Pacific Journal of Mathematics

ABSOLUTELY DIVERGENT SERIES AND ISOMORPHISM OF SUBSPACES

William Henry Ruckle

Vol. 58, No. 2 April 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

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WILLIAM H. RUCKLE

We consider the relation between the following two statements for E and F a pair of normed spaces.

(SI) For each absolutely divergent series $\Sigma_n x_n$ in E there is a continuous linear mapping T from E into F such that $\Sigma_n Tx_n$ diverges absolutely.

(LI) The finite dimensional subspaces of E are uniformly isomorphic to subspaces of F under isomorphisms which extend to all of E without increase of norm.

Our main result is that (SI) implies (LI) when F is isometric to $F \times F$ with a certain type of norm. We also observe that if a normed space E is not isomorphic to a subspace of an $L_p(\mu)$ space, then for each r with $1 \leqq r < \infty$ there is a series $\Sigma_n x_n$ in E such that $\Sigma_n \|Tx_n\|^r < \infty$ for each continuous linear mapping T from E into l_p but $\Sigma_n \|x_n\|^r = \infty$.

It is not hard to show that (LI) $\Rightarrow$ (SI) (Proposition 4.1). The main thrust of our work is to prove that (SI) $\Rightarrow$ (LI) in some important cases when F has infinite dimension. (Theorems 4.2 and 4.6). Our most important result is Theorem 4.6 which roughly maintains that (SI) $\Rightarrow$ (LI) if F is uniformly isometric to $F \times F$ in a way which we shall later clarify (Definition 4.5). The condition we need on F is satisfied for most familiar Banach spaces (e.g. l_p, $L_p[0, 1]$, $(1 \leqq p \leqq \infty)$, $C[0, 1]$).

Sections §2 and §3 are devoted to a basic study of properties (SI) and (LI) respectively. In §5 we relate our work here with that of other authors and state some problems.

2. Series immersion.

DEFINITION 2.1. A normed space E is said to be *series immersed* in a normed *space* F if the following statement holds:

(SI) For each absolutely divergent series $\Sigma_n x_n (\Sigma_n \|x_n\| = \infty)$ in E there is a continuous linear mapping T from E into F such that $\Sigma_n Tx_n$ diverges absolutely.

If E is series immersed in F then each subspace of E is also. An easy perturbation argument shows that E the completion of E is also series immersed in F.

PROPOSITION. 2.2. *The normed space E is series immersed in the normed space F if and only if the following condition holds for* (one) *all* $p \geqq 1$.

(SI_p) For each absolutely p-divergent series $\Sigma_n x_n$ ($\Sigma_n \|x_n\|^p = \infty$) in E there is a continuous linear mapping T from E into F such that $\Sigma_n Tx_n$ is absolutely p-divergent.

Proof. (SI) $\Rightarrow$ (SI_p) for all p. Suppose (SI_p) fails to hold for some p. Then there is a series $\Sigma_n x_n$ in E such that $\Sigma_n \|x_n\|^p = \infty$ but $\Sigma_n \|Tx_n\|^p < \infty$ for all T in $L(E, F)$. Let (a_n) be a sequence of nonnegative numbers in $l^q(1/p + 1/q = 1)$ such that $\Sigma_n a_n \|x_n\| = \infty$. Then $\Sigma_n a_n x_n$ diverges absolutely but $\Sigma_n T(a_n x_n)$ converges absolutely for all T in $L(E, F)$ so that (SI) fails to hold.

(SI_p) $\Rightarrow$ (SI). Suppose (SI) fails to hold. Then there is $\Sigma_n x_n$ in E with $\Sigma_n \|x_n\| = \infty$ but $\Sigma_n \|Tx_n\| < \infty$ for all T in $L(E, F)$. Let $y_n = x_n / \|x_n\|^{1/q}$ for each n where $1/p + 1/q = 1$. Then

$$\sum_n \|y_n\|^p = \sum_n (\|x_n\|^{(q-1)/q})^p = \sum_n \|x_n\| = \infty$$

but for each T in $L(E, F)$

$$\begin{aligned}\sum_n \|Ty_n\|^p &= \sum_n \left[\frac{\|Tx_n\|}{\|x_n\|^{1/q}}\right]^p \\ &\leqq \sum_n \left\{\left[\frac{\|Tx_n\| \, \|T\|^{1/q}}{\|Tx_n\|^{1/q}}\right]^p : \|Tx_n\| \neq 0\right\} \\ &\leqq \|T\|^{p/q} \sum_n \|Tx_n\| < \infty.\end{aligned}$$

Thus (SI_p) fails to hold.

