Pacific Journal of Mathematics

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Vol. 59, No. 1 May 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 59, No. 1, 1975

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An analog of the Dirichlet-Jordan theorem and a uniqueness theorem are established for dual trigonometric series equations when the right hand sides of the dual equations are given functions of bounded variation. In the usual fashion there are two series in these equations one of which has coefficients, say, $\{\mathfrak{j}/_n n\}$ or $\{\mathfrak{j}_n/n - 1/2\}$, and the other coefficients $\{\mathfrak{j}_n\}$. In the first series we establish ordinary convergence and in the second Abel-Poisson convergence. In general $\mathfrak{j}_n \neq o(1)$ and the second series does not converge in the ordinary sense on any set of positive measure. A best possible estimate on growth conditions for $\{\mathfrak{j}_n\}$ needed for uniqueness is given. In the proof a mixed boundary value problem of potential theory is associated with the dual series. Conformal mapping replaces this potential problem with one in which a Dirichlet boundary condition can be associated with the dual series. Analysis of this new problem provides the denouement.

1.0. *Problem statement.* Perhaps the most important theorem in the application of Fourier series is the celebrated result of Dirichlet and Jordan [4, p. 114; 26, p. 57] describing the behavior of trigonometric expansions of functions of bounded variation. I establish here an analog of this theorem and the uniqueness of the expansions for the four dual trigonometric equations given below which are the canonical forms to which all classical dual trigonometric series can be reduced by elementary transformations [19, p. 150]. Let $\mathfrak{p}$ denote the interval $0 < x < \pi$ and b be a fixed point in $\mathfrak{p}$. Let $\mathfrak{l}$ denote the interval $0 < x < b$ and $\mathfrak{r}$ the interval $b < x < \pi$. Let $f(x)$ and $g(x)$ be functions of bounded variation for which we use the notation $\hat{f}(x) = (f(x+0) + f(x-0))/2$. The dual trigonometric series to be studied are:

$$\text{(1A)}\qquad \lim_{r=1-0}\sum_{n=1}^{\infty}\left(\frac{\mathfrak{j}_n}{n-1/2}\cos(n-1/2)x\right)r^n = \hat{f}(x),\ x \in \mathfrak{l}\,,$$

$$\text{(1B)}\qquad \lim_{r=1-0}\sum_{n=1}^{\infty}(\mathfrak{j}_n\cos(n-1/2)x)r^n = \hat{g}(x),\ x \in \mathfrak{r}\,;$$

$$\text{(2A)}\qquad \lim_{r=1-0}\sum_{n=1}^{\infty}\left(\frac{\mathfrak{j}_n}{n-1/2}\sin(n-1/2)x\right)r^n = \hat{f}(x),\ x \in \mathfrak{l}\,,$$

$$\text{(2B)} \qquad \lim_{r=1-0} \sum_{n=1}^{\infty} (j_n \sin (n - 1/2)x) r^n = \hat{g}(x),\ x \in \mathrm{r}\ ;$$

$$\text{(3A)} \qquad \lim_{r=1-0} \sum_{n=1}^{\infty} \left(\frac{j_n}{n} \sin nx\right) r^n = \hat{f}(x),\ x \in \mathrm{I}\ ,$$

$$\text{(3B)} \qquad \lim_{r=1-0} \sum_{n=1}^{\infty} (j_n \sin nx) r^n = \hat{g}(x),\ x \in \mathrm{r};$$

$$\text{(4A)} \qquad j_1 + \lim_{r=1-0} \sum_{n=1}^{\infty} \left(\frac{j_{n+1}}{n} \cos nx\right) r^n = \hat{f}(x),\ x \in \mathrm{I}\ ,$$

$$\text{(4B)} \qquad \lim_{r=1-0} \sum_{n=1}^{\infty} (j_{n+1} \cos nx) r^n = \hat{g}(x),\ x \in \mathrm{r}\ .$$

1.1. *Background.* These dual series have primarily been examined in connection with applications, especially in mechanical engineering as explained in [19] (see [1; 2; 9; 12; 20; 16] and references [6; 8] in [10] for more recent applications). Understandably this has led to the development of formal answers and special methods with little information on the limitations needed to insure their validity, cf., [19; 4; 7; 10; 24] and references [1; 2; 9–17] in [10]. This paper was motivated by the desire to present a unified approach to these equations and to answer basic mathematical questions of existence, uniqueness, and behavior raised earlier. The need for this was made more urgent by Srivastav's interesting formal construction [21] showing eq. (4A–B) does not have a unique solution and the discussion that has occurred for some solutions e.g., reference [1] in [10] and *Math. Rev.* 37 (1969), #5632.

