L)$ denotes the set of points of right (left) increase of A . Indeed, since $\tau_{A_s-} \leq s \leq \tau_{A_s}$, if A_s is in K there exists $v \neq s$ such that $\tau_{A_s-} < v < \tau_{A_s}$. But then

$$A_v = \inf \{u : \tau_u > v\} = A_s.$$

On the other hand, if $A_v = A_s$ for $v \neq s$, $\tau_{A_s-} \leq v \wedge s$ and $\tau_{A_s} \geq v \vee s$. Hence A_s is in K .

But we also know that almost surely

$$R \subset \{s : X_s \in F\} \subset R \cup L$$

(see [1]-V-3.8). Moreover $R \cup L - R \cap L$ is a countable set. Therefore $1_{K^c}(A_s) = 1_F(X_s)$ for almost all s , and

$$\tau_t^c = \int_{\tau_0}^{\tau_t} 1_F(X_s) ds \text{ a.s.}$$

Moreover, since $\tau_0 = T_F$ a.s. X_s is in F^c for all $s < \tau_0$ a.s. and so the result.

Using this representation, we have the following theorem.

THEOREM 4.2. *Let τ^c be the continuous part of τ . Then τ^c is a continuous additive functional of the process \hat{X} . In particular, τ_t^c is in \mathcal{F}_t for all t and*

$$(4.5) \quad \tau_{t+s}^c = \tau_t^c + \tau_s^c \circ \hat{\theta}_t,$$

almost surely for all t, s .

Proof. It is clear that the function $t \rightarrow \tau_t^c$ is nondecreasing and continuous and $\tau_0^c = 0$ almost surely. Moreover for all $t \geq \hat{\xi} = A_\infty$

$$\tau_t^c = \tau_{\hat{\xi}}^c = \int_0^\infty 1_F(X_s) ds,$$

and $\tau_t^c = 0$ for all t a.s. P^Δ .

Now, let us consider

$$B_t = \int_0^t 1_F(X_s) ds.$$

B is a perfect continuous additive functional of X and by its strong additivity property we have

$$\begin{aligned} \tau_{t+s}^c &= B_{\tau_{t+s}} \\ &= B_{\tau_t + \tau_s \circ \hat{\theta}_t} \\ &= B_{\tau_t} + B_{\tau_s \circ \hat{\theta}_t}[\hat{\theta}_t] \\ &= B_{\tau_t} + B_{\tau_s} \circ \hat{\theta}_t \\ &= \tau_t^c + \tau_s^c \circ \hat{\theta}_t \quad \text{a.s.} \end{aligned}$$

All that remains to be proved is the measurability of τ_t^c . Let $D = \text{supp } B$. Since $B_m = 0$ a.s., $T_D \geq \tau_0$ a.s. and

$$D = \{x : P^x[T_D = 0] = 1\} \subset F.$$

Furthermore

$$u_B^\alpha(x) = \int_0^\infty e^{-\alpha t} P^x(X_t \in F) dt \leq \frac{1}{\alpha}.$$

In this situation, for each finite measure μ on $(E_\Delta, \mathcal{E}_\Delta)$, there exists a

sequence g_n in $b\mathcal{E}_\Delta^{**}$ such that $B_t^n = \int_0^t g_n(X_s) dA_s$ converges to B_t on $[0, \infty)$ almost surely P^μ , the convergence being uniform on each compact subinterval (see [3]). Hence

$$B_{\tau_i}^n 1_{\{\tau_i < \infty\}} \rightarrow \tau_i^c 1_{\{\tau_i < \infty\}} \quad \text{a.s. } P^\mu.$$

But

$$\begin{aligned} B_{\tau_i}^n 1_{\{\tau_i < \infty\}} &= \int_0^{\tau_i} g_n(X_s) dA_s 1_{\{\tau_i < \infty\}} \\ &= \int_0^t g_n(\hat{X}_s) ds 1_{[0, \hat{\xi}]}(t). \end{aligned}$$