PROPOSITION 2.3. (SI) *holds if and only if*:
(SI′) *There is $M > 0$ such that*

$$\sum_{x \in A} \|x_n\| \leqq M \sup\left\{\sum_{x \in A} \|Tx\| : T \in U(E, F)\right\}$$

for every finite subset A of E. Here $U(E, F)$ denotes the unit ball of $L(E, F)$.

Proof. Let $l(E)$ consist of all (x_n) in E for which

$$\|(x_n)\| = \sum_n \|x_n\| < \infty.$$

Let $\sigma_F(E)$ consist of all (x_n) in E for which

$$|||\, (x_n)\, ||| = \sup\left\{\sum_{n=1}^{\infty} \|Tx_n\| : T \in U(E, F)\right\} < \infty.$$

If E is complete, as we may assume, $l(E)$ and $\sigma_F(E)$ are Banach spaces with their respective norms and

$$|||\, (x_n)\, ||| \leqq \|(x_n)\|, \ (x_n) \text{ in } \sigma_F(E).$$

(SI) $\Rightarrow$ (SI$'$). If (SI) holds then $\sigma_F(E) = l(E)$. Thus the norms $\|\ \|$ and $|||\ \ |||$ are equivalent so there is $M > 0$ such that

$$\|(x_n)\| \leqq M\ |||\, (x_n)\, |||\ ,\ (x_n) \in l(E). \tag{2–1}$$

But (2–1) implies (SI$'$) in the special case when (x_n) consists of only finitely many nonzero vectors.

(SI$'$) $\Rightarrow$ (SI). Condition (SI$'$) implies that (2–1) holds for sequences which are finitely nonzero. Since such sequences are dense in $\sigma_F(E)$ it follows that (2–1) holds for all (x_n) in $\sigma_F(E)$. Hence $l(E) = \sigma_F(E)$. If $\Sigma_n \|Tx_n\| < \infty$ for each T in $U(E, F)$ then $(x_n) \in \sigma_F(E)$ by the Uniform Boundedness Principle. Therefore, (SI) holds.

We can imitate the proof of 2.3 using $l_p(E)$ the space of all (x_n) in E for which

$$\|(x_n)\|_p = \left(\sum_n \|x_n\|^p\right)^{1/p} < \infty$$

and $\sigma_{F,p}(E)$ the space of all (x_n) in E for which

$$|||\, (x_n)\, |||_p = \sup\left\{\left(\sum_n \|Tx_n\|^p\right)^{1/p} : T \in U(E, F)\right\} < \infty$$

to obtain the following statement.

PROPOSITION 2.4. (SI$_p$) *holds if and only if*
(SI$'_p$) *There is* $M_p > 0$ *such that*

$$\sum_{x \in A} \|x\|^p \leqq M_p \sup\left\{\sum_{x \in A} \|Tx\|^p : T \in U(E, F)\right\}$$

for every finite subset A of E.

3. Local immersion.

DEFINITION 3.1. A normed space E is said to be *locally immersed* in a normed space F if the following condition holds:

(LI) There is a number $K \geqq 1$ such that for each finite dimensional subspace G of E there is a continuous linear mapping T in $U(E, F)$ such that

$$\|Tx\| \geqq K\|x\| \qquad x \in G.$$

In other words (LI) means there is an isomorphism T from G into E with $\|T\|\,\|T^{-1}\| \leqq K$ which extends without change of norm to all of E.

PROPOSITION 3.2. (a) (LI) *is equivalent to the following statement*:

(LI′) *There is a number $D \geqq 1$ such that for each finite subset A of E there is T in $U(E, F)$ such that*

$$\|Tx\| \geqq D\|x\| \qquad x \in A.$$

(b) *If D satisfies* (LI″) *then $D + \epsilon = K$ satisfies* (LI) *for each $\epsilon > 0$.* (c) *If K satisfies* (LI) *then $K = D$ satisfies* (LI′).