Shepherd [18] established rigorously the existence of a solution to (3A–B) with $b = \pi/2$, $f = \sin mx$, and $g = -m \sin mx$. His procedure requires an explicit solution of an infinite vector equation $Ap = q$ where p and q are infinite column vectors and $A_{mn} = (m + n - 1/2)^{-1}$ (more recent applications of this method are found in [8; 6]). By extending the method along lines suggested in [18], one can obtain a special case of the results given in Theorem 3 for eq. (3A–B), but further generalizations appear difficult because of the explicit quantities involved in the proof and the fact that the inverse of A is not unique [14; 22]. In [23] a general formulation of both dual integral equations and dual series was given, but the results are formal and applications to specific series have not been forthcoming. In [11] we established an existence and uniqueness theory for dual orthogonal equations in Hilbert space applicable to dual series associated with potential problems with mixed boundary conditions of the second and third kind, cf. [5], but as pointed out in [11] this technique breaks down for the series studied here in which one of the mixed conditions is a

Dirichlet condition.

1.2. *Outline and comments.* The idea of the proof is this. We associate with each dual trigonometric series a mixed boundary value problem in a semi-infinite strip which we conformally map onto a rectangle to obtain a separated variable solution, since in this form we can use the full armamenatarium of the theory of Fourier series to directly analyse the solution. The rectangle is mapped back onto the original domain in such a way that the dual series can now, in a certain sense, be associated with a Dirichlet condition—an association which is the denouement. To make the required connections between solutions requires uniqueness theorems which we are able to present in a simple manner by means of Wolf's reflection principle [25].

It might occur to the reader that in §2.0 it would be easier to map the above mentioned strip onto a half-plane and obtain a Keldysh-Sedova problem. However, this would require more restrictive continuity assumptions on f and g [13, p. 347] and, more importantly, would not permit use of the theory of Fourier series in so direct a fashion as can be done with a separated variable solution in a rectangle.

The gist of the paper can be obtained by reading Theorems 1–3 whose statements involve little technical detail. Roughly, our most important result is that each of the eqs. (iA–B) has one, and only one, solution such that

$$\sum_{n=1}^{N} |j_n| = o(N^{3/2}) \tag{5}$$

and for establishing uniqueness this estimate is best possible. As shown in Theorem 3 a general theory of dual series cannot limit itself, as has been the case previously, to ordinary summation, since in general the series in (iB) diverge almost everywhere in the ordinary sense even for very smooth functions f and g, e.g., $f \equiv 1$ and $g \equiv 0$ in eq. (2A–B). There is merit, I believe, in pointing this out.

1.3. *Notation.* Standard notation for Fourier series is used [4; 26]. $\bar{R}$ will denote the closure of R. Integrals and measure are in the sense of Lebesgue. In the set forming symbol, say $\{z: A(z)\}$, we often suppress the bound variable when the meaning is clear and write $\{A(z)\}$. Abel-Poisson summability will be called summability A. We denote by $\mathfrak{l}_0$ the set of points $\{(x, 0): x \in \mathfrak{l}\}$ with $\mathfrak{r}_0$ and $\mathfrak{p}_0$ defined similarly. Let $f(x, y)$ be defined in a set R and along some open arc p in the boundary of R. We say f is continuous at $(x_0, y_0) \in p$ (adding 'relative to R' if needed for clarity) if for each sequence $\{(x_n, y_n)\}$ in $R \cup P$ for which $(x_n, y_n) \rightarrow (x_0, y_0)$, one has $f(x_n, y_n) \to f(x_0, y_0)$.

2.0. *A Dirichlet-Jordan theorem.* In our proof we make use of the following theorem which paraphrases results in [25].