So $B_{\tau_i}^n 1_{\{\tau_i < \infty\}}$ is in $\hat{\mathcal{F}}_t$ for $\{\tau_i < \infty\} = \{\hat{X}_t \in F\} = \{t < \hat{\xi}\}$. Then $\tau_i^c 1_{[0, \hat{\xi}]}(t)$ is in $\hat{\mathcal{F}}_t$ for all t . Now since τ_i^c is continuous and constant on $\{t \geq \hat{\xi}\}$,

$$\tau_i^c = \int_0^t 1_{[0, \hat{\xi}]}(s) d\tau_s^c.$$

Hence

$$\tau_i^c = \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} [\tau_{(k/2^n)t}^c - \tau_{((k-1)/2^n)t}^c] 1_{[0, \hat{\xi}]} \left(\frac{k}{2^n} t \right).$$

But

$$[\tau_{(k/2^n)t}^c - \tau_{((k-1)/2^n)t}^c] 1_{[0, \hat{\xi}]} \left(\frac{k}{2^n} t \right)$$

is in $\hat{\mathcal{F}}_{(k/2^n)t} \subseteq \hat{\mathcal{F}}_t$ and so τ_i^c is in $\hat{\mathcal{F}}_t$ for all t . This completes the proof of Theorem 4.2.

Let us turn now to the purely discontinuous part of the process τ .

If \mathcal{B} is the collection of Borel sets on $(0, \infty)$, let us define for B in \mathcal{B} and t in $[0, \infty)$,

$$(4.6) \quad M_t(B) = |\{s \in (0, t] : \tau_s - \tau_{s-} \in B\}|,$$

i.e., the number of points s in $(0, t]$ such that $\tau_s - \tau_{s-}$ is in B . Clearly for each ω in Ω and B in \mathcal{B} bounded away from zero [i.e., $B \subseteq (1/n, \infty)$ for some $n < \infty$] the paths $t \rightarrow M_t(B)(\omega)$ are right continuous step functions with jumps of size 1. Also $M_0(B) = 0$ and $M_{\hat{\xi}}(B) = M_\infty(B)$. Now we have the following lemma.

LEMMA 4.3. For each ω in $\{\hat{\xi} > 0\}$ and t in $[0, \hat{\xi}(\omega))$, $M_t(B)(\omega)$ is a σ -finite measure on \mathcal{B} . Moreover for all positive Borel functions g on $[0, \infty)$ such that $g(0) = 0$ and for all $t < \hat{\xi}$,

$$(4.7) \quad \sum_{0 < s \leq t} g(\tau_s - \tau_{s-}) = \int_{(0, \infty)} g(u) M_t(du).$$

In particular,

$$(4.8) \quad \tau_t^j = \int_{(0, \infty)} u M_t(du).$$

Proof. Clearly $M_t(B)$ is a counting measure such that $M_t[(1/n, \infty)] < \infty$ since $\tau_t < \infty$. Now if $g = 1_B$ for B in \mathcal{B} , both sides of the equality (4.7) are equal to $M_t(B)$ and so by the monotone class theorem, (4.7) holds for all positive Borel functions g on $[0, \infty)$ with $g(0) = 0$, the latter condition preserving the countability of the sum of the left-hand side.

From now on let us fix ω in $\{\hat{\xi} > 0\} - N$, where N is the set of measure zero appearing in the statement of Corollary 3.5. It follows from the proof of Theorem 3.2, that we can suppose without loss of generality that $\tau_0 \equiv 0$ and $\hat{\xi} \equiv \hat{\xi}(\omega)$ since we are now only concerned with the measure P^ω . From the general theory of Levy processes we have the following theorem.

THEOREM 4.4. *Under the Assumption 3.4, there exists N in \mathcal{F} with $P^\mu(N) = 0$ for all finite measures μ on $(E_\Delta, \mathcal{E}_\Delta)$, such that for all ω in $\{\hat{\xi} > 0\} - N$, for all sets B in \mathcal{B} , the process $M_t(B)$, $0 \leq t < \hat{\xi}(\omega)$, is a Levy process of Poisson type (possibly with infinite parameter) on $(\Omega, \mathcal{F}^0, P^\omega)$. In particular,*

$$(4.9) \quad E^\omega [e^{-\alpha M_t(B)}] = e^{-(1-e^{-\alpha})E^\omega N_t(B)}.$$

Proof. We will only sketch the proof since this result is well known (see, for instance, [5]-I where it is treated in full detail). It is not too difficult to see that $M_t(B)$ is measurable with respect to $\mathcal{H}_t = \sigma\{\tau_s : 0 < s \leq t\}$ for all B in \mathcal{B} and $t < \hat{\xi}(\omega)$. And so this process has independent increments by Theorem 3.2. It is also continuous in probability since

$$P^\omega [M_{t-}(B) < M_t(B)] \leq P^\omega(\tau_{t-} < \tau_t) = 0$$

by Theorem 3.3.