Proof. (LI′) $\Rightarrow$ (LI); and (b). If G and ϵ are given, let D be determined by (LI′) and let $\delta \leqq \epsilon/(K^2 + K\epsilon)$. Let A be a δ-net in the unit sphere of G. If T is given by (LI′) for A and x in G has norm one there is y in A with $\|y - x\| < \delta$. Since $\|T\| = 1$ we see

$$\begin{aligned}\|Tx\| &\geqq \|Ty\| - \|T(y - x)\| \\ &\geqq \frac{1}{K} - \delta \geqq \frac{1}{K} - \frac{\epsilon}{K^2 + K\epsilon} \\ &= \frac{1}{K + \epsilon}.\end{aligned}$$

Thus

$$(K + \epsilon)\|Tx\| \geqq \|x\|$$

for x in G.

(LI) $\Rightarrow$ (LI′); and (c). Obvious.

Condition (LI) is commonly described by saying that E is finitely represented in F.

4. Series immersion and local immersion.

PROPOSITION 4.1. *If E is locally immersed in F then E is series immersed in F. If D satisfies* (LI′) *then $M = D$ satisfies* (SI′).

Proof. The second statement is obvious and implies the first.

If E is a finite dimensional normed space then E is series immersed in every normed space, but locally immersed in normed spaces of greater dimenion. Thus the converse of 4.1 is not generally true, but it is true in certain important cases.

The first case, which we treat in Proposition 4.2 is an easy isometric condition.

PROPOSITION 4.2. *If E is series immersed in F and for some $p \geqq 1$* (SI'_p) *holds with $M_p = (1+\epsilon)$ for each $\epsilon > 0$, then E is locally immersed in F. In this case $D = (1+\epsilon)$ satisfies* (LI′) *for each $\epsilon > 0$.*

Proof. For A a finite subset of E, and $\epsilon > 0$ let

$$\delta = \frac{|A|}{|A| - \left(\frac{\epsilon}{1+\epsilon}\right)} - 1 \tag{4–1}$$

where $|A|$ denotes the number of points in A. Since (SI'_p) holds with $M_p = 1 + \frac{1}{2}\delta$ there is T in $U(E, F)$ such that

$$\begin{aligned} |A| &= \sum_{x \in A} \left\| \frac{x}{\|x\|} \right\|^p \\ &\leqq (1+\delta) \sum_{x \in A} \left\| \frac{Tx}{\|x\|} \right\|^p . \end{aligned} \tag{4–2}$$

If

$$\left\| \frac{Tx}{\|x\|} \right\| < \frac{1}{1+\epsilon}$$

for some x in A then since $T \in U(E, F)$ we would have

$$\sum_{x \in A} \left\| \frac{Tx}{\|x\|} \right\|^p < (|A| - 1) + \left(\frac{1}{1+\epsilon} \right)^p$$
$$\leqq |A| - 1 + \frac{1}{1+\epsilon} = |A| - \left(\frac{\epsilon}{1+\epsilon} \right)$$

so that

$$\frac{|A|}{|A| - \frac{\epsilon}{1+\epsilon}} < 1 + \delta$$

which contradicts (4.1). Therefore, we conclude

$$(1+\epsilon)\|Tx\| \geqq \|x\|$$

for each x in A.

Let θ be a norm on R^2 with the following two properties

$$\theta(0, 1) = \theta(1, 0) = 1$$

$\theta(a_1 b_1 a_2 b_2) \leqq \max\{|a_1|, |a_2|\}\,\theta(b_1, b_2)$. We extend θ to a norm on R^n for all n by the iterative formula

(4–3) $$\theta(a_1, a_2, \cdots, a_n) = \theta(\theta(a_1, a_2, \cdots, a_{n-1}), a_n).$$

For convenience we shall speak of θ as an *iterative functional.* It is easy to see that $(\Sigma_{i=1}^n |a_i|^p)^{1/p}$ $(1 \leqq p < \infty)$ and $\max\{|a_i|: i \leqq i \leqq n\}$ are examples of iterative functionals. See [7] for other examples.

