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**FIXED POINTS FOR ORIENTATION PRESERVING
HOMEOMORPHISMS OF THE PLANE WHICH INTERCHANGE
TWO POINTS**

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FIXED POINTS FOR ORIENTATION PRESERVING HOMEOMORPHISMS OF THE PLANE WHICH INTERCHANGE TWO POINTS

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Let T be an orientation preserving homeomorphism defined on a subset of the plane which interchanges two points, P and Q . Let Γ be a simple curve joining P and Q and let Ω be a simply connected set contained in the domain and range of T such that $\Gamma \subset \Omega$, $T(\Gamma) \subset \Omega$, $T^{-1}(\Gamma) \subset \Omega$. Then T has a fixed point in Ω . A corollary concerning fixed points of homeomorphisms on S^2 follows.

The proof would be trivial if T were necessarily an element of a flow on the plane, however an example given in this paper shows that this need not be the case.

If, in particular, T is defined in the whole plane or if its domain and range are the same halfplane, then the existence (but no constructive information about the location) of a fixed point could also be derived from classical results of Brouwer [2] (see for instance Proposition 0($a = b$) and Proposition 1.1 of S. A. Andrea [1]).

1. The theorem. We use \mathcal{R} to denote the real numbers and \mathcal{R}^2 for the coordinate plane. A curve is a continuous function whose domain is a compact interval of \mathcal{R} and whose range is a subset of \mathcal{R}^2 . If $[a, b]$ is an interval within the domain of the curve Φ , we use $\Phi[a, b]$ as a shorthand for $\{\Phi(t): t \in [a, b]\}$; $\Phi(a, b)$ and $\Phi[a, b)$ have analogous meanings. The terms close curve, simple curve, and simple closed curve have the standard meanings.

For the following lemmas we fix two simple closed curves, Φ_1 and $\Phi_2: [0, 3] \rightarrow \mathcal{R}^2$. The first, Φ_1 , is the triangle defined by

$$\Phi_1(t) = \begin{cases} (2t - 1, 2t) & \text{for } 0 \leq t \leq 1 \\ (1, 4 - 2t) & \text{for } 1 \leq t \leq 2 \\ (5 - 2t, 0) & \text{for } 2 \leq t \leq 3 \end{cases}$$

Referring to Figure 1, $\Phi_1[0, 1]$ is the segment MH , $\Phi_1[1, 2]$ is the segment HK , and $\Phi_1[2, 3]$ is the segment KM .

The second curve, Φ_2 , is defined so that the following conditions are satisfied:

(I) $\Phi_2[0, 1]$ is the segment from $L = (\lambda, 0)$ (with $\lambda > 0$) to $M = (-1, 0)$; one has therefore: $\Phi_2(\rho) = 0 = (0, 0)$ for a suitable ρ with $0 < \rho < 1$.

(II) $\Phi_2[0, 3]$ has winding number -1 about each of its interior points (just as Φ_1 has); and therefore $\Phi_2[1, 3]$ has winding number $-1/2$ about the origin (just as $\Phi_1[0, 2]$ has).

(III) $\Phi_2(1, 2]$ is disjoint from $\Phi_1[1, 3]$. In Figure 1, $\Phi_2[1, 2]$ is represented by the curve MN , which except for M is disjoint from HK and KM .

(IV) $\Phi_2[2, 3]$ is disjoint from $\Phi_1[2, 3]$. In Figure 1, $\Phi_2[2, 3]$ is represented by the curve NL , which except perhaps for L , is disjoint from KM .

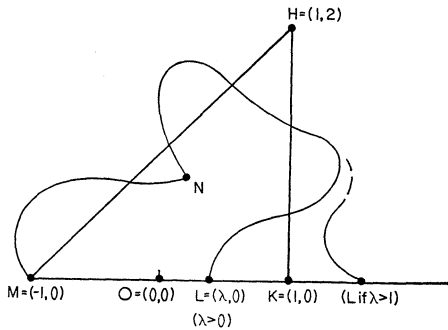


FIGURE 1

LEMMA 1. With Φ_1 and Φ_2 as above, the closed curve $\Psi(t) = \Phi_1(t) - \Phi_2(t)$, $0 \leq t \leq 3$, has winding number -1 about the origin.