Therefore if B in \mathcal{B} is bounded away from zero, by the Poisson law of rare events, there exists $\lambda < \infty$ such that

$$E^\omega [e^{-\alpha M_t(B)}] = e^{-\lambda(1-e^{-\alpha})}.$$

Hence $\lambda = E^\omega M_t(B)$.

If B is arbitrary, let $B_n = B \cap (1/n, \infty)$. Then $M_t(B_n)$ increases to $M_t(B)$ and $E^\omega M_t(B_n)$ increases to $E^\omega M_t(B)$. Hence

$$E^\omega [e^{-\alpha M_t(B)}] = e^{-(1-e^{-\alpha})E^\omega M_t(B)}.$$

Using this result it is easy to see that if we define

$$(4.10) \quad \nu_t(B)(\omega) = E^\omega M_t(B),$$

then for all ω in $\{\hat{\xi} > 0\} - N$, for all $t < \hat{\xi}(\omega)$, $\nu_t(\cdot)(\omega)$ is a σ -finite measure on \mathcal{B} which is finite on the sets in \mathcal{B} bounded away from zero.

Moreover for each ω in $\{\hat{\xi} > 0\} - N$ and for each B in \mathcal{B} bounded away from zero, the function $t \rightarrow \nu_t(B)(\omega)$ is increasing and continuous on $[0, \hat{\xi}(\omega))$.

Regrouping all the results we have about the structure of the process τ , we can state the following theorem.

THEOREM 4.5. *Under the Assumption 3.4, there exists N in $\hat{\mathcal{F}}$ with $P^\mu(N) = 0$ for all finite measures μ on $(E_\Delta, \mathcal{E}_\Delta)$ such that for all ω in $\{\hat{\xi} > 0\} - N$, and for all t in $[0, \hat{\xi}(\omega))$,*

$$(4.12) \quad \tau_t = \tau_t^c + \int_{(0, \infty)} u M_t(du) \text{ a.s. } P^\omega,$$

where τ_t^c is a continuous additive functional of \hat{X} and $M_t(B)$ is a Levy process of Poisson type for each set B in \mathcal{B} .

Moreover if $\nu_t(B)(\omega) = E^\omega M_t(B)$, then $\nu_t(\cdot)(\omega)$ is a Levy measure and

$$(4.13) \quad E^\omega(e^{-\alpha \tau_t}) = \exp \left[-\alpha \tau_t^c(\omega) - \int_{(0, \infty)} (1 - e^{-\alpha u}) \nu_t(du)(\omega) \right].$$

Proof. All we have to prove is (4.13). Since τ_t^c is $\hat{\mathcal{F}}_t$ measurable, we have

$$E^\omega(e^{-\alpha \tau_t}) = e^{-\alpha \tau_t^c(\omega)} E^\omega [e^{-\alpha \int_{(0, \infty)} u M_t(du)}] \text{ a.s.}$$

Since both sides of the equality are continuous in t and α , by subtracting another set of measure zero, the equality holds for all t and α almost surely in ω .

Now it follows from the general theory of Levy processes that for B_k , $1 \leq k \leq n$, disjoint sets in \mathcal{B} bounded away from zero, $M_t(B_k)$ $1 \leq k \leq n$ are independent random variables and so,

$$\begin{aligned}
E^\omega \exp \left[-\alpha \int_{(0,\infty)} u M_t(du) \right] &= \lim_{n \rightarrow \infty} E^\omega \exp \left[-\alpha \sum_{k=1}^{n2^n-1} \frac{k}{2^n} M_t \left[\left(\frac{k}{2^n}, \frac{k+1}{2^n} \right) \right] \right] \\
&= \lim_{n \rightarrow \infty} \prod_{k=1}^{n2^n-1} E^\omega \exp \left[-\alpha \frac{k}{2^n} M_t \left[\left(\frac{k}{2^n}, \frac{k+1}{2^n} \right) \right] \right] \\
&= \lim_{n \rightarrow \infty} \exp \left[-\sum_{k=1}^{n2^n-1} \left(1 - e^{-\alpha(k/2^n)} \right) \nu_t \left[\left(\frac{k}{2^n}, \frac{k+1}{2^n} \right) \right] (\omega) \right] \\
&= \exp \left[-\int_{(0,\infty)} (1 - e^{-\alpha u}) \nu_t(du)(\omega) \right].
\end{aligned}$$

From this equation, we see that

$$\int_{(0,\infty)} (1 - e^{-\alpha u}) \nu_t(du)(\omega) < \infty,$$

and this implies that $\nu_t(\cdot)(\omega)$ is a Levy measure.