THEOREM W. *Let D be a domain in the upper half plane which contains an open segment $\mathfrak{c}$ of the real axis on its boundary. Each point of $\mathfrak{c}$ contains a circular neighborhood whose upper half lies in D. Let $\phi(x, y)$ be a function harmonic in D, $\phi \to 0$ as $y \to +0$ for $x \in \mathfrak{c}$, and $\phi = o(y^{-2})$ as $y \to +0$ uniformly in x for $x \in \mathfrak{c}$. Then ϕ can be analytically continued into the domain D^* symmetric to D with respect to the real axis. If ψ is conjugate to ϕ and $\psi = o(y^{-2})$ as $y \to +0$ uniformly in x for $x \in \mathfrak{c}$, then $F = \psi + i\phi$ can be analytically continued into D^*.*

REMARK 1. This is based on Theorems C and D in [25] in which there is allowed an exceptional subset T of $\mathfrak{c}$ on which ϕ need not tend to zero as $y \to +0$. If T is not the empty set, it is easy to see that these theorems are in need of modification, vid., *Math. Rev.* 9(1948), p. 420. Our use is limited to the case in which T is the empty set.

We proceed to our main result.

THEOREM 1. *Let f and g be functions of bounded variation on $\bar{\mathfrak{l}}$ and $\bar{\mathfrak{r}}$ respectively. Then there is one, and only one, solution $(j_1, j_2, \cdots)$ satisfying* (5) *to each of the dual trigonometric equations* (1A–B), $i = 1, 2, 3, 4$.

Proof. It is sufficient to give the proof for eq. (1A–B), since the proof for the other three dual equations is practically identical. Let R_z be the half strip $\{x \in \mathfrak{p}; y > 0\}$ and S_z the subset of $\bar{R}_z$ in which $y > 0$. We associate with (1A–B) the boundary value problem P_z: find a function $T(x, y)$ harmonic in S_z, bounded in $\bar{R}_z$, and satisfying the boundary conditions

$$T_x = 0 \text{ on } \{x = 0; y > 0\} \text{ and } T = 0 \text{ on } \{x = \pi; y > 0\}\,, \tag{6}$$

$$\lim T = \hat{f}(x) \text{ as } y \to +0 \text{ for } x \in \mathfrak{l}\,, \tag{7}$$

$$\lim T_y = \hat{g}(x) \text{ as } y \to +0 \text{ for } x \in \mathfrak{r}\,. \tag{8}$$

REMARK 2. A boundary condition written in the form of (6) implies T_x is continuous relative to R_z at each point $(0, y)$, $y > 0$, whereas (7) only implies for each $x \in \mathfrak{l}$ that T is continuous on the right as function of y at $y = 0$.

In the $w(= u + iv)$-plane let λ be the interval $0 < u < 1$ and ρ the interval $0 < v < \kappa$ where κ is a positive constant. Let λ_0 be the set $\{(u, 0): u \in \lambda\}$, ρ_0 the set $\{(1, v): v \in \rho\}$, and R_w the rectangle $\{u \in \lambda; v \in \rho\}$. It is well known [13, p. 202] that there exists a function, say $w(z)$, conformally mapping R_z onto R_w with the correspondences $0 \to 0$, $b \to 1$, $\pi \to 1 + i\kappa$, and $\infty \to i\kappa$ for a proper choice of κ. For future use we note the following properties of w[13, Ch. II §§1 1.29, 3.35, and 3.37]:

(A) w is continuous on $\bar{\mathfrak{r}}_0$ and continuously differentiable on $\mathfrak{r}_0 \cup (0, \pi)$;

(B) in the neighborhood of $z = b$, $w = (z - b)^{1/2}t(z) + 1$ where $t(z)$ is a function analytic at $z = b$ and $t(b) \neq 0$;

(C) the inverse function $z(w)$ is twice continuously differentiable on $\bar{\rho}_0$.

We define $\phi(u)$ on $\bar{\lambda}$ and $\gamma(v)$ on $\bar{\rho}$ by $\phi = f(x(u, 0))$ and $\gamma = g(x(1, v))$. Since $x(u, 0)$ is bounded and increasing on $\bar{\lambda}$, ϕ is a function of bounded variation on $\bar{\lambda}$. Similarly, γ is a function of bounded variation on $\bar{\rho}$.

Let S_w be the subset of $\bar{R}_w$ in which $u < 1$ and $v > 0$. We define a new potential problem P_w: find a function $\tau(u, v)$ harmonic in S_w, bounded in $\bar{R}_w$, and satisfying the boundary conditions

$$(9) \qquad \tau_u = 0 \text{ on } \{u = 0; v \in \rho\} \text{ and } \tau = 0 \text{ on } \{u \in \lambda; v = \kappa\} ,$$

$$(10) \qquad \lim \tau = \hat{\phi}(u) \text{ as } v \longrightarrow +0 \text{ for } u \in \lambda ,$$

$$(11) \qquad \lim \tau_u = -\hat{\gamma}(v)|z'(w)| \text{ as } u \longrightarrow 1 - 0 \text{ for } v \in \rho .$$

Since z' is continuous on $\bar{\rho}_0$ as a function of v (Property C), it is an obvious verification to show T is a solution to P_z if, and only if, τ defined by $\tau(u, v) = T(x, y)$ is a solution to P_w.

