

# Pacific Journal of Mathematics

## **A NOTE ON QUADRATIC FORMS OVER PYTHAGOREAN FIELDS**

ROGER P. WARE

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A theorem of T. A. Springer states that if  $F$  is a field of characteristic not two and  $L$  is an extension field of  $F$  of odd degree then any anisotropic quadratic form over  $F$  remains anisotropic over  $L$ . A weaker version (and an immediate consequence) of this theorem says that the natural map  $r : W(F) \rightarrow W(L)$ , from the Witt ring of  $F$  to the Witt ring of  $L$ , is injective. This note investigates the relationship between these statements in the case that  $L$  is a finite Galois extension of a pythagorean field  $F$ . Specifically, it is shown that if  $r$  is injective then any anisotropic quadratic form over  $F$  remains anisotropic over  $L$  and if, in addition,  $L$  is pythagorean then the extension must be of odd degree. An example is provided of a Galois extension of even degree with  $r$  injective.

Notations and terminology in this paper will follow [4]. Thus by a field  $F$  we shall mean one of characteristic different from two and  $W(F)$  will denote the Witt ring of anisotropic quadratic forms over  $F$ . If  $F \subset L$  is an extension of field then  $r_{L/F} : W(F) \rightarrow W(L)$  will denote the induced homomorphism of Witt rings. When there is no possibility of confusion we shall simply write  $r$  in place of  $r_{L/F}$ . In general, the mapping  $r$  will fail to be injective. However, if  $F \subset L$  is an extension of odd degree then the above mentioned theorem of Springer will imply the injectivity of  $r$  [4, Chapter 7, §2]. In the case of ordered (= formally real) fields, information about the kernel of  $r$  can be used to yield information about extending orderings. Specifically, every ordering on  $F$  extends to an ordering on  $L$  if and only if  $\text{Ker } r$  is a nil ideal of  $W(F)$  [3, Corollary 2.11]. One can use this, together with Springer's theorem, to recover the fact that if  $F \subset L$  is an extension of odd degree with  $F$  formally real then every ordering on  $F$  extends to  $L$ . Moreover, if  $F$  is pythagorean then  $W(F)$  has no nonzero nilpotent elements [4, Theorems 3.3 and 6.1, pp. 236 and 248] so for any extension  $L$  of  $F$ ,  $r : W(F) \rightarrow W(L)$  is injective if and only if every ordering on  $F$  extends to  $L$ .

**PROPOSITION 1.** *Let  $F \subset L$  be a finite Galois extension of degree  $n$  with  $L$  pythagorean. If  $r : W(F) \rightarrow W(L)$  is injective then  $n$  is odd.*

*Proof.* Let  $G$  be the Galois group of the extension  $F \subset L$ , let  $H$  be

a 2-Sylow subgroup of  $G$ , and let  $K = L^H$  be the fixed field of  $H$ . Then  $K$  is also pythagorean [4, Exercise 17, p. 254].

If  $F$  is not formally real then every element of  $K$  is a square in  $K$  (i.e.  $K$  is “quadratically closed”). Thus, from Galois theory,  $H$  must be trivial and hence  $G$  is a group of odd order.

Now assume  $F$  is formally real and let  $<$  be an ordering on  $F$ . Since  $r: W(F) \rightarrow W(L)$  is injective,  $<$  extends to  $L$  (and to  $K$ ). Moreover by [2, Exercise 2, p. 289],  $<$  extends to exactly  $[L:F]$  orderings on  $L$  and to  $t \leq [K:F]$  orderings on  $K$  (compare [3, Proposition 5.12]). Let  $<_1, <_2, \dots, <_m, m \leq t$ , be the orderings on  $K$  which extend  $<$  and which also extend to  $L$ . Since  $K \subset L$  is a Galois extension, it again follows that each  $<_i$  extends exactly  $[L:K]$  different ways to  $L$ . Thus  $[L:F] = m[L:K]$ , which implies that  $m = [K:F]$ . Hence  $m = t$  so that every extension of  $<$  to  $K$  also extends to  $L$ . But every ordering on  $K$  is the extension of some ordering on  $F$ , so it follows that every ordering on  $K$  extends to  $L$ . Since  $K$  is a pythagorean field, the mapping  $r_{L/K}: W(K) \rightarrow W(L)$  is injective. If the Galois group  $H$  of the extension  $K \subset L$  is not trivial then there will exist a nonsquare  $a$  in  $K$  with  $\sqrt{a}$  in  $L$ . Then  $\langle 1, -a \rangle$  is an anisotropic form over  $K$  whose class in  $W(K)$  is a nonzero element in the kernel of  $r_{L/K}$ . Thus  $H$  is also trivial in this case, i.e.  $n$  is odd.

**COROLLARY.** *Let  $F \subset L$  be a finite Galois extension of degree  $n$  with  $L$  pythagorean. If every ordering on  $F$  extends to  $L$  then  $n$  is odd.*

*Proof.* By [4, Exercise 17, p. 254],  $F$  is also pythagorean.

The following modification of a construction due to Manfred Knebusch shows that the hypothesis that  $L$  be pythagorean is essential in Proposition 1 and its corollary.

**EXAMPLE.** A Galois extension  $F \subset L$  of formally real fields with  $F$  pythagorean (actually euclidean),  $[L:F]$  even, and  $r: W(F) \rightarrow W(L)$  injective.

