

Pacific Journal of Mathematics

**A COUNTEREXAMPLE IN THE THEORY OF DEFINABLE
AUTOMORPHISMS**

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As it is well known, the groups of definable automorphisms of two elementary equivalent relational structures satisfy the same \forall_1 -statements. We show that this does not hold in general for \forall_2 -statements, thus correcting an error in the literature.

0. An automorphism φ of a model \mathfrak{M} is said to be definable if there is a formula H of the (first-order) language of \mathfrak{M} and elements $a_1, \dots, a_n \in M$, such that for all $x, y \in M$

$$\mathfrak{M} \models H(x, y, a_1, \dots, a_n) \quad \text{iff} \quad \varphi(x) = y.$$

Let $\text{Def Aut}(\mathfrak{M})$ denote the group of definable automorphisms of \mathfrak{M} (see [5]).

In [1] it is remarked that if \mathfrak{M} and \mathfrak{N} are elementary equivalent, then $\text{Def Aut}(\mathfrak{M})$ and $\text{Def Aut}(\mathfrak{N})$ are universally equivalent. In this note we show that this is the best possible result. We give an example of an $\forall\exists$ -statement, which holds in $\text{Def Aut}(\mathfrak{M})$ but not in $\text{Def Aut}(\mathfrak{M}')$, where \mathfrak{M} and \mathfrak{M}' are two elementary equivalent models. In fact our \mathfrak{M}' is an elementary extension of \mathfrak{M} . This disproves Theorems 1,2 in [3] (p. 109).

We construct our example from the Prüfer group $\mathbf{Z}(3^\infty)$ and investigate definability using the method of Ehrenfeucht games.

1. Our example is as follows. H is the (group theoretical) statement

$$\forall x \exists y \ x = y^2.$$

We define \mathfrak{M} to be $(M, Z, \omega, <, f)$, where M is the disjoint union of Z and ω . Z is the underlying set of the Prüfergroup $\mathbf{Z}(3^\infty)$, which is defined as

$$\left\{ \frac{n}{3^m} \mid n, m \in \mathbf{Z} \right\} / \mathbf{Z}.$$

$<$ is the natural ordering of ω , the set of natural numbers. f is a binary function defined by

$$f(n, z) := z + \frac{1}{3^n}Z \quad \text{if } n \in \omega \quad \text{and } z \in Z$$

$$(+ \text{ stands for the addition in } Z(3^\omega))$$

$$:= 0 \quad (\in \omega) \quad \text{otherwise.}$$

We shall denote by f_n the function

$$\lambda z f(n, z): Z \rightarrow Z \quad (n \in \omega).$$

Every automorphism of \mathfrak{M} operates on ω as the identity and therefore commutes with each f_n . The f_n generate the group of all translations of $Z(3^\omega)$, and so it is easily seen (see e.g. [4] p. 43) that the automorphisms of \mathfrak{M} are just those permutations of M , which leave ω fixed and operate on Z like a translation. Since the f_n are definable, all automorphisms are definable and hence

$$\text{Def Aut}(\mathfrak{M}) \cong Z(3^\omega) \models H.$$

Let $\mathfrak{M}' = (M', Z', W', <', f')$ be an elementary extension of \mathfrak{M} such that $W' \neq \omega$. We claim that $\text{Def Aut}(\mathfrak{M}') \not\models H$.

2. First we look at definability in \mathfrak{M} .

Every element x of $Z(3^\omega)$ has a unique representation

$$x = \sum_{i=1}^{\infty} \frac{k_i}{3^i} Z, \quad k_i \in \{-1, 0, 1\}, \quad \text{almost all } k_i = 0.$$

We define

$$|x| := \sum_{i=1}^{\infty} |k_i|, \quad v(x) := \max \{i \mid k_i \neq 0\} \quad \text{and}$$

$$\bar{v}(x) := \min \{i \mid k_i \neq 0\}$$

We note that

- (i) $|-x| = |x|$
- (ii) $|x + y| \leq |x| + |y|$
- (iii) $|x + y| = |x| + |y|$ if $v(x) < \bar{v}(y)$
- (iv) $v(x + y) \leq \max(v(x), v(y))$

Let I_n be the set of all partial functions φ from Z in Z with the following property:

$\text{dom } \varphi$ is finite and for all $a, b \in \text{dom } \varphi$

$$|a - b| \leq 2^n \quad \text{iff} \quad |\varphi(a) - \varphi(b)| \leq 2^n$$

and in this case $a - b = \varphi(a) - \varphi(b)$.

Clearly $I_{n+1} \subset I_n$ and if $\varphi \in I_0$, $a, b \in \text{dom } \varphi$ and $f_m(a) = b$ then $f_m(\varphi(a)) = \varphi(b)$.