Proof. It is clear that $\Psi[0, 3]$ is a closed curve with $\Psi(t) \neq (0, 0)$ for all t , so that the winding number of Ψ about the origin is defined. The idea of the following proof is to deform Ψ without touching the origin, into a curve which obviously has winding number -1 about the origin.

Let

$$\Xi(u, v) = \Phi_1(u) - \Phi_2(v).$$

From conditions (I) (II) (III) (IV) it follows rather easily that $\Xi^{-1}(0)$ is a subset of the hatched area in Figure 2. For our purpose it is enough to prove that the origin is never in the range of $\Xi(u, v)$ restricted to the dotted region in Figure 2, i.e. the region bounded by the segment AG and the piecewise linear curve $\Sigma = ABCDEFG$ where $A = (0, 0)$, $B = (0, \rho)$, $C = (2, \rho)$, $D = (2, 1)$, $E = (5/2, 1)$, $F = (5/2, 3)$, $G = (3, 3)$. In details: for $0 \leq u \leq 1$, $0 \leq v \leq \rho$, $\Phi_1(u)$ is a point of the segment MH while $\Phi_2(v)$ is on the segment LO , hence $\Phi_1(u) \neq \Phi_2(v)$; for $0 < u \leq 2$, $\rho \leq v \leq 1$, $\Phi_1(u)$ is on one of the segments MH ,

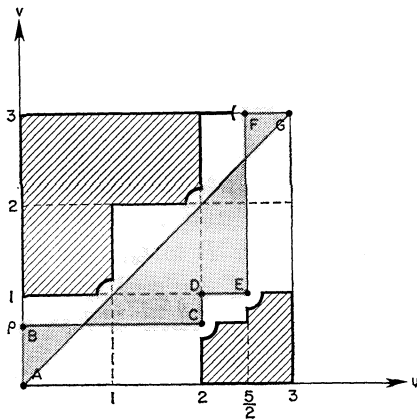


FIGURE 2

FIGURE 2

HK but not at M (since $u > 0$) while $\Phi_2(v)$ is on the segment OM , hence again $\Phi_1(u) \neq \Phi_2(v)$; for $1 \leq u \leq 5/2, 1 \leq v \leq 2$, $\Phi_1(u)$ is on one of the segments HK, KO while $\Phi_2(v)$ is a point of the curve MN which by condition (III) cannot intersect HK, KM except at M , but M is not on HK, KO , hence again $\Phi_1(u) \neq \Phi_2(v)$; for $2 \leq u \leq 3, 2 \leq v < 3$ the same conclusion follows from condition (IV); finally for $5/2 \leq u \leq 3, v = 3$, $\Phi_1(u)$ is on the segment OM while $\Phi_2(v) = L$, hence $\Phi_1(u) \neq \Phi_2(v)$ everywhere in the dotted region.

The diagonal $\Delta(t) = (t, t), 0 \leq t \leq 3$, is obviously homotopic to $\Sigma = ABCDEFG$ staying within the dotted region in which, as just proved, $\Xi(u, v)$ is never the origin. Hence

$$\Psi(t) = \Phi_1(t) - \Phi_2(t) = (\Xi \circ \Delta)(t)$$

is homotopic to $(\Xi \circ \Sigma)(t)$ never hitting the origin, and therefore the winding numbers about the origin are the same.

It only remains to check that $\Xi \circ \Sigma$ has winding number -1 about the origin, which follows just by adding the winding numbers of $\Xi \circ AB, \Xi \circ BC, \Xi \circ CD, \Xi \circ DE, \Xi \circ EF, \Xi \circ FG$ which turn out to be respectively $0, -1/2$ (because of condition (II)), $0, 0, -1/2$ (because of condition (II)), 0 .

LEMMA 2. *Let Φ_1 and Φ_2 be defined as above. Let T be a homeomorphism defined on Φ_1 as well as in its interior and such that $T(\Phi_1(t)) = \Phi_2(t), 0 \leq t \leq 3$. Then T has a fixed point which is contained in the intersection of the interiors of Φ_1 and Φ_2 .*

Proof. Assume T has no fixed point in the intersection of the interiors of the simple closed curves Φ_1, Φ_2 . Then

$$H_s(t) = s\Phi_1(t) - T(s\Phi_1(t))$$

is a homotopy from the constant $-T(0)$ map to $\Phi_1(t) - \Phi_2(t)$ which never hits the origin. This contradicts Lemma 1.

