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# FIXED POINTS FOR ORIENTATION PRESERVING HOMEOMORPHISMS OF THE PLANE WHICH INTERCHANGE TWO POINTS

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# FIXED POINTS FOR ORIENTATION PRESERVING HOMEOMORPHISMS OF THE PLANE WHICH INTERCHANGE TWO POINTS

EMILIO GAGLIARDO AND CLIFFORD KOTTMAN

**Let $T$ be an orientation preserving homeomorphism defined on a subset of the plane which interchanges two points, $P$ and $Q$. Let $\Gamma$ be a simple curve joining $P$ and $Q$ and let $\Omega$ be a simply connected set contained in the domain and range of $T$ such that $\Gamma \subset \Omega$, $T(\Gamma) \subset \Omega$, $T^{-1}(\Gamma) \subset \Omega$. Then $T$ has a fixed point in $\Omega$. A corollary concerning fixed points of homeomorphisms on $S^2$ follows.**

**The proof would be trivial if $T$ were necessarily an element of a flow on the plane, however an example given in this paper shows that this need not be the case.**

If, in particular, $T$ is defined in the whole plane or if its domain and range are the same halfplane, then the existence (but no constructive information about the location) of a fixed point could also be derived from classical results of Brouwer [2] (see for instance Proposition $0(a \Rightarrow b)$ and Proposition 1.1 of S. A. Andrea [1]).

1. **The theorem.** We use $\mathscr{R}$ to denote the real numbers and $\mathscr{R}^2$ for the coordinate plane. A curve is a continuous function whose domain is a compact interval of $\mathscr{R}$ and whose range is a subset of $\mathscr{R}^2$. If $[a, b]$ is an interval within the domain of the curve $\Phi$, we use $\Phi[a, b]$ as a shorthand for $\{\Phi(t): t \in [a, b]\}$; $\Phi(a, b)$ and $\Phi[a, b)$ have analogous meanings. The terms close surve, simple curve, and simple closed curve have the standard meanings.

For the following lemmas we fix two simple closed curves, $\Phi_1$ and $\Phi_2: [0, 3] \rightarrow \mathscr{R}^2$. The first, $\Phi_1$, is the triangle defined by

$$\Phi_1(t) = \begin{cases} (2t-1, 2t) & \text{for} \quad 0 \leqq t \leqq 1 \\ (1, 4-2t) & \text{for} \quad 1 \leqq t \leqq 2 \\ (5-2t, 0) & \text{for} \quad 2 \leqq t \leqq 3 \end{cases}$$

Referring to Figure 1, $\Phi_1[0, 1]$ is the segment $MH$, $\Phi_1[1, 2]$ is the segment $HK$, and $\Phi_1$ $[2, 3]$ is the segment $KM$.

The second curve, $\Phi_2$, is defined so that the following conditions are satisfied:

(I) $\Phi_2[0, 1]$ *is the segment from* $L = (\lambda, 0)$ (with $\lambda > 0$) to $M = (-1, 0)$; one has therefore: $\Phi_2(\rho) = 0 = (0, 0)$ for a suitable $\rho$ with $0 < \rho < 1$.

(II) $\Phi_2[0, 3]$ has winding number $-1$ about each of its interior points (just as $\Phi_1$ has); and therefore $\Phi_2[1, 3]$ has winding number $-1/2$ about the origin (just as $\Phi_1[0, 2]$ has).

(III) $\Phi_2(1, 2]$ is disjoint from $\Phi_1[1, 3]$. In Figure 1, $\Phi_2[1, 2]$ is represented by the curve $MN$, which except for $M$ is disjoint from $HK$ and $KM$.

(IV) $\Phi_2[2, 3)$ is disjoint from $\Phi_1[2, 3]$. In Figure 1, $\Phi_2[2, 3]$ is represented by the curve $NL$, which except perhaps for $L$, is disjoint from $KM$.

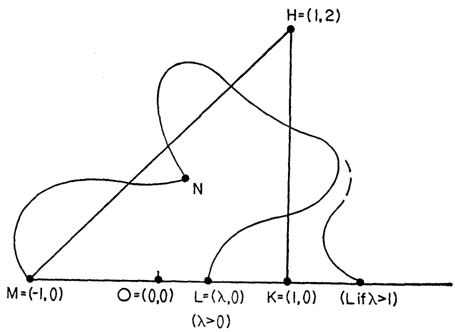


FIGURE 1

LEMMA 1. *With* $\Phi_1$ *and* $\Phi_2$ *as above, the closed curve* $\Psi(t) = \Phi_1(t) - \Phi_2(t)$, $0 \leqq t \leqq 3$, *has winding number* $-1$ *about the origin.*

*Proof.* It is clear that $\Psi[0, 3]$ is a closed curve with $\Psi(t) \neq (0, 0)$ for all $t$, so that the winding number of $\Psi$ about the origin is defined. The idea of the following proof is to deform $\Psi$ without touching the origin, into a curve which obviously has winding number $-1$ about the origin.