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Zvi Artstein and John Allen Burns, <i>Integration of compact set-valued functions</i>	297
Mark Benard, <i>Characters and Schur indices of the unitary reflection group [321]³</i>	309
Simeon M. Berman, <i>A new characterization of characteristic functions of absolutely continuous distributions</i>	323
Monte Boisen and Philip B. Sheldon, <i>Pre-Prüfer rings</i>	331
Hans-Heinrich Brungs, <i>Three questions on duo rings</i>	345
Iracema M. Bund, <i>Birnbau-Orlicz spaces of functions on groups</i>	351
John D. Elwin and Donald R. Short, <i>Branched immersions between 2-manifolds of higher topological type</i>	361
Eric Friedlander, <i>Extension functions for rank 2, torsion free abelian groups</i>	371
Jon Froemke and Robert Willis Quackenbush, <i>The spectrum of an equational class of groupoids</i>	381
Barry J. Gardner, <i>Radicals of supplementary semilattice sums of associative rings</i>	387
Shmuel Glasner, <i>Relatively invariant measures</i>	393
George Rudolph Gordh, Jr. and Sibe Mardesic, <i>Characterizing local connectedness in inverse limits</i>	411
Siegfried Graf, <i>On the existence of strong liftings in second countable topological spaces</i>	419
Stanley P. Gudder and D. Strawther, <i>Orthogonally additive and orthogonally increasing functions on vector spaces</i>	427
Darald Joe Hartfiel and Carlton James Maxson, <i>A characterization of the maximal monoids and maximal groups in β_X</i>	437
Robert E. Hartwig and S. Brent Morris, <i>The universal flip matrix and the generalized faro-shuffle</i>	445
William Emery Haver, <i>Mappings between ANRs that are fine homotopy equivalences</i>	457
J. Bockett Hunter, <i>Moment sequences in l^p</i>	463
Barbara Jeffcott and William Thomas Spears, <i>Semimodularity in the completion of a poset</i>	467
Jerry Alan Johnson, <i>A note on Banach spaces of Lipschitz functions</i>	475
David W. Jonah and Bertram Manuel Schreiber, <i>Transitive affine transformations on groups</i>	483
Karsten Juul, <i>Some three-point subset properties connected with Menger's characterization of boundaries of plane convex sets</i>	511
Ronald Brian Kirk, <i>The Haar integral via non-standard analysis</i>	517
Justin Thomas Lloyd and William Smiley, <i>On the group of permutations with countable support</i>	529
Erwin Lutwak, <i>Dual mixed volumes</i>	531
Mark Mahowald, <i>The index of a tangent 2-field</i>	539
Keith Miller, <i>Logarithmic convexity results for holomorphic semigroups</i>	549
Paul Milnes, <i>Extension of continuous functions on topological semigroups</i>	553
Kenneth Clayton Pietz, <i>Cauchy transforms and characteristic functions</i>	563
James Ted Rogers Jr., <i>Whitney continua in the hyperspace $C(X)$</i>	569
Jean-Marie G. Rolin, <i>The inverse of a continuous additive functional</i>	585
William Henry Ruckle, <i>Absolutely divergent series and isomorphism of subspaces</i>	605
Rolf Schneider, <i>A measure of convexity for compact sets</i>	617
Alan Henry Schoenfeld, <i>Continuous measure-preserving maps onto Peano spaces</i>	627
V. Merriline Smith, <i>Strongly superficial elements</i>	643
Roger P. Ware, <i>A note on quadratic forms over Pythagorean fields</i>	651
Roger Allen Wiegand and Sylvia Wiegand, <i>Finitely generated modules over Bezout rings</i>	655
Martin Ziegler, <i>A counterexample in the theory of definable automorphisms</i>	665