We show that the family I has the back and forth property: Let $\varphi \in I_{n+1}$ and $a \in Z \setminus \text{dom } \varphi$. We want to extend φ to $\varphi' \in I_n$ with $\text{dom } \varphi' = \text{dom } \varphi \cup \{a\}$.

There are two possible cases

(1) There is $b \in \text{dom } \varphi$ such that $|a - b| \leq 2^n$. Define $\varphi'(a) := \varphi(b) + (a - b)$. Then $\varphi' \in I_n$. For let e.g. $c \in \text{dom } \varphi$ and $|c - a| \leq 2^n$. It follows from (i) and (ii)

$$|b - c| \leq |a - b| + |c - a| \leq 2^n + 2^n = 2^{n+1}$$

hence

$$\varphi(b) - \varphi(c) = b - c. \quad \text{It follows}$$

$$\varphi(c) - \varphi'(a) = c - a.$$

(2) For all $b \in \text{dom } \varphi$ $|a - b| > 2^n$.

Choose $a' \in Z$ such that $|a'| > 2^n$ and $\bar{v}(a') > v(\varphi(b))$ for all $b \in \text{dom } \varphi$. Define $\varphi'(a) := a'$. From (iii) it follows that for all $b \in \text{dom } \varphi$

$$|\varphi'(a) - \varphi(b)| > 2^n.$$

Since $\varphi^{-1} \in I_n$ iff $\varphi \in I_n$ it is clear that for all $\varphi \in I_{n+1}$ and $a \in Z$ there is an extension φ' of φ such that $\varphi' \in I_n$, $a \in \text{rg } \varphi'$.

Let H_n be the set of all formulas (of the language of \mathfrak{M}), which contain at most n quantifiers and where all function symbols are applied to variables only. It is shown in [2] that, if $\varphi \in I_n$, $a_1, \dots, a_k \in \text{dom } \varphi$, $b_1, \dots, b_e \in \omega$ and $H \in H_n$

$$\mathfrak{M} \models H(a_1, \dots, a_k, b_1, \dots, b_e) \quad \text{iff}$$

$$\mathfrak{M} \models H(\varphi(a_1), \dots, \varphi(a_k), b_1, \dots, b_e).$$

This is a consequence of the back and forth property.

Let now $z \in Z$, $|z| > 2^n$ and g be the translation

$$\lambda x(x + z): Z \rightarrow Z.$$

We show that g is not definable using a formula in H_n .

Assume that there is a $H \in H_n$, $a_1, \dots, a_k \in Z$, $b_1, \dots, b_e \in \omega$ such that for all $a, b \in Z$

$$\mathfrak{M} \models H(a, b, a_1, \dots, a_k, b_1, \dots, b_e) \text{ iff } g(a) = b.$$

Choose $c \in Z$ such that $|c| > 2^n$ and $\bar{v}(c)$ and $\bar{v}(2c)$ are greater than all $v(a_i)$ and $v(z)$. (Choose a c of the form $\sum_{i=m}^{m'} 1/3^i$). Define $\varphi(a_i) := a_i$ ($i = 1, \dots, k$), $\varphi(c) := c$ and $\varphi(z + c) := z - c$. It is easily seen that $\varphi \in I_n$. For

$$\begin{aligned} |(z + c) - a_i| &= |z - a_i| + |c| > 2^n \text{ by (iii), (iv)} \\ |(z - c) - a_i| &= |z - a_i| + |-c| > 2^n \text{ (by (i))} \\ |(z + c) - c| &> 2^n \\ |(z - c) - c| &= |z - 2c| = |z| + |2c| > 2^n \text{ (by (iii)).} \end{aligned}$$

Therefore we have, $\mathfrak{M} \models H(c, z - c, a_1, \dots, b_e)$

since $\mathfrak{M} \not\models H(c, z + c, a_1, \dots, b_e)$

Whence $z - c = z + c$ and we have the contradiction $c = 0$.

We prove now that $\mathcal{M}' \models H$. First note that

$$\left| \frac{1}{2} \cdot \frac{1}{3_m} \right| = \left| \sum_{i=1}^m \frac{-1}{3^i} \right| = m.$$

This and the last result imply that for all $m > 2^n$, $H(x, y, x_1, \dots, x_r) \in H_n$, $a_1, \dots, a_r \in M$ $H(x, y, a_1, \dots, a_r)$ does not define an automorphism ψ of \mathfrak{M} such that $\psi^2 = f_m \cup id_\omega$. This is expressible by a set of sentences which hold also in \mathfrak{M}' . If we choose $m \in W' \setminus \omega$, we have for all $n \in \omega$ $m' > 2^n$, hence $f'_m \cup id_\omega$ is a definable automorphism such that there is no definable automorphism ψ $\psi^2 = f'_m \cup id_\omega$. Whence $\mathcal{M}' \models H$.

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