LEMMA 4.3. *For θ given as above let*

(4–4) $$\theta'(b_1, b_2) = \sup\{a_1 b_1 + a_2 b_2 : \theta(a_1, a_2) \leqq 1\}.$$

Then for θ', the iterative functional determined by θ' we have for all n,

$$\theta'(b_1, \cdots, b_n) = \sup\left\{ \sum_{i=1}^n a_i b_i : \theta(a_1, \cdots, a_n) \leqq 1 \right\}.$$

Proof. We use induction. Assume the statement holds for $n - 1$. Then

$$\begin{aligned}\theta'(b_1, b_2, \cdots, b_n) &= \theta'(\theta'(b_1, b_2, \cdots, b_{n-1}), b_n)\\ &= \sup\{\theta'(b_1, b_2, \cdots, b_{n-1})c_1 + b_n c_2 \colon \theta(c_1, c_2) \leqq 1\}\\ &= \sup\left\{c_1 \sum_{i=1}^{n-1} d_i b_i + b_n c_2 \colon \theta(d_1, \cdots, d_{n-1}) \leqq 1,\ \theta(b_1, b_2) \leqq 1\right\}\\ &= \sup\left\{\sum_{i=1}^{n-1} a_i b_i + a_n c_n \colon \theta(a_1, \cdots, a_{n-1}) \leqq c_1,\ \theta(c_1, a_n) \leqq 1\right\}\\ &= \sup\left\{\sum_{i=1}^{n} a_i b_i \colon \theta(a_1, \cdots, a_m) \leqq 1\right\}.\end{aligned}$$

LEMMA 4.4. *Let θ be an iterative functional, and let θ' be determined by (4–4). If $c_i \geqq 0$ $i = 1, 2, \cdots, n$ there are $a_i \geqq 0$ and $b_i \geqq 0$ $i = 1, 2, \cdots,$ with $a_i b_i = c_i$ and*

$$\theta(a_i)\theta'(b_i) = \sum_{i=1}^{n} c_i.$$

Proof. We use induction on n, first proving our assertion when $n = 2$.

We may assume $c_1 + c_2 = 1$. Let

$$S = \{(u, v, x, y) \colon 0 \leqq u, v, x, y \leqq 1,\ \theta(u, v) = \theta'(x, y) = ux + vy = 1\}$$

$$S_1 = \{(u, v) \colon 0 \leqq u,\ v \leqq 1,\ \theta(u, v) = 1\}.$$

Then S_1 is connected in R^2 and

$$S = \cup\{S_{(u,v)} \colon (u, v) \in S_1\}$$

where

$$S_{(u,v)} = \{(u, v, x, y) \colon 0 \leqq x,\ y \leqq 1,\ \theta'(x, y) = ux + vy = 1\}.$$

It is easy to see that each $S_{(u,v)}$ is convex, hence connected. We shall prove S is connected. Let S_0 be any component of S. Then S_0 and $S \sim S_0$ are compact so both $P(S_0)$ and $P(S \sim S_0)$ are compact subsets of S_1. Here $P(u, v, x, y) = (u, v)$. For each (u, v) in S_1 either $S_{(u,v)}$ is contained entirely in S_0 or in $S \sim S_0$ so $P(S_0)$ and $P(S \sim S_0)$ are disjoint. But $S_1 = P(S_0) \cup P(S \sim S_0)$ so that $P(S \sim S_0)$ is empty and $P(S_0) = S_1$. Therefore, $S_0 = S$ so S is connected. The function

$$f(u, v, x, y) = ux$$

is continuous and assumes the values 0 and 1 on S. Hence there is (a_1, a_2, b_1, b_2) in S such that $a_1 a_2 = a$. This establishes our lemma when $n = 2$.