We shall show that a solution to P_w is

$$(12) \qquad \tau(u, v) = \sum_{n=1}^{\infty} \frac{g_n}{n} \sin\left(\frac{n\pi v}{\kappa}\right) \cosh\left(\frac{n\pi u}{\kappa}\right) \operatorname{csch}\left(\frac{n\pi}{\kappa}\right) + \frac{f_0}{2}\left(\frac{\kappa - v}{\kappa}\right)$$
$$+ \sum_{n=1}^{\infty} f_n \cos(n\pi u) \sinh(n\pi(\kappa - v)) \operatorname{csch}(n\pi\kappa) ,$$

$$(13) \qquad g_n = \frac{-2}{\pi} \int_0^{\kappa} \gamma(v) \left[\left|\frac{dz}{dw}\right|\right]_{u=1} \sin\left(\frac{n\pi v}{\kappa}\right) dv , \quad n = 1, 2, \cdots ,$$

$$(14) \qquad f_n = 2 \int_0^1 \phi(u) \cos(n\pi u) du , \qquad n = 0, 1, \cdots .$$

Since ϕ and $\gamma|z'|$ are functions of bounded variation, $f_n = O(n^{-1})$ and $g_n = O(n^{-1})$ [26, p. 48]. Thus τ is harmonic in S_w and satisfies (9). By the Abel-Poisson sum theorem [26, p. 97], τ also satisfies the boundary conditions (10) and (11). It remains to show τ is bounded. The first series in (12) is dominated by a series with constant terms $O(n^{-2})$ so that it is bounded on $\bar{R}_w$. If we set $\phi(-0) = \phi(+0)$ and $\phi(1+0) = \phi(1-0)$, the second series in (12) tends to the bounded function $\hat{\phi}(u)$ as $v \longrightarrow +0$ for $u \in \bar{\lambda}$. By the positiveness of summation A [26, p. 98], this series is bounded on $\bar{R}_w$. Thus τ is a solution to P_w.

We introduce $\eta(v) = \tau(1-0, v)$ for $v \in \bar{\rho}$. For future reference note that $\eta(v(x, 0))$ is in $L^2(\mathfrak{r})$ because $\eta(v)$ is a bounded continuous function on ρ and $v(x, 0)$ is continuous on $\bar{\mathfrak{r}}$. We introduce a third boundary value problem Q_w: find a function $\nu(u, v)$ harmonic in S_w, bounded in $\bar{R}_w$, and satisfying the boundary conditions (9), (10), and $\nu \longrightarrow \eta(v)$ as $u \longrightarrow 1-0$ for $v \in \rho$. Clearly τ is a solution to Q_w. Let us show that it is the only solution. If Q_w had more than one solution, there would exist a solution, say θ, satisfying Q_w with $\phi \equiv 0$ and $\eta \equiv 0$. Theorem W implies θ is continuous along λ_0 and ρ_0. Applying the maximum principle [17, p. 105] it follows that $\theta \equiv 0$ which shows Q_w has at most one solution.

REMARK 3. The uniqueness of the solution to Q_w is the key to the proof of Theorem 1, since from this uniqueness we obtain existence.

We proceed to our last boundary value problem Q_z: find a function $U(x, y)$ harmonic in S_z, bounded in $\bar{R}_z$, satisfying (6), (7) and $U \longrightarrow \eta(v(x, 0))$ as $y \longrightarrow +0$ for $x \in \mathfrak{r}$. Let $h(x)$ be defined by

$$(15) \qquad h(x) = f(x),\ x \in \bar{\mathfrak{l}},\ \text{and}\ h(x) = \eta v(x, 0)),\ b < x \leqq \pi\ .$$

Then Q_z has a solution

$$(16) \qquad U(x, y) = \sum_{n=1}^{\infty} [j_n/(n - 1/2)] \exp(-(n - 1/2)y) \cos(n - 1/2)x$$

$$(17) \qquad \frac{j_n}{n - 1/2} = \frac{2}{\pi}\int_0^{\pi} h(x) \cos(n - 1/2)x\, dx\ .$$

Since $h \in L^2(\mathfrak{p})$, it follows after a bit of algebra that $\{j_n\}$ satisfies (5). Clearly, U is a solution to Q_z if, and only if, ν defined by $\nu(u, v) = U(x, y)$ is a solution to Q_w. Since $\tau \equiv \nu$, U is a solution to P_z from which follows the validity of (1A–B).

