Pacific Journal of Mathematics

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Vol. 58, No. 2 April 1975

PACIFIC JOURNAL OF MATHEMATICS
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As it is well known, the groups of definable automorphisms of two elementary equivalent relational structures satisfy the same $\forall_1$-statements. We show that this does not hold in general for $\forall_2$-statements, thus correcting an error in the literature.

0. An automorphism φ of a model $\mathfrak{M}$ is said to be definable if there is a formula H of the (first-order) language of $\mathfrak{M}$ and elements $a_1, \cdots, a_n \in M$, such that for all $x, y \in M$

$$\mathfrak{M} \models H(x, y, a_1, \cdots, a_n) \quad \text{iff} \quad \varphi(x) = y.$$

Let Def Aut ($\mathfrak{M}$) denote the group of definable automorphisms of $\mathfrak{M}$ (see [5]).

In [1] it is remarked that if $\mathfrak{M}$ and $\mathfrak{N}$ are elementary equivalent, then Def Aut($\mathfrak{M}$) and Def Aut($\mathfrak{N}$) are universally equivalent. In this note we show that this is the best possible result. We give an example of an $\forall\exists$-statement, which holds in Def Aut($\mathfrak{M}$) but not in Def Aut($\mathfrak{M}'$), where $\mathfrak{M}$ and $\mathfrak{M}'$ are two elementary equivalent models. In fact our $\mathfrak{M}'$ is an elementary extension of $\mathfrak{M}$. This disproves Theorems 1,2 in [3] (p. 109).

We construct our example from the Prüfer group $\mathbf{Z}(3^\infty)$ and investigate definability using the method of Ehrenfeucht games.

1. Our example is as follows. H is the (group theoretical) statement

$$\forall x \exists y \; x = y^2.$$

We define $\mathfrak{M}$ to be $(M, Z, \omega, <, f)$, where M is the disjoint union of Z and ω. Z is the underlying set of the Prüfergroup $\mathbf{Z}(3^\infty)$, which is defined as

$$\left\{ \frac{n}{3^m} \,\middle|\, n, m \in \mathbf{Z} \right\} / \mathbf{Z}.$$

$<$ is the natural ordering of ω, the set of natural numbers. f is a binary function defined by

$$f(n,z):=z+\frac{1}{3^n}\mathbf{Z} \quad \text{if} \quad n\in\omega \quad \text{and} \quad z\in Z$$

$$(+\text{ stands for the addition in } \mathbf{Z}(3^\infty))$$

$$:=0\ (\in\omega) \quad \text{otherwise.}$$

We shall denote by f_n the function

$$\lambda z f(n,z)\colon Z\to Z \qquad (n\in\omega).$$

Every automorphism of $\mathfrak{M}$ operates on ω as the identity and therefore commutes with each f_n. The f_n generate the group of all translations of $\mathbf{Z}(3^\infty)$, and so it is easily seen (see e.g. **[4]** p. 43) that the automorphisms of $\mathfrak{M}$ are just those permutations of M, which leave ω fixed and operate on Z like a translation. Since the f_n are definable, all automorphisms are definable and hence

$$\text{Def Aut}(\mathfrak{M})\cong\mathbf{Z}(3^\infty)\models H.$$

Let $\mathfrak{M}'=(M',Z',W',<',f')$ be an elementary extension of $\mathfrak{M}$ such that $W'\neq\omega$. We claim that Def Aut$(\mathfrak{M}')\not\models H$.

2. First we look at definability in $\mathfrak{M}$.

Every element x of $\mathbf{Z}(3^\infty)$ has an unique representation

$$x=\sum_{i=1}^{\infty}\frac{k_i}{3^i}\mathbf{Z},\ k_i\in\{-1,0,1\}, \text{ almost all } k_i=0.$$

We define

$$|x|:=\sum_{i=1}^{\infty}|k_i|,\ v(x):=\max\{i\mid k_i\neq0\} \quad \text{and}$$

$$\bar{v}(x):=\min\{\mathrm{i}\mid k_i\neq0\}$$

We note that

(i) $|-x|=|x|$
(ii) $|x+y|\leqq|x|+|y|$
(iii) $|x+y|=|x|+|y|$ if $v(x)<\bar{v}(y)$
(iv) $v(x+y)\leqq\max(v(x),v(y))$

Let I_n be the set of all partial functions φ from Z in Z with the following property:

dom φ is finite and for all $a,b\in\text{dom}\,\varphi$

$$|a-b|\leqq2^n \quad \text{iff} \quad |\varphi(a)-\varphi(b)|\leqq2^n$$

and in this case $a - b = \varphi(a) - \varphi(b)$.

Clearly $I_{n+1} \subset I_n$ and if $\varphi \in I_0$, $a, b \in \operatorname{dom}\varphi$ and $f_m(a) = b$ then $f_m(\varphi(a)) = \varphi(b)$.