Now assume the lemma holds for $n - 1$. Given $\Sigma_{i=1}^n c_i$, we use the preceeding case to find (a_0, a_n), (b_0, b_n) with $a_0 b_0 = \Sigma_{i=1}^{n-1} c_i$, $a_n b_n = c_n$ and $\Sigma_{i=1}^n \theta(a_0, a_n)\,\theta'(b_0, b_n) = \Sigma_{i=1}^n c_i$. We then use the inductive hypothesis to find $a_1, \cdots, a_{n-1}$; $b_1, \cdots, b_{n-1}$ with $c_i = a_i b_i$ and $a_0 b_0 = \theta(a_1, \cdots, a_{n-1})\,\theta'(b_1, \cdots, b_{n-1})$. We may assume $\theta(a_1, \cdots, a_{n-1}) = a_0$ and $\theta'(b_1, \cdots, b_{n-1}) = b_0$. Then $a_i b_i = c_i$ for each i and

$$\theta(a_1, \cdots, a_n)\,\theta'(b_1, \cdots, b_m) = \theta(a_0, a_n)\,\theta'(b_0, b_n) = \sum_{i=1}^n c_i.$$

DEFINITION 4.5. A Banach space E is called *unconditionally resolveable* if E is isometric to $E \times E$ with the norm

$$\|(x, y)\| = \theta((\|x\|\ \|y\|)) \tag{4–5}$$

where θ is a norm on R^2 with the property that

$$\theta(a_1 b_1, a_2 b_2) \leqq \max\{|a_1|, |a_2|\}\theta(b_1, b_2).$$

Since $E \times E$ with the norm (4.5) is isometric to $E \times E$ with the norm

$$\theta\left(\left(\frac{\|x\|}{r_1}, \frac{\|y\|}{r_2}\right)\right)$$

for all $r_1 r_2 > 0$ we may assume $\theta(1, 0) = \theta(0, 1) = 1$. If E is unconditionally resolveable then for each n it is isometric to the n-fold product $E \times E \times \cdots \times E$ with the norm

$$\|(x_1, \cdots, x_n)\| = \theta((\|x_1\|, \cdots, \|x_n\|))$$

where θ is the iterative functional determined by θ.

THEOREM 4.6. (a) *If a normed space E is series immersed in a Banach space F, which is unconditionally resolveable, then E is locally immersed in F.* (b) *If M satisfies* (SI′) *then* $(1 + \epsilon)M$ *satisfies* (LI′) *for each* $\epsilon > 0$.

Proof. Suppose M satisfies (SI′). Given A a finite subset of E and $\epsilon > 0$ we imitate the development of inequality (5) on p. 1021 of **[5]**, but using $1/(1 + \epsilon)M$ instead of $1/2\lambda$, to obtain a number r numbers $c_i > 0$ with $\Sigma_{i=1}^r c_i = 1$ and mappings $T_1, \cdots, T_r$ in $U(E, F)$ such that

$$\sum_{i=1}^{r} c_i \|T_i(x)\| > \frac{1}{(1+\epsilon)M} \|x\|, \qquad x \in A. \tag{4–6}$$

(Note that inequality (5) cited is misprinted; it should read

$$\sum_{i=1}^{r} c_i \|T_r(x_n)\| > \frac{1}{2\lambda} \|x_n\|; \qquad n = 1, 2, \cdots, k.)$$

Next we use Lemma 4.4 to find $(d_1, \cdots, d_r)$ with

$$\theta(d_1, \cdots, d_r) = \theta'(c_1/d_1, c_2/d_2, \cdots, c_r/d_r) = 1.$$

Here θ is the iterative functional for which E is isometric to $E \times E \times \cdots \times E$ (r factors) with the norm

$$\|(x_n)\| = \theta(\|x_n\|).$$

We define T_A from E into $E \times \cdots \times E$ by

$$T_A(x) = (d_i T_i(x)).$$

Then if $\|x\| \leqq 1$

$$\begin{aligned} \|T_A x\| &= \theta((d_i \|T_i(x)\|)) \\ &\leqq \sup_i \|T_i(x)\| \, \theta(d_i) \leqq 1 \end{aligned}$$

so $\|T_A\| \leqq 1$. Moreover, if $x \in A$

$$\begin{aligned} \|T_A x\| &= \theta((d_i \|T_i(x)\|)) \\ &= \theta((d_i \|T_i(x)\|)) \, \theta'(c_i/d_i) \\ &\leqq \sum_i c_i \|T_i(x)\| > \frac{1}{(1+\epsilon)M} \|x\|. \end{aligned}$$

Therefore, $(1+\epsilon)M = D$ satisfies (LI′).

If F is any Banach space $l^p(F)$ is unconditionally resolveable. Thus the l^p spaces $1 \leqq p < \infty$ are unconditionally resolveable as are $m, C[0, 1]$ and $L_p[0, 1]$, $1 \leqq p < \infty$.