THEOREM. *Let T be an orientation preserving homeomorphism defined in a subset of the plane and interchanging two points P, Q . Let Γ be a simple curve joining P to Q , and Ω a simply connected set contained in the domain of T as well as in its range and such that $\Gamma \subset \Omega, T(\Gamma) \subset \Omega, T^{-1}(\Gamma) \subset \Omega$. Then T has a fixed point in Ω .*

Proof. We show that the situation of Lemma 2 must occur. We may assume that the plane is coordinatized so that: $P = (1, 0)$, $Q = (0, 0)$, and Γ is the segment PQ . We may also assume (replacing P, Q with another pair of points if necessary) that T interchanges no pair of points of Γ between P and Q . Let ϕ be the parametrization of Γ given by $\phi(t) = (1 - t, 0), 0 \leq t \leq 1$. Define:

$$t_1 = \inf \{t: \phi[0, t] \cap T\phi[0, t] \neq \emptyset \text{ or } \phi[0, t] \cap T^{-1}\phi[0, t] \neq \emptyset\}$$

(the set is nonempty since it contains the number 1). It is clear that either $\phi(t_1) \in T\phi[0, t_1]$ or $\phi(t_1) \in T^{-1}\phi[0, t_1]$. First, we will assume only one of these events occurs. Later, we will consider the case when both inclusions are valid.

By replacing T by T^{-1} if necessary, we may assume the second of the two inclusions, that is $\phi(t_1) \in T^{-1}\phi[0, t_1]$, or equivalently $T\phi(t_1) \in \phi[0, t_1]$. Let $M = \phi(t_1), H = T^{-1}(M), L = T(M)$, and define $t_0 \leq t_1$ to be the scalar such that $\phi(t_0) = L$. We may assume $t_0 < t_1$, otherwise $\phi(t_1)$ is a fixed point. Define $t_{-1} = \sup \{t \leq t_0: T^{-1}\phi(t) \in T\phi[0, t_1]\}$, and let $N = T^{-1}\phi(t_{-1})$. Finally, choose t_* so that $T\phi(t_*) = N$ and let $K = \phi(t_*)$. The situation is summarized in Figure 3. Now the three paths

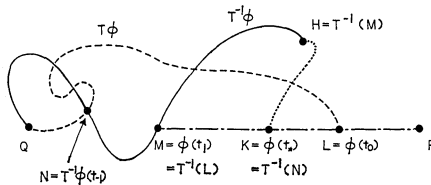


FIGURE 3

$\phi(t)$	$t_0 \leq t \leq t_1$	from L to M
$T^{-1}\phi(t_0 - t)$	$0 \leq t \leq t_0 - t_{-1}$	from M to N

and

$T\phi(t)$	$t_* \leq t \leq t_0$	from N to L
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form a simple closed curve which turns out to be contained in Ω (because it is composed by portions of Γ , $T(\Gamma)$, $T^{-1}(\Gamma)$) together with its interior (because Ω is simply connected). Let $\Phi_2: [0, 3] \rightarrow \mathcal{R}^2$ be a parametrization of this curve so that $\Phi_2[0, 1]$ is the path LM , $\Phi_2[1, 2]$ is the path MN , and $\Phi_2[2, 3]$ is the path NL . Define $\Phi_1(t) = T^{-1}\Phi_2(t)$ for $0 \leq t \leq 3$; notice that $\Phi_1[0, 1] = T^{-1}\phi[t_0, t_1]$ is a path from M to H , $\Phi_1[1, 2]$ is a path (not necessarily in Ω) from H to K , and $\Phi_1[2, 3] = \phi[t_*, t_1]$ is a path from K to M . Since T is a homeomorphism, Φ_1 is also a simple closed curve, and by applying the Schoenflies Theorem and introducing a new coordinate system on the plane (which may have the opposite orientation of the old one) we may assume that Φ_1 is identical to the triangle defined before Lemma 1 and that Φ_2 satisfies condition (I) for an appropriate choice of $\lambda > 0$. It only remains to show that Φ_2 satisfies conditions (II), (III), and (IV). Since $T\Phi_1 = \Phi_2$ and T preserves orientation, condition (II) is immediate. Condition (IV) follows from the choice of t_1 . It is easily seen that the set $C_{MN} = \Phi_2(1, 2]$, which is the path from M to N , is disjoint from $\Phi_1[2, 3]$, so to verify condition (III) it suffices to show C_{MN} is disjoint from $C_{HK} = \Phi_1(1, 2] = T^{-1}C_{MN}$, which is the path from H to K . To do this, observe that the path $\phi[t_{-1}, t_1]$ followed by the path C_{MN} is a simple curve, hence its image under T^{-1} , C_{NMHK} is also free of self-intersections. But the sets C_{MN} and C_{HK} are disjoint portions of the set C_{NMHK} .

