

Pacific Journal of Mathematics

**A COUNTEREXAMPLE IN THE THEORY OF DEFINABLE
AUTOMORPHISMS**

MARTIN ZIEGLER

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As it is well known, the groups of definable automorphisms of two elementary equivalent relational structures satisfy the same \forall_1 -statements. We show that this does not hold in general for \forall_2 -statements, thus correcting an error in the literature.

0. An automorphism φ of a model \mathfrak{M} is said to be definable if there is a formula H of the (first-order) language of \mathfrak{M} and elements $a_1, \dots, a_n \in M$, such that for all $x, y \in M$

$$\mathfrak{M} \models H(x, y, a_1, \dots, a_n) \quad \text{iff} \quad \varphi(x) = y.$$

Let $\text{Def Aut}(\mathfrak{M})$ denote the group of definable automorphisms of \mathfrak{M} (see [5]).

In [1] it is remarked that if \mathfrak{M} and \mathfrak{N} are elementary equivalent, then $\text{Def Aut}(\mathfrak{M})$ and $\text{Def Aut}(\mathfrak{N})$ are universally equivalent. In this note we show that this is the best possible result. We give an example of an $\forall\exists$ -statement, which holds in $\text{Def Aut}(\mathfrak{M})$ but not in $\text{Def Aut}(\mathfrak{M}')$, where \mathfrak{M} and \mathfrak{M}' are two elementary equivalent models. In fact our \mathfrak{M}' is an elementary extension of \mathfrak{M} . This disproves Theorems 1,2 in [3] (p. 109).

We construct our example from the Prüfer group $\mathbf{Z}(3^\infty)$ and investigate definability using the method of Ehrenfeucht games.

1. Our example is as follows. H is the (group theoretical) statement

$$\forall x \exists y \ x = y^2.$$

We define \mathfrak{M} to be $(M, Z, \omega, <, f)$, where M is the disjoint union of Z and ω . Z is the underlying set of the Prüfergroup $\mathbf{Z}(3^\infty)$, which is defined as

$$\left\{ \frac{n}{3^m} \mid n, m \in \mathbf{Z} \right\} / \mathbf{Z}.$$

$<$ is the natural ordering of ω , the set of natural numbers. f is a binary function defined by

$$f(n, z) := z + \frac{1}{3^n} \mathbf{Z} \quad \text{if } n \in \omega \quad \text{and } z \in Z$$

(+ stands for the addition in $\mathbf{Z}(3^\omega)$)

$$:= 0 \quad (\in \omega) \quad \text{otherwise.}$$

We shall denote by f_n the function

$$\lambda z f(n, z): Z \rightarrow Z \quad (n \in \omega).$$

Every automorphism of \mathfrak{M} operates on ω as the identity and therefore commutes with each f_n . The f_n generate the group of all translations of $\mathbf{Z}(3^\omega)$, and so it is easily seen (see e.g. [4] p. 43) that the automorphisms of \mathfrak{M} are just those permutations of M , which leave ω fixed and operate on Z like a translation. Since the f_n are definable, all automorphisms are definable and hence

$$\text{Def Aut}(\mathfrak{M}) \cong \mathbf{Z}(3^\omega) \upharpoonright H.$$

Let $\mathfrak{M}' = (M', Z', W', <', f')$ be an elementary extension of \mathfrak{M} such that $W' \neq \omega$. We claim that $\text{Def Aut}(\mathfrak{M}') \neq H$.

2. First we look at definability in \mathfrak{M} .

Every element x of $\mathbf{Z}(3^\omega)$ has a unique representation

$$x = \sum_{i=1}^{\infty} \frac{k_i}{3^i} \mathbf{Z}, \quad k_i \in \{-1, 0, 1\}, \quad \text{almost all } k_i = 0.$$

We define

$$|x| := \sum_{i=1}^{\infty} |k_i|, \quad v(x) := \max \{i \mid k_i \neq 0\} \quad \text{and}$$

$$\bar{v}(x) := \min \{i \mid k_i \neq 0\}$$

We note that

- (i) $|-x| = |x|$
- (ii) $|x + y| \leq |x| + |y|$
- (iii) $|x + y| = |x| + |y|$ if $v(x) < \bar{v}(y)$
- (iv) $v(x + y) \leq \max(v(x), v(y))$

Let I_n be the set of all partial functions φ from Z in Z with the following property:

$\text{dom } \varphi$ finite and for all $a, b \in \text{dom } \varphi$

$$|a - b| \leq 2^n \quad \text{iff} \quad |\varphi(a) - \varphi(b)| \leq 2^n$$

and in this case $a - b = \varphi(a) - \varphi(b)$.

Clearly $I_{n+1} \subset I_n$ and if $\varphi \in I_0$, $a, b \in \text{dom } \varphi$ and $f_m(a) = b$ then $f_m(\varphi(a)) = \varphi(b)$.