We show that the family I has the back and forth property: Let $\varphi \in I_{n+1}$ and $a \in Z \setminus \operatorname{dom}\varphi$. We want to extend φ to $\varphi' \in I_n$ with $\operatorname{dom}\varphi' = \operatorname{dom}\varphi \cup \{a\}$.

There are two possible cases

(1) There is $b \in \operatorname{dom}\varphi$ such that $|a - b| \leqq 2^n$. Define $\varphi'(a) := \varphi(b) + (a - b)$. Then $\varphi' \in I_n$. For let e.g. $c \in \operatorname{dom}\varphi$ and $|c - a| \leqq 2^n$. It follows from (i) and (ii)

$$|b - c| \leqq |a - b| + |c - a| \leqq 2^n + 2^n = 2^{n+1}$$

hence

$$\varphi(b) - \varphi(c) = b - c. \quad \text{It follows}$$

$$\varphi(c) - \varphi'(a) = c - a.$$

(2) For all $b \in \operatorname{dom}\varphi$ $|a - b| > 2^n$.
Choose $a' \in Z$ such that $|a'| > 2^n$ and $\bar{v}(a') > v(\varphi(b))$ for all $b \in \operatorname{dom}\varphi$. Define $\varphi'(a) := a'$. From (iii) it follows that for all $b \in \operatorname{dom}\varphi$

$$|\varphi'(a) - \varphi(b)| > 2^n.$$

Since $\varphi^{-1} \in I_n$ iff $\varphi \in I_n$ it is clear that for all $\varphi \in I_{n+1}$ and $a \in Z$ there is an extension φ' of φ such that $\varphi' \in I_n$, $a \in rg\varphi'$.

Let H_n be the set of all formulas (of the language of $\mathfrak{M}$), which contain at most n quantifiers and where all function symbols are applied to variables only. It is shown in [2] that, if $\varphi \in I_n$, $a_1, \cdots, a_k \in \operatorname{dom}\varphi$ $b_1, \cdots, b_e \in \omega$ and $H \in H_n$

$$\mathfrak{M} \models H(a_1, \cdots, a_k, b_1, \cdots, b_e) \quad \text{iff}$$

$$\mathfrak{M} \models H(\varphi(a_1), \cdots, \varphi(a_k), b_1, \cdots, b_e).$$

This is a consequence of the back and forth property.

Let now $z \in Z$, $|z| > 2^n$ and g be the translation

$$\lambda x(x + z) \colon Z \to Z.$$

We show that g is not definable using a formula in H_n.

Assume that there is a $H \in H_n$, $a_1, \cdots, a_k \in Z$, $b_1, \cdots, b_e \in \omega$ such that for all $a, b \in Z$

$$\mathfrak{M} \models H(a, b, a_1, \cdots, a_k, b_1, \cdots, b_2) \quad \text{iff} \quad g(a) = b.$$

Choose $c \in Z$ such that $|c| > 2^n$ and $\bar{v}(c)$ and $\bar{v}(2c)$ are greater than all $v(a_i)$ and $v(z)$. (Choose a c of the form $\Sigma_{i=m}^{m'} 1/3^i$). Define $\varphi(a_i) := a_i$ $(i = 1, \cdots, k)$, $\varphi(c) := c$ and $\varphi(z + c) := z - c$. It is easily seen that $\varphi \in I_n$. For

$$|(z + c) - a_i| = |z - a_i| + |c| > 2^n \quad \text{by (iii), (iv))}$$

$$|(z - c) - a_i| = |z - a_i| + |-c| > 2^n \quad \text{(by (i))}$$

$$|(z + c) - c| > 2^n$$

$$|(z - c) - c| = |z - 2c| = |z| + |2c| > 2^n \quad \text{(by (iii))}.$$

Therefore we have, $\mathfrak{M} \models H(c, z - c, a_1, \cdots, b_e)$

$$\text{since} \quad \mathfrak{M} \not\models H(c, z + c, a_1, \cdots, b_e)$$

Whence $z - c = z + c$ and we have the contradiction $c = 0$.

We prove now that $\mathcal{M}' \models H$. First note that

$$\left| \frac{1}{2} \cdot \frac{1}{3_m} \right| = \left| \sum_{i=1}^{m} \frac{-1}{3^i} \right| = m.$$

This and the last result imply that for all $m > 2^n$, $H(x, y, x_1, \cdots, x_r) \in H_n$, $a_1, \cdots, a_r \in M$ $H(x, y, a_1, \cdots, a_r)$ does not define an automorphism ψ of $\mathfrak{M}$ such that $\psi^2 = f_m \cup id_\omega$. This is expressible by a set of sentences which hold also in $\mathfrak{M}'$. If we choose $m \in W' \backslash \omega$, we have for all $n \in \omega$ $m' > 2^n$, hence $f'_m \cup id_W$, is a definable automorphism such that there is no definable automorphism ψ $\psi^2 = f'_m \cup id_{W'}$. Whence $\mathcal{M} \not\models H$.

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Received May 27, 1974.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics

Vol. 58, No. 2 April, 1975