We now examine uniqueness. If a second solution existed, there would be constants $(k_1, k_2, \cdots)$ satisfying (5) such that the function

$$W(x, y) = \sum_{n=1}^{\infty} [(k_n/(n - 1/2)] \cos(n - 1/2)x \exp(-(n - 1/2)y)$$

satisfies P_z, except perhaps for boundedness on $\bar{R}_z$, with f and g set equal to zero. To establish uniqueness it suffices (as will be seen) to show W is continuous along $\bar{\mathfrak{p}}_0$. To this purpose we introduce the function X conjugate to W,

$$X(x, y) = C - \sum_1^\infty [k_n/(n - 1/2)] \sin (n - 1/2)x \exp (-(n - 1/2)y) ,$$

where C is a constant. We set $F(z) = X + iW$ and $G = F_y$ so that

$$G(z) = -i \sum_1^\infty k_n \exp (i(n - 1/2)z) .$$

Let us now show that $G = o(y^{-3/2})$ as $y \to +0$ uniformly in x for $x \in \bar{\mathfrak{p}}$. Indeed for $x \in \bar{\mathfrak{p}}$ and $y \leqq 2$, one finds $|G|$ is bounded by $e \sum |k_n| r^n$ where $r = e^{-y}$. Now by virtue of (5) there is a sequence of nonnegative constants $\{\delta_n\}$ such that $\delta_n = o(1)$ and

$$\frac{\sum_1^\infty |k_n| r^n}{1 - r} = \sum_{n=1}^\infty \left[\sum_{m=1}^n |k_m| \right] r^n \leqq \sum_{n=1}^\infty \delta_n n^{3/2} r^n = o([1 - r]^{-5/2}) .$$

This establishes the required growth for G. Since $W_y \to 0$ as $y \to +0$ for $x \in \mathfrak{r}$, it follows from Theorem W that G is analytic on $\mathfrak{r}_0$ so that X is constant on $\mathfrak{r}_0$. Choose C such that $X = 0$ on $\mathfrak{r}_0$. Let $H = i(z - b)^{1/2}F$, and write $H = H_1 + iH_2$. Clearly, H is analytic in R_z and $H_2(x, y) \to 0$ as $y \to +0$ for $x \in \mathfrak{l} \cup \mathfrak{r}$. By modifying the argument used above for G it follows from (5) that $F = o(y^{-1/2})$ as $y \to +0$ uniformly in x for $x \in \bar{\mathfrak{p}}$. This implies: $H_2(b, y) \to 0$ as $y \to +0$; $H = o(y^{-1/2})$ uniformly in x for $x \in \bar{\mathfrak{p}}$. Whence by Theorem W, H is analytic on $\mathfrak{p}_0$. Therefore W is continuous on $\mathfrak{p}_0$. Since W can be reflected through $\{x = 0; y > 0\}$ and $\{x = \pi; y > 0\}$ similar, but easier, arguments show W is continuous at $z = 0$ and $z = \pi$. Applying the maximum principle [17, p. 75] we obtain $W \equiv 0$. Since $\{\cos (n - 1/2)x\}$ is a complete orthogonal set, $k_n = 0$ for $n = 1, 2, \cdots$.

3.0. *Behavior of the expansions.* We describe certain characteristics of the solutions and the ways in which our results can be considered best possible. Let us denote by $S_i(x)$ the series in eq. (iA), e.g.,

$$S_1(x) = \sum_1^\infty [j_n/(n - 1/2)] \cos (n - 1/2)x .$$

THEOREM 2. *Let us assume the hypothesis of Theorem* 1. *Then* $S_i(x)$, $i = 1, 2, 3, 4$, *converges to an absolutely continuous function on* $\bar{\mathfrak{r}}$, *has two sided continuity at* $x = b$, *in particular,*

$$S_i(b) = f(b - 0) ; \tag{18}$$

the coefficients $(j_1, j_2, \cdots)$ *satisfy*

$$j_n = O(1) ; \tag{19}$$

summation A in eq. (iA) *can be replaced by ordinary summation.*

Proof. From Properties A and B of the mapping function, we see that $v(x, 0)$ is absolutely continuous on $\bar{\mathfrak{r}}$. Now let us show that $\eta(v)$ is absolutely continuous on $\bar{\rho}$. First we consider the series

$$I(v) = \frac{\pi}{\kappa} \sum_{n=1}^{\infty} g_n \coth\left(\frac{n\pi}{\kappa}\right) \cos\left(\frac{n\pi v}{\kappa}\right), \ v \in \bar{\rho} .$$

