A NOTE ON QUADRATIC FORMS OVER PYTHAGOREAN FIELDS

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A theorem of T. A. Springer states that if $F$ is a field of characteristic not two and $L$ is an extension field of $F$ of odd degree then any anisotropic quadratic form over $F$ remains anisotropic over $L$. A weaker version (and an immediate consequence) of this theorem says that the natural map $r: W(F) \rightarrow W(L)$, from the Witt ring of $F$ to the Witt ring of $L$, is injective. This note investigates the relationship between these statements in the case that $L$ is a finite Galois extension of a pythagorean field $F$. Specifically, it is shown that if $r$ is injective then any anisotropic quadratic form over $F$ remains anisotropic over $L$ and if, in addition, $L$ is pythagorean then the extension must be of odd degree. An example is provided of a Galois extension of even degree with $r$ injective.

Notations and terminology in this paper will follow [4]. Thus by a field $F$ we shall mean one of characteristic different from two and $W(F)$ will denote the Witt ring of anisotropic quadratic forms over $F$. If $F \subseteq L$ is an extension of field then $r_{L/F}: W(F) \rightarrow W(F)$ will denote the induced homomorphism of Witt rings. When there is no possibility of confusion we shall simply write $r$ in place of $r_{L/F}$. In general, the mapping $r$ will fail to be injective. However, if $F \subseteq L$ is an extension of odd degree then the above mentioned theorem of Springer will imply the injectivity of $r$ [4, Chapter 7, §2]. In the case of ordered (= formally real) fields, information about the kernel of $r$ can be used to yield information about extending orderings. Specifically, every ordering on $F$ extends to an ordering on $L$ if and only if $\text{Ker} \ r$ is a nil ideal of $W(F)$ [3, Corollary 2.11]. One can use this, together with Springer’s theorem, to recover the fact that if $F \subseteq L$ is an extension of odd degree with $F$ formally real then every ordering on $F$ extends to $L$. Moreover, if $F$ is pythagorean then $W(F)$ has no nonzero nilpotent elements [4, Theorems 3.3 and 6.1, pp. 236 and 248] so for any extension $L$ of $F$, $r: W(F) \rightarrow W(F)$ is injective if and only if every ordering on $F$ extends to $L$.

**Proposition 1.** Let $F \subseteq L$ be a finite Galois extension of degree $n$ with $L$ pythagorean. If $r: W(F) \rightarrow W(L)$ is injective then $n$ is odd.

**Proof.** Let $G$ be the Galois group of the extension $F \subseteq L$, let $H$ be
a 2-Sylow subgroup of $G$, and let $K = L^u$ be the fixed field of $H$. Then $K$ is also pythagorean [4, Exercise 17, p. 254].

If $F$ is not formally real then every element of $K$ is a square in $K$ (i.e. $K$ is "quadratically closed"). Thus, from Galois theory, $H$ must be trivial and hence $G$ is a group of odd order.

Now assume $F$ is formally real and let $<$ be an ordering on $F$. Since $r : W(F) \to W(L)$ is injective, $<$ extends to $L$ (and to $K$). Moreover by [2, Exercise 2, p. 289], $<$ extends to exactly $[L : F]$ orderings on $L$ and to $t \equiv [K : F]$ orderings on $K$ (compare [3, Proposition 5.12]). Let $<_{1}, <_{2}, \cdots , <_{m}, m \equiv t$, be the orderings on $K$ which extend $<$ and which also extend to $L$. Since $K \subset L$ is a Galois extension, it again follows that each $<_{i}$ extends exactly $[L : K]$ different ways to $L$. Thus $[L : F] = m[L : K]$, which implies that $m = [K : F]$. Hence $m = t$ so that every extension of $<$ to $K$ also extends to $L$. But every ordering on $K$ is the extension of some ordering on $F$, so it follows that every ordering on $K$ extends to $L$. Since $K$ is a pythagorean field, the mapping $r_{L/K} : W(K) \to W(L)$ is injective. If the Galois group $H$ of the extension $K \subset L$ is not trivial then there will exist a nonsquare $a$ in $K$ with $\sqrt{a}$ in $L$. Then $\langle 1, - a \rangle$ is an anisotropic form over $K$ whose class in $W(K)$ is a nonzero element in the kernel of $r_{L/K}$. Thus $H$ is also trivial in this case, i.e. $n$ is odd.

**Corollary.** Let $F \subset L$ be a finite Galois extension of degree $n$ with $L$ pythagorean. If every ordering on $F$ extends to $L$ then $n$ is odd.

**Proof.** By [4, Exercise 17, p. 254], $F$ is also pythagorean.

The following modification of a construction due to Manfred Knebusch shows that the hypothesis that $L$ be pythagorean is essential in Proposition 1 and its corollary.

**Example.** A Galois extension $F \subset L$ of formally real fields with $F$ pythagorean (actually euclidean), $[L : F]$ even, and $r : W(F) \to W(L)$ injective.