We show that the family I has the back and forth property: Let $\varphi \in I_{n+1}$ and $a \in Z \setminus \text{dom } \varphi$. We want to extend φ to $\varphi' \in I_n$ with $\text{dom } \varphi' = \text{dom } \varphi \cup \{a\}$.

There are two possible cases

(1) There is $b \in \text{dom } \varphi$ such that $|a - b| \leq 2^n$. Define $\varphi'(a) := \varphi(b) + (a - b)$. Then $\varphi' \in I_n$. For let e.g. $c \in \text{dom } \varphi$ and $|c - a| \leq 2^n$. It follows from (i) and (ii)

$$|b - c| \leq |a - b| + |c - a| \leq 2^n + 2^n = 2^{n+1}$$

hence

$$\varphi(b) - \varphi(c) = b - c. \quad \text{It follows}$$

$$\varphi(c) - \varphi'(a) = c - a.$$

(2) For all $b \in \text{dom } \varphi$ $|a - b| > 2^n$.

Choose $a' \in Z$ such that $|a'| > 2^n$ and $\bar{v}(a') > v(\varphi(b))$ for all $b \in \text{dom } \varphi$. Define $\varphi'(a) := a'$. From (iii) it follows that for all $b \in \text{dom } \varphi$

$$|\varphi'(a) - \varphi(b)| > 2^n.$$

Since $\varphi^{-1} \in I_n$ iff $\varphi \in I_n$ it is clear that for all $\varphi \in I_{n+1}$ and $a \in Z$ there is an extension φ' of φ such that $\varphi' \in I_n$, $a \in \text{rg } \varphi'$.

Let H_n be the set of all formulas (of the language of \mathfrak{M}), which contain at most n quantifiers and where all function symbols are applied to variables only. It is shown in [2] that, if $\varphi \in I_n$, $a_1, \dots, a_k \in \text{dom } \varphi$, $b_1, \dots, b_e \in \omega$ and $H \in H_n$

$$\mathfrak{M} \models H(a_1, \dots, a_k, b_1, \dots, b_e) \quad \text{iff}$$

$$\mathfrak{M} \models H(\varphi(a_1), \dots, \varphi(a_k), b_1, \dots, b_e).$$

This is a consequence of the back and forth property.

Let now $z \in Z$, $|z| > 2^n$ and g be the translation

$$\lambda x(x + z): Z \rightarrow Z.$$

We show that g is not definable using a formula in H_n .

Assume that there is a $H \in H_n$, $a_1, \dots, a_k \in Z$, $b_1, \dots, b_e \in \omega$ such that for all $a, b \in Z$

$$\mathfrak{M} \models H(a, b, a_1, \dots, a_k, b_1, \dots, b_e) \quad \text{iff} \quad g(a) = b.$$

Choose $c \in Z$ such that $|c| > 2^n$ and $\bar{v}(c)$ and $\bar{v}(2c)$ are greater than all $v(a_i)$ and $v(z)$. (Choose a c of the form $\sum_{i=m}^m 1/3^i$). Define $\varphi(a_i) := a_i$ ($i = 1, \dots, k$), $\varphi(c) := c$ and $\varphi(z+c) := z-c$. It is easily seen that $\varphi \in I_n$. For

$$\begin{aligned} |(z+c) - a_i| &= |z - a_i| + |c| > 2^n && \text{by (iii), (iv)} \\ |(z-c) - a_i| &= |z - a_i| + |-c| > 2^n && \text{by (i)} \\ |(z+c) - c| &> 2^n \\ |(z-c) - c| &= |z - 2c| = |z| + |2c| > 2^n && \text{(by (iii)).} \end{aligned}$$

Therefore we have, $\mathfrak{M} \models H(c, z-c, a_1, \dots, b_e)$

$$\text{since} \quad \mathfrak{M} \not\models H(c, z+c, a_1, \dots, b_e)$$

Whence $z-c = z+c$ and we have the contradiction $c = 0$.

We prove now that $\mathcal{M}' \models H$. First note that

$$\left| \frac{1}{2} \cdot \frac{1}{3^m} \right| = \left| \sum_{i=1}^m \frac{-1}{3^i} \right| = m.$$

This and the last result imply that for all $m > 2^n$, $H(x, y, x_1, \dots, x_r) \in H_n$, $a_1, \dots, a_r \in M$ $H(x, y, a_1, \dots, a_r)$ does not define an automorphism ψ of \mathfrak{M} such that $\psi^2 = f_m \cup id_\omega$. This is expressible by a set of sentences which hold also in \mathfrak{M}' . If we choose $m \in W' \setminus \omega$, we have for all $n \in \omega$ $m' > 2^n$, hence $f'_m \cup id_w$ is a definable automorphism such that there is no definable automorphism ψ $\psi^2 = f'_m \cup id_w$. Whence $\mathcal{M}' \models H$.

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Received May 27, 1974.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

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