Choose  $n \geq 5$  and let  $K$  be a formally real field on which the alternating group  $A_n$  acts as a group of automorphisms (e.g.  $K = \mathbf{R}(x_1, \dots, x_n)$ ). Let  $k = K^{A_n}$  be the fixed field and let  $\tilde{k}$  be the quadratic closure of  $k$ , i.e. the compositum of all Galois extensions of  $k$  with degree a power of 2 [4, p. 219]. Then  $\tilde{k}$  is a Galois extension of  $k$  and since  $[K:k]$  is not a power of two,  $K$  is not contained in  $\tilde{k}$ . Thus  $\tilde{k} \cap K \neq K$  is a Galois extension of  $k$  so Galois theory and the simplicity of  $A_n$  imply that  $\tilde{k} \cap K = k$ .

Now let  $R$  be a real closure ([2], [4], [5]) of the formally real field  $K$  and let  $F = R \cap \tilde{k}$ . Then we also have  $F \cap K = k$ . Moreover,  $F$  is formally real and it is easy to see that any  $a$  in  $F$  is either a square in  $F$

or the negative of a square in  $F$ . In particular,  $F$  is pythagorean and has exactly one ordering. From Sylvester's law of inertia we have  $W(F) \cong Z$  (cf. [4, pp. 42–43]).

Let  $L = FK$  be the compositum of  $F$  and  $K$  in  $R$ . Then  $L$  is a formally real Galois extension of  $F$  with Galois group  $A_n$  [5, Theorem 4, p. 196]. In particular,  $[L : F]$  is even. Finally, any signature  $\sigma_{<} : W(L) \rightarrow Z$  arising from an ordering  $<$  on the formally real field  $L$  (see [4, pp. 42–43], [3, p. 211]) will provide a splitting for the map  $r : W(F) \rightarrow W(L)$ .

**PROPOSITION 2.** *Let  $F$  be a pythagorean field and  $L$  a finite Galois extension of  $F$ . Then the following statements are equivalent;*

- (1)  $r : W(F) \rightarrow W(L)$  is injective.
- (2) If  $q$  is an anisotropic quadratic form over  $F$  then  $q_L = L \otimes_F q$  is anisotropic over  $L$ .

*Proof.* (1)  $\Rightarrow$  (2). If  $F$  is not formally real then  $F$  is quadratically closed so all anisotropic forms over  $F$  are one dimensional. Hence the implication is obvious in this case.

Now assume  $F$  is formally real and let  $Tr^*$  denote Scharlau's transfer map relative to the  $F$ -linear trace map  $Tr_{L/F}$  (which associates to each quadratic form  $q$  over  $L$  the  $F$ -quadratic form  $Tr_{L/F} \circ q$ ) [4, Chapter 7, §1, 6], [3, §5]. Then for any anisotropic form  $q$  over  $F$ , there is an isometry  $L \otimes_F Tr^*(q_L) q_L \perp \cdots \perp q_L \cong [L : F] \cdot q_L$ , where  $[L : F] \cdot q_L = q_L \perp \cdots \perp q_L$ ,  $[L : F]$  times [4, Theorem 6.1, p. 212] compare [3, Corollary 5.10]). Since the mapping  $r : W(F) \rightarrow W(L)$  is injective this means that  $Tr^*(q_L)$  is isometric to  $[L : F] \cdot q$  over  $F$ . But  $F$  is a formally real pythagorean field, so by (the proof of) [4, Theorem 3.3],  $[L : F] \cdot q$  is anisotropic over  $F$ . Therefore  $Tr^*(q_L)$  is anisotropic over  $F$  so that, in particular,  $q_L$  is anisotropic over  $L$ .

The implication (2)  $\Rightarrow$  (1) is immediate.

It seems to be an open question whether, for an arbitrary extension  $F \subset L$ , the injectivity of  $r : W(F) \rightarrow W(L)$  implies that anisotropic forms over  $F$  remain so over  $L$ . However, for a certain class of pythagorean fields the answer is affirmative. Let  $F$  be a formally real field, let  $X$  be the set of orderings on  $F$ , and for  $a$  in  $F$ , let  $V(a) = \{< \text{in } X \mid a > 0\}$ . Then the family  $V(a)_{a \in F}$  generates a compact, Hausdorff, totally disconnected topology on  $X$  [3, Lemma 3.3, Theorem 3.18]. The field  $F$  satisfies the Strong Approximation Property (SAP) if given any two disjoint closed subsets  $U, V$  of  $X$  there is an element  $a$  in  $F$  which is positive at the orderings in  $U$  and negative at the orderings in  $V$  (cf. [1, Definition 1.4], [3, Corollary 3.21]).

**PROPOSITION 3.** *Let  $F$  be a formally real pythagorean field satisfying SAP and let  $L$  be any extension field of  $F$ . If  $r : W(F) \rightarrow W(L)$  is*

*injective then any anisotropic quadratic form over  $F$  remains anisotropic over  $L$ .*

*Proof.* In view of [1, Theorem 5.3 (1)], any anisotropic form  $q$  over  $F$  can be written  $q = \langle a_1, \dots, a_n \rangle$  where either all the  $a_i$ 's are positive or all the  $a_i$ 's are negative with respect to some ordering  $<$  on  $F$ . If  $r: W(F) \rightarrow W(L)$  is injective then  $<$  extends to  $L$  so an equation  $a_1x_1^2 + \dots + a_nx_n^2 = 0$  with each  $x_i$  in  $L$  is impossible.

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Received March 13, 1974. Partially supported by NSF Grant GP-37781.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

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