Let

$$\Xi(u, v) = \Phi_1(u) - \Phi_2(v) .$$

From conditions (I) (II) (III) (IV) it follows rather easily that $\Xi^{-1}(0)$ is a subset of the hatched area in Figure 2. For our purpose it is enough to prove that the origin is never in the range of $\Xi(u, v)$ restricted to the dotted region in Figure 2, i.e. the region bounded by the segment $AG$ and the piecewise linear curve $\Sigma = ABCDEFG$ where $A = (0, 0)$, $B = (0, \rho)$, $C = (2, \rho)$, $D = (2, 1)$, $E = (5/2, 1)$, $F = (5/2, 3)$, $G = (3, 3)$. In details: for $0 \leqq u \leqq 1$, $0 \leqq v \leqq \rho$, $\Phi_1(u)$ is a point of the segment $MH$ wnile $\Phi_2(v)$ is on the segment $LO$, hence $\Phi_1(u) \neq \Phi_2(v)$; for $0 < u \leqq 2$, $\rho \leqq v \leqq 1$, $\Phi_1(u)$ is on one of the segmehts $MH$,

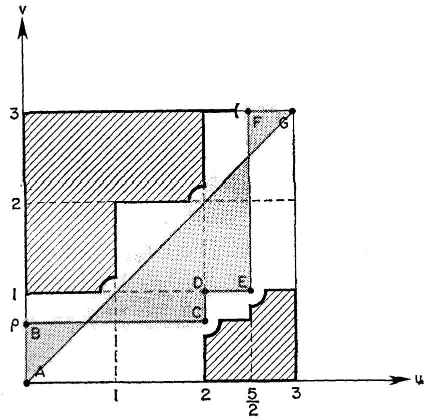


FIGURE 2

FIGURE 2

$HK$ but not at $M$ (since $u > 0$) while $\Phi_2(v)$ is on the segment $OM$, hence again $\Phi_1(u) \neq \Phi_2(v)$; for $1 \leqq u \leqq 5/2$, $1 \leqq v \leqq 2$, $\Phi_1(u)$ is on one of the segments $HK$, $KO$ while $\Phi_2(v)$ is a point of the curve $MN$ which by condition (III) cannot intersect $HK$, $KM$ except at $M$, but $M$ is not on $HK$, $KO$, hence again $\Phi_1(u) \neq \Phi_2(v)$; for $2 \leqq u \leqq 3$, $2 \leqq v < 3$ the same conclusion follows from condition (IV); finally for $5/2 \leqq u \leqq 3$, $v = 3$, $\Phi_1(u)$ is on the segment $OM$ while $\Phi_2(v) = L$, hence $\Phi_1(u) \neq \Phi_2(v)$ everywhere in the dotted region.

The diagonal $\Delta(t) = (t, t)$, $0 \leqq t \leqq 3$, is obviously homotopic to $\Sigma = ABCDEFG$ staying within the dotted region in which, as just proved, $\Xi(u, v)$ is never the origin. Hence

$$\Psi(t) = \Phi_1(t) - \Phi_2(t) = (\Xi \circ \Delta)(t)$$

is homotopic to $(\Xi \circ \Sigma)(t)$ never hitting the origin, and therefore the winding numbers about the origin are the same.

It only remains to check that $\Xi \circ \Sigma$ has winding number $-1$ about the origin, which follows just by adding the winding numbers of $\Xi \circ AB$, $\Xi \circ BC$, $\Xi \circ CD$, $\Xi \circ DE$, $\Xi \circ EF$, $\Xi \circ FG$ which turn out to be respectively $0$, $-1/2$ (because of condition (II)), $0$, $0$, $-1/2$ (because of condition (II)), $0$.

LEMMA 2. *Let $\Phi_1$ and $\Phi_2$ be defined as above. Let $T$ be a homeomorphism defined on $\Phi_1$ as well as in its interior and such that $T(\Phi_1(t)) = \Phi_2(t)$, $0 \leqq t \leqq 3$. Then $T$ has a fixed point which is contained in the intersection of the interiors of $\Phi_1$ and $\Phi_2$.*

*Proof.* Assume $T$ has no fixed point in the intersection of the interiors of the simple closed curves $\Phi_1$, $\Phi_2$. Then

$$H_s(t) = s\Phi_1(t) - T(s\Phi_1(t))$$

is a homotopy from the constant $-T(0)$ map to $\Phi_1(t) - \Phi_2(t)$ which never hits the origin. This contradicts Lemma 1.