Since $g_n = O(n^{-1})$, it follows that $I \in L^2(\rho)$ and the series can be integrated term by term [26, p. 59]. The resulting function, $J = \int_0^v I dv$, is absolutely continuous and $J(0) = 0$. Next we consider the series

$$F(v) = f_0[(\kappa - v)/2\kappa] + \sum_{n=1}^{\infty}(-1)^n f_n \sinh(n\pi(\kappa - v)) \operatorname{csch}(n\pi\kappa) ,$$

Clearly, $\eta = F + J$. Since $f_n = O(n^{-1})$, F is an infinitely differentiable function on $\delta \leqq v \leqq \kappa$ for $\delta > 0$. After a little algebra one can write

$$F(v) = \frac{f_0}{2} + \sum_{n=1}^{\infty} (-1)^n f_n e^{-n\pi v} - \frac{f_0 v}{2\kappa} + \sum_{n=1}^{\infty} f_n p_n(v)$$

where $p_n^{(m)} = O(e^{-n})$, $m = 0, 1$, uniformly in v for $v \in \bar{\rho}$, and further $p_n(0) = 0$. Therefore utilizing the fact that $\cos n\pi u$ is symmetric about $u = 1$ and the Abel-Poisson summation theorem [26, p. 97] one concludes that $F(+0) = f(b - 0)$ and F is continuous on $\bar{\rho}$. Let $\Phi(u)$ denote the even extension of ϕ with period 2 so that

$$f_0/2 + \sum_{n=1}^{\infty} f_n \cos(n\pi u) = \hat{\Phi}(u) , \quad -\infty < u < \infty .$$

At the point $u = 1$, $\hat{\Phi}$ has a symmetrical derivative equal to zero so that Fatou's theorem [4, p. 160] implies

$$\lim \sum_{n=1}^{\infty} ((-1)^n f_n n) r^n = 0 \quad \text{as} \quad r \longrightarrow 1 - 0 .$$

Thus $F'(v)$ is continuous on $\bar{\rho}$. Consequently, $\eta(v)$ is absolutely continuous on $\bar{\rho}$. Since $v(x, 0)$ is increasing and absolutely continuous on $\bar{\mathfrak{r}}$, $\eta(v(x, 0))$ is absolutely continuous on $\bar{\mathfrak{r}}$ as a function of x [15, p. 195]. From the computations above, h has two sided continuity at $x = b$, so that by the Dirichlet-Jordan theorem (18) is valid as well as the convergence of S_1 on $\bar{\mathfrak{r}}$. Since h is of bounded variation on $\bar{\mathfrak{p}}$, (19) follows from (17). Finally, by Littlewood's Tauberian theorem [26, p. 81] we can replace summation A in (1A) by ordinary summation.

We turn our attention to examining the extent to which the results in Theorems 1 and 2 can be regarded as best possible.

Under the hypothesis of Theorem 1, we know that h is of bounded variation on $\bar{\mathfrak{p}}$. Therefore, if $j_n = o(1)$, it follows from (17) that h is continuous on $\mathfrak{p}$ [26, p. 60], which is false in general. If $j_n \neq o(1)$, then by the Cantor-Lebesgue theorem [4, p. 174] the series in (iB) do not converge in the ordinary sense on any set of positive measure.

Let us consider eq. (2A–B) with $b = 1$, $f \equiv 1$, and $g \equiv 0$. For this case an easy computation shows

$$j_n = \frac{2}{\pi} + \frac{2}{\pi}\int_1^\pi h'(x)\cos(n - 1/2)x\,dx\ . \tag{20}$$

Since h is absolutely continuous on $\bar{\mathfrak{r}}$, it follows that $h' \in L(\mathfrak{r})$ [15, p. 268]. Therefore, by the Riemann-Lebesgue theorem [26, p. 45] the integral in (20) tends to zero as $n \longrightarrow \infty$. Hence, the estimate (19) cannot be improved.