Choose $n \cong 5$ and let $K$ be a formally real field on which the alternating group $A_{n}$ acts as a group of automorphisms (e.g. $K = R(x_{1}, \cdots , x_{n})$). Let $k = K^{\sigma}$ be the fixed field and let $\bar{k}$ be the quadratic closure of $k$, i.e. the compositum of all Galois extensions of $k$ with degree a power of 2 [4, p. 219]. Then $\bar{k}$ is a Galois extension of $k$ and since $[K : k]$ is not a power of two, $K$ is not contained in $k$. Thus $\bar{k} \cap K \neq K$ is a Galois extension of $k$ so Galois theory and the simplicity of $A_{n}$ imply that $\bar{k} \cap K = k$.

Now let $R$ be a real closure ([2],[4],[5]) of the formally real field $K$ and let $F = R \cap \bar{k}$. Then we also have $F \cap K = k$. Moreover, $F$ is formally real and it is easy to see that any $a$ in $F$ is either a square in $F$
or the negative of a square in \( F \). In particular, \( F \) is pythagorean and has exactly one ordering. From Sylvester's law of inertia we have \( W(F) \equiv Z \) (cf. [4, pp. 42–43]).

Let \( L = FK \) be the compositum of \( F \) and \( K \) in \( R \). Then \( L \) is a formally real Galois extension of \( F \) with Galois group \( A_n \) [5, Theorem 4, p. 196]. In particular, \( [L : F] \) is even. Finally, any signature \( \sigma_c : W(L) \to Z \) arising from an ordering \( < \) on the formally real field \( L \) (see [4, pp. 42–43], [3, p. 211]) will provide a splitting for the map \( r : W(F) \to W(L) \).

**Proposition 2.** Let \( F \) be a pythagorean field and \( L \) a finite Galois extension of \( F \). Then the following statements are equivalent:

1. \( r : W(F) \to W(L) \) is injective.
2. If \( q \) is an anisotropic quadratic form over \( F \) then \( q_L = L \otimes_F q \) is anisotropic over \( L \).

**Proof.** (1) \( \Rightarrow \) (2). If \( F \) is not formally real then \( F \) is quadratically closed so all anisotropic forms over \( F \) are one dimensional. Hence the implication is obvious in this case.

Now assume \( F \) is formally real and let \( Tr^* \) denote Scharlau's transfer map relative to the \( F \)-linear trace map \( Tr_{L/F} \) (which associates to each quadratic form \( q \) over \( L \) the \( F \)-quadratic form \( Tr_{L/F} \circ q \) [4, Chapter 7, §1, 6], [3, §5]). Then for any anisotropic form \( q \) over \( F \), there is an isometry \( L \otimes_F Tr^*(q_L)q_L \perp \cdots \perp q_L \equiv [L : F] \cdot q_L \), where \( [L : F] \cdot q_L = q_L \perp \cdots \perp q_L, [L : F] \) times [4, Theorem 6.1, p. 212] compare [3, Corollary 5.10]). Since the mapping \( r : W(F) \to W(L) \) is injective this means that \( Tr^*(q_L) \) is isometric to \( [L : F] \cdot q \) over \( F \). But \( F \) is a formally real pythagorean field, so by (the proof of ) [4, Theorem 3.3], \( [L : F] \cdot q \) is anisotropic over \( F \). Therefore \( Tr^*(q_L) \) is anisotropic over \( F \) so that, in particular, \( q_L \) is anisotropic over \( L \).

The implication (2) \( \Rightarrow \) (1) is immediate.

It seems to be an open question whether, for an arbitrary extension \( F \subset L \), the injectivity of \( r : W(F) \to W(L) \) implies that anisotropic forms over \( F \) remain so over \( L \). However, for a certain class of pythagorean fields the answer is affirmative. Let \( F \) be a formally real field, let \( X \) be the set of orderings on \( F \), and for \( a \) in \( F \), let \( V(a) = \{ < \in X \mid a > 0 \} \). Then the family \( V(a)_{a,F} \) generates a compact, Hausdorff, totally disconnected topology on \( X \) [3, Lemma 3.3, Theorem 3.18]. The field \( F \) satisfies the Strong Approximation Property (SAP) if given any two disjoint closed subsets \( U, V \) of \( X \) there is an element \( a \) in \( F \) which is positive at the orderings in \( U \) and negative at the orderings in \( V \) (cf. [1, Definition 1.4], [3, Corollary 3.21]).

**Proposition 3.** Let \( F \) be a formally real pythagorean field satisfying SAP and let \( L \) be any extension field of \( F \). If \( r : W(F) \to W(L) \) is
injective then any anisotropic quadratic form over \( F \) remains anisotropic over \( L \).

**Proof.** In view of [1, Theorem 5.3 (1)], any anisotropic form \( q \) over \( F \) can be written \( q = \langle a_1, \ldots, a_n \rangle \) where either all the \( a_i \)'s are positive or all the \( a_i \)'s are negative with respect to some ordering \( < \) on \( F \). If \( r: W(F) \to W(L) \) is injective then \( < \) extends to \( L \) so an equation \( a_1 x_1^2 + \cdots + a_n x_n^2 = 0 \) with each \( x_i \) in \( L \) is impossible.

**References**


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