THEOREM. *Let $T$ be an orientation preserving homeomorphism defined in a subset of the plane and interchanging two points $P$, $Q$. Let $\Gamma$ be a simple curve joining $P$ to $Q$, and $\Omega$ a simply connected set contained in the domain of $T$ as well as in its range and such that $\Gamma \subset \Omega$, $T(\Gamma) \subset \Omega$, $T^{-1}(\Gamma) \subset \Omega$. Then $T$ has a fixed point in $\Omega$.*

*Proof.* We show that the situation of Lemma 2 must occur. We may assume that the plane is coordinatized so that: $P = (1, 0)$, $Q = (0, 0)$, and $\Gamma$ is the segment $PQ$. We may also assume (replacing $P$, $Q$ with another pair of points if necessary) that $T$ interchanges no pair of points of $\Gamma$ between $P$ and $Q$. Let $\phi$ be the parametrization of $\Gamma$ given by $\phi(t) = (1 - t, 0)$, $0 \leqq t \leqq 1$. Define:

$$t_1 = \inf\{t\colon \phi[0, t] \cap T\phi[0, t] \neq \varnothing \quad \text{or} \quad \phi[0, t] \cap T^{-1}\phi[0, t] \neq \varnothing\}$$

(the set is nonempty since it contains the number 1). It is clear that either $\phi(t_1) \in T\phi[0, t_1]$ or $\phi(t_1) \in T^{-1}\phi[0, t_1]$. First, we will assume only one of these events occurs. Later, we will consider the case when both inclusions are valid.

By replacing $T$ by $T^{-1}$ if necessary, we may assume the second of the two inclusions, that is $\phi(t_1) \in T^{-1}\phi[0, t_1]$, or equivalently $T\phi(t_1) \in \phi[0, t_1]$. Let $M = \phi(t_1)$, $H = T^{-1}(M)$, $L = T(M)$, and define $t_0 \leqq t_1$ to be the scalar such that $\phi(t_0) = L$. We may assume $t_0 < t_1$, otherwise $\phi(t_1)$ is a fixed point. Define $t_{-1} = \sup\{t \leqq t_0\colon T^{-1}\phi(t) \in T\phi[0, t_1]\}$, and let $N = T^{-1}\phi(t_{-1})$. Finally, choose $t_*$ so that $T\phi(t_*) = N$ and let $K = \phi(t_*)$. The situation is summarized in Figure 3. Now the three paths

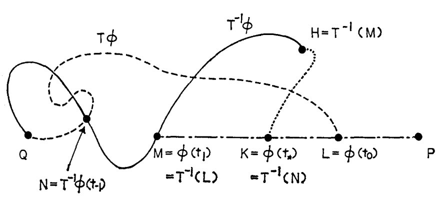


FIGURE 3

$$\begin{array}{llll} \phi(t) & t_0 \leqq t \leqq t_1 & \text{from } L & \text{to } M \\ T^{-1}\phi(t_0 - t) & 0 \leqq t \leqq t_0 - t_{-1} & \text{from } M & \text{to } N \end{array}$$

and

$$\begin{array}{llll} T\phi(t) & t_* \leqq t \leqq t_0 & \text{from } N & \text{to } L \end{array}$$

form a simple closed curve which turns out to be contained in $\Omega$ (because it is composed by portions of $\Gamma$, $T(\Gamma)$, $T^{-1}(\Gamma)$) together with its interior (because $\Omega$ is simply connected). Let $\Phi_2\colon [0, 3] \to \mathscr{R}^2$ be a parametrization of this curve so that $\Phi_2[0, 1]$ is the path $LM$, $\Phi_2[1, 2]$ is the path $MN$, and $\Phi_2[2, 3]$ is the path $NL$. Define $\Phi_1(t) = T^{-1}\Phi_2(t)$ for $0 \leqq t \leqq 3$; notice that $\Phi_1[0, 1] = T^{-1}\phi[t_0, t_1]$ is a path from $M$ to $H$, $\Phi_1[1, 2]$ is a path (not necessarily in $\Omega$) from $H$ to $K$, and $\Phi_1[2, 3] = \phi[t_*, t_1]$ is a path from $K$ to $M$. Since $T$ is a homeomorphism, $\Phi_1$ is also a simple closed curve, and by applyidg the Schoenflies Theorem and introducing a new coordinate system on the plane (which may have the opposite orientation of the old one) we may assume that $\Phi_1$ is identical to the triangle defined before Lemma 1 and that $\Phi_2$ satisfies condition (I) for an appropriate choice of $\lambda > 0$. It only remains to show that $\Phi_2$ satisfies conditions (II), (III), and (IV). Since $T\Phi_1 = \Phi_2$ and $T$ preserves orientation, condition (II) is immediate. Condition (IV) follows from the choice of $t_1$. It is easily seen that the set $C_{MN} = \Phi_2(1, 2]$, which is the path from $M$ to $N$, is disjoint from $\Phi_1[2, 3]$, so to verify condition (III) it suffices to show $C_{MN}$ is disjoint from $C_{HK} = \Phi_1(1, 2] = \mathrm{T}^{-1}C_{MN}$, which is the path from $H$ to $K$. To do this, observe that the path $\phi[t_{-1}, t_1]$ followed by the path $C_{MN}$ is a simple curve, hence its image under $T^{-1}$, $C_{NMHK}$ is also free of self-intersections. But the sets $C_{MN}$ and $C_{HK}$ are disjoint portions of the set $C_{NMHK}$.