Finally, we show that (5) is a best possible estimate for uniqueness, i.e., if (5) is replaced by

$$\sum_1^N |j_n| = O(N^{3/2}) \tag{21}$$

uniqueness fails. Counterexamples can be constructed for each of the four dual equations. The simplest example we have found is for eq. (2A–B). Consider the function

$$T(x, y) = \left\{\frac{(\cos^2 x + \sinh^2 y)^{1/2} - \cos x \cosh y}{2(\cos^2 x + \sinh^2 y)}\right\}^{1/2},\ (x, y) \in R_z\ ,$$

where the square roots are taken nonnegative. It is easy to verify that T is harmonic in S_z and satisfies the boundary conditions $T = 0$ on $\{x = 0; y > 0\}$, $T_x = 0$ on $\{x = \pi; y > 0\}$,

$$\lim T = 0 \text{ for } x \in \mathfrak{l} \text{ and } \lim T_y = 0 \text{ for } x \in \mathfrak{r} \text{ as } y \longrightarrow +0$$

with $b = \pi/2$. It is also clear that

$$\lim T = (-\cos x)^{-1/2} \text{ as } y \longrightarrow +0 \text{ for } x \in \mathfrak{r}\ .$$

After verifying in the fashion of §2 that an appropriate uniqueness theorem holds, it follows that T has the representation

$$T(x, y) = \sum_1^\infty [j_n/(n - 1/2)] \sin(n - 1/2)x \exp(-(n - 1/2)y)$$

$$\frac{j_n}{n - 1/2} = \frac{2}{\pi}\int_{\pi/2}^\pi \frac{\sin(n - 1/2)x}{(-\cos x)^{1/2}}dx\ .$$

If we set $t = x - \pi/2$ and use the addition formula for sines, we find that $j_n/(n - 1/2)$ is the sum of four integrals with constant multipliers whose absolute values are independent of n. One of these integrals is

$$I = \int_0^{\pi/2} \frac{\sin nt \cos(t/2)}{(\sin t)^{1/2}} dt .$$

Now

$$\frac{1}{\sin t} = \frac{1 + t^2 G(t)}{t} \text{ and } \cos \frac{t}{2} = 1 + L(t),\ 0 \leqq t \leqq \frac{\pi}{2}$$

where G and L are analytic. Thus

$$I = \int_0^{\pi/2} \frac{\sin nt}{\sqrt{t}} dt + O(n^{-1}) .$$

Further, if we set $\alpha_n = \sqrt{(n\pi/2)}$, then

$$\int_0^{\pi/2} \frac{\sin nt}{\sqrt{t}} dt = \frac{2}{\sqrt{n}} \int_0^{\alpha_n} \sin \xi^2 d\xi = O(n^{-1/2})$$

because of well known properties of Fresnel integrals. The three remaining integrals in the sum for $j_n/(n - 1/2)$ can similarly be shown to be $O(n^{-1/2})$. Thus $\{j_n\}$ satisfies (21) and is a nonzero solution to eq. (2A–B) with $f \equiv 0$ and $g \equiv 0$. In summary we have established

THEOREM 3. *Theorems* 1 *and* 2 *are sharp in the following sense. Theorem* 1 *is false if summation A in* (iB) *is replaced for any i, $i = 1, 2, 3, 4$, by ordinary summation on any subset of* $\mathfrak{r}$ *of positive measure. The growth estimate* (5) *is best possible for if* (5) *is replaced by* (21) *the uniqueness assertion in Theorem* 1 *is false. The estimate* (19) *is best possible for if* (19) *is replaced by $j_n = o(1)$, then Theorem* 2 *is false.*

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Received August 12, 1973 and in revised form May 6, 1974. It is noted with appreciation that this paper was written while the author was a guest of the University of California at Berkeley. Professor Henry Helson is thanked for his admonitions which eliminated needless and tedious calculations. I am grateful to Professor Frantisek Wolf for his insightful suggestions, in particular, pointing out the relevance of the reflection principle in [25] which allowed the proofs to be shortened and the conclusions to be sharpened. This work was supported in part by the U. S. Army Research Office under grant No. DAH CO4 74G0140.

COLORADO STATE UNIVERSITY
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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 59, No. 1 May, 1975