COROLLARY 4.7. *If a normed space E is series immersed in a Banach space F and F is locally immersed in an unconditionally resolveable space G and G is locally immersed in F then E is locally immersed in F.*

5. Remarks and problems.

5.1. In [5] it is proven that a normed space E is not isomorphic to a subspace of an $L_p(\mu)$ space $1 \leqq p < \infty$ if and only if there is a series $\Sigma_n x_n$ in E which diverges absolutely but such that $\Sigma_n \|Tx_n\| < \infty$ for each continuous linear mapping from E into l_p. This proof is based on the facts that l_p is unconditionally resolveable (with $\theta(a, b) = (|a|^p + |b|^p)^{1/p}$) and that a normed space is locally immersed in l_p if and only if it is isomorphic to a subspace of $L_p(\mu)$ for some μ (e.g. see Proposition 7.1 of [6]). In [9] P. Saphar announced a result which implies that E is not isomorphic to a subspace of $L_p(\mu)$ $1 \leqq p < \infty$ if and only if there is a series $\Sigma_n x_n$ in E such that $\Sigma_n \|Tx_n\|^p < \infty$ for each T in $L(E, l_p)$ but $\Sigma_n \|x_n\|^p = \infty$. By Proposition 2.2 these two statements are equivalent and in fact equivalent to the following more general statement:

THEOREM. *A normed space E is not isomorphic to a subspace of $L_p(\mu)$ $(1 \leqq p < \infty)$ if and only if for one* (each) *r with $1 \leqq r < \infty$ there is a series $\Sigma_n x_n$ in E such that $\Sigma_n \|Tx_n\|^r < \infty$ for each T in $L(E, l_p)$ but $\Sigma_n \|x_n\|^r = \infty$.*

As a corollary to this theorem we get the following strengthening of the Dvoretzky-Rogers theorem [2] due to Pietsch [8] and perhaps also to Grothendieck [3].

COROLLARY. *For every infinite dimensional normed space E and every r with $1 \leqq r < \infty$ there is a series $\Sigma_n x_n$ in E such that $\Sigma_n |x'(x_n)|^r < \infty$ for all continuous linear functionals x' on E but $\Sigma_n \|x_n\|^r = \infty$.*

Of course, this corollary also follows from the Dvoretzky-Rogers Theorem and Proposition 2.2.

Problem 5.2. To what extent may we weaken the requirement in Theorem 4.6 that F be unconditionally resolveable? Is the theorem true if F is merely infinite dimensional? Isomorphic to its square?

Problem 5.3. What Banach spaces are unconditionally resolveable? Is the following conjecture true: If F is isomorphic to $F \times F$, then there is an equivalent norm on F for which it is unconditionally resolveable?

5.4. If E is series immersed (resp. locally immersed) in F and F is series immersed (resp. locally immersed) in E we write $E \sim_S F$ (resp. $E \sim_L F$). Both $\sim_S$ and $\sim_L$ are equivalence relations. Every $\sim_L$

equivalence class is contained in an $\sim_S$ equivalence class. The finite dimensional spaces constitute the smallest $\sim_S$ equivalence class, but two finite dimensional spaces are $\sim_L$ equivalent if and only if they have the same dimension. By a theorem of Dvoretzky [1], the class of infinite dimensional Hilbert spaces constitute an $\sim_L$ equivalence class which is minimal among infinite dimensional spaces. It is easy to see that this class is also an $_{\bar{S}}$ equivalence class, which is the second "smallest." By the theorem stated in 5.1 the $L_p(\mu)$ spaces constitute distinct $\sim_L$ and $\sim_S$ equivalence classes.

5.5. The argument $(\mathrm{SI}_p) \Rightarrow (\mathrm{SI})$ in Proposition 2.2 holds for all $p > 0$, but the argument $(\mathrm{SI}) \Rightarrow (\mathrm{SI}_p)$ holds only for $p \geqq 1$.

Problem. Does $(\mathrm{SI}) \Rightarrow (\mathrm{SI}_p)$ for all $p > 0$?

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Received February 22, 1974.

Clemson University

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics

Vol. 58, No. 2 April, 1975