We return now to the case that both $\phi(t_1) \in T\phi[0, t_1]$ and $\phi(t_1) \in T^{-1}\phi[0, t_1]$. In this case we refer to Figure 4. Let $M = \phi(t_1)$ and define $t_0$ and $t_*$ so that $M = T^{-1}\phi(t_0) = T\phi(t_*)$. We have $t_* \neq t_0$, since equality would violate the assumption made in the third sentence of this proof. Replacing $T$ by $T^{-1}$ if necessary, we may assume $t_* < t_0$. Let $H = \phi(t_*)$, $K = \phi(t_0)$ and $N = T(K)$. The two paths

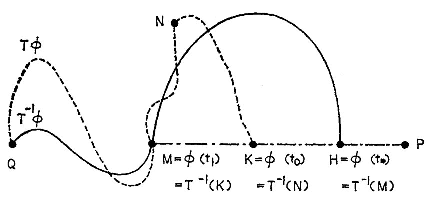


FIGURE 4

$$\phi(t) \quad t_0 \leqq t \leqq t_1 \quad \text{from} \quad K \quad \text{to} \quad M$$

and

$$T\phi(t) \quad t_* \leqq t \leqq t_1 \quad \text{from} \quad M \quad \text{through} \quad N \quad \text{to} \quad K$$

form a simple closed curve; let $\Phi_2\colon [0, 3] \to \mathscr{R}^2$ be a parametrization of this curve so that $\Phi_2[0, 1]$ is the path $KM$, $\Phi_2[1, 2]$ is the path

$MN$, and $\Phi_2[2, 3]$ is the path $NK$. Define $\Phi_1(t) = T^{-1}\Phi_2(t)$, $0 \leqq t \leqq 3$. The remainder of the proof in this case is analogous to the proof of the first case.

**2. Remarks and examples.** The proof of the theorem would become trivial if the hypotheses guaranteed the existence of a closed curve from $P$ through $Q$ to $P$ which is transformed into itself. But this is not always true, for if $T_1\colon \mathscr{R}^2 \to \mathscr{R}^2$ is defined by $T_1(x, y) = (-x, -y + \sin x)$ then $T_1$ is an orientation-preserving homeomorphism on $\mathscr{R}^2$ which interchanges the points $P = (\pi, 0)$ and $Q = (-\pi, 0)$; however, *there exists no bounded connected set containing $P$ and $Q$ which is transformed into itself.* It is interesting to note that this implies that $T_1$ is *not* an element of any flow on $\mathscr{R}^2$.

To see that the orientation-preserving hypothesis is necessary, consider the homeomorphism $T_2\colon \mathscr{R}^2 \to \mathscr{R}^2$ defined by

$$T_2(x, y) = \begin{cases} (-x, y) & \text{if } |x| \geqq \dfrac{\pi}{2} \\ (-x, y + \cos x) & \text{if } |x| < \dfrac{\pi}{2} \end{cases}$$

$T_2$ interchanges every pair of points $\{(x, y), (-x, y)\}$ for which $|x| \geqq \pi/2$, but has no fixed point.

We conclude with a simple corollary to the Theorem (with $\Omega = \mathscr{R}^2$). Our notation follows that of [3].

COROLLARY. *Let $T\colon S^2 \to S^2$ be a homeomorphism such that $T$ is of Brouwer degree* 1, *and $T$ interchanges two points. Then $T$ has two fixed points.*

*Proof.* Standard results in algebraic topology (see [3] page 124, Exercise 3) show that $T$ has at least one fixed point, say $U$. Now $S^2 \sim U$ is homeomorphic to $\mathscr{R}^2$; let $h\colon S^2 \sim U \to \mathscr{R}^2$ be a homeomorphism. Then $h \circ T \circ h^{-1}$ is a homeomorphism of $\mathscr{R}^2$ which interchanges two points, and Theorem 34, page 122 of [3] shows that it also preserves orientation. Thus $h \circ T \circ h^{-1}$ has a fixed point, say $W$, and $V = h^{-1}(W)$ is another fixed point of $T$.

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