We return now to the case that both $\phi(t_1) \in T\phi[0, t_1]$ and $\phi(t_1) \in T^{-1}\phi[0, t_1]$. In this case we refer to Figure 4. Let $M = \phi(t_1)$ and define t_0 and t_* so that $M = T^{-1}\phi(t_0) = T\phi(t_*)$. We have $t_* \neq t_0$, since equality would violate the assumption made in the third sentence of this proof. Replacing T by T^{-1} if necessary, we may assume $t_* < t_0$. Let $H = \phi(t_*)$, $K = \phi(t_0)$ and $N = T(K)$. The two paths

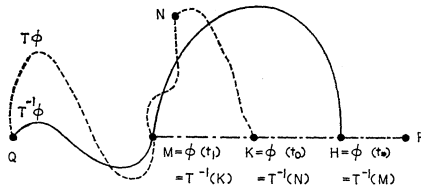


FIGURE 4

$\phi(t)$ $t_0 \leq t \leq t_1$ from K to M

and

$T\phi(t)$ $t_* \leq t \leq t_1$ from M through N to K

form a simple closed curve; let $\Phi_2: [0, 3] \rightarrow \mathcal{R}^2$ be a parametrization of this curve so that $\Phi_2[0, 1]$ is the path KM , $\Phi_2[1, 2]$ is the path

MN , and $\Phi_2[2, 3]$ is the path NK . Define $\Phi_1(t) = T^{-1}\Phi_2(t)$, $0 \leq t \leq 3$. The remainder of the proof in this case is analogous to the proof of the first case.

2. **Remarks and examples.** The proof of the theorem would become trivial if the hypotheses guaranteed the existence of a closed curve from P through Q to P which is transformed into itself. But this is not always true, for if $T_1: \mathcal{R}^2 \rightarrow \mathcal{R}^2$ is defined by $T_1(x, y) = (-x, -y + \sin x)$ then T_1 is an orientation-preserving homeomorphism on \mathcal{R}^2 which interchanges the points $P = (\pi, 0)$ and $Q = (-\pi, 0)$; however, *there exists no bounded connected set containing P and Q which is transformed into itself*. It is interesting to note that this implies that T_1 is *not* an element of any flow on \mathcal{R}^2 .

To see that the orientation-preserving hypothesis is necessary, consider the homeomorphism $T_2: \mathcal{R}^2 \rightarrow \mathcal{R}^2$ defined by

$$T_2(x, y) = \begin{cases} (-x, y) & \text{if } |x| \geq \frac{\pi}{2} \\ (-x, y + \cos x) & \text{if } |x| < \frac{\pi}{2} \end{cases}$$

T_2 interchanges every pair of points $\{(x, y), (-x, y)\}$ for which $|x| \geq \pi/2$, but has no fixed point.

We conclude with a simple corollary to the Theorem (with $\Omega = \mathcal{R}^2$). Our notation follows that of [3].

COROLLARY. *Let $T: S^2 \rightarrow S^2$ be a homeomorphism such that T is of Brouwer degree 1, and T interchanges two points. Then T has two fixed points.*

Proof. Standard results in algebraic topology (see [3] page 124, Exercise 3) show that T has at least one fixed point, say U . Now $S^2 \sim U$ is homeomorphic to \mathcal{R}^2 ; let $h: S^2 \sim U \rightarrow \mathcal{R}^2$ be a homeomorphism. Then $h \circ T \circ h^{-1}$ is a homeomorphism of \mathcal{R}^2 which interchanges two points, and Theorem 34, page 122 of [3] shows that it also preserves orientation. Thus $h \circ T \circ h^{-1}$ has a fixed point, say W , and $V = h^{-1}(W)$ is another fixed point of T .

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