STIELTJES DIFFERENTIAL-BOUNDARY OPERATORS III,
MULTIVALUED OPERATORS-LINEAR RELATIONS

ALLAN M. KRALL

This article deals with a multivalued differential-boundary operator on a nondense domain regarding it as a linear relation. The adjoint relation is derived. It is shown that these dual relations have the same form as exhibited in earlier papers where the operators involved were uniquely defined on dense domains. Self-adjoint relations are considered on the Hilbert space $L_2^*[0, 1]$. The connection with self-adjoint operators defined on subspaces of $L_2^*[0, 1]$ is made.

I. Introduction. This article is a continuation of [8] and [9]. The notation is the same. We review it briefly. $X$ is the Banach space $L^p[0, 1]$, $1 \leq p < \infty$, consisting of all $n$-dimensional vectors

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

under the norm

$$\|y\| = \left( \int_0^1 \left( \sum_{i=1}^n |y_i|^p \right)^{p/2} dt \right)^{1/p}.$$  

$X^*$ is the dual space $L^q[0, 1]$, $1/p + 1/q = 1$.

$A$ and $B$ are $m \times n$ matrices, $m \leq 2n$, satisfying rank $(A: B) = m$. $C$ and $D$ are $(2n - m) \times n$ matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is nonsingular. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}$ is given by $\begin{pmatrix} A^* & -C^* \\ B^* & D^* \end{pmatrix}$, where $\tilde{A}$ and $\tilde{B}$ are $m \times n$ matrices satisfying rank $(\tilde{A}: \tilde{B}) = m$, and $\tilde{C}$ and $\tilde{D}$ are $(2n - m \times n)$ matrices. Hence the large matrices above may be multiplied together in the usual component-like manner.

$K$ is a regular $m \times n$ matrix valued function of bounded variation satisfying $dK(0) = 0$, $dK(1) = 0$. $K_\xi$ is a regular $r \times n$ matrix valued function of bounded variation satisfying $dK_\xi(0) = 0$, $dK_\xi(1) = 0$.

$H$ is a regular $n \times (2m - m)$ matrix valued function of bounded variation satisfying $dH(0) = 0$, $dH(1) = 0$. $H_\xi$ is a regular $n \times s$ matrix valued function of bounded variation satisfying $dH_\xi(0) = 0$, $dH_\xi(1) = 0$. $P$ is a continuous $n \times n$ matrix.

Now let $\mathcal{D}$ denote those elements $y \in X$ satisfying

1. For each $y$ there is an $s \times 1$ matrix valued constant $\psi$ such that
is absolutely continuous.

2. \( ly = (y + H[Cy(0) + Dy(1)] + H_i \psi)' + Py \) exists a.e. and is in \( X \).

3. \( Ay(0) + \int_0^1 dK(t)y(t) + By(1) = 0, \quad \int_0^1 dK_i(t)y(t) = 0. \)

The purpose of this article is to discuss the expression \( l \) on \( \mathcal{D} \) as a linear relation \( L \), defined by its graph

\[
L = \{(y, ly) : y \in \mathcal{D} \} \subset X \times X.
\]

Note that \( ly \) may be multivalued. If \( H_i \) possesses a linear combination of columns which is absolutely continuous, then \( ly \) is unique only modulo such combinations. Note also that \( \mathcal{D} \) may not be dense in \( X \). If \( K_i \) possesses linear combinations of rows which are absolutely continuous then \( \mathcal{D} \) is orthogonal to those combinations.

In search of the adjoint of \( L \), we encounter the following problem even if \( ly \) is uniquely defined: If \( \mathcal{D} \) is dense in \( X \), \( y \in D \), and \( f \in X^* \), let \( [y, f] \) denote \( f(y) \). Then \( \mathcal{D}^* \), the domain of \( L^* \), in \( X^* \) is given by

\[
\mathcal{D}^* = \{f : [ly, f] = [y, g] \text{ for some } g \in X^* \text{ and } y \in \mathcal{D}\}.
\]

If \( l^* \) denotes the form of the adjoint, then \( l^*f = g \) is uniquely defined. For if \( l^*f = h \) as well, then \([y, g - h] = 0 \) for all \( y \in \mathcal{D} \). If \( y \to y_0 \), then \([y, g - h] \to [y_0, g - h] = 0 \), and \( g - h = 0 \) in \( X^* \).

However, if \( \mathcal{D} \) is not dense in \( X \), then \( l^*f = g \) is defined only modulo \( \mathcal{D}^\perp \) (Kelley and Namioka [7; p. 120]). If \( d^\perp \in \mathcal{D}^\perp \) and \( f \in \mathcal{D}^\perp \), then

\[
[l_y, f] = [y, g + d^\perp] = [y, g].
\]

Hence \( l^*f = g + d^\perp \) for all \( d^\perp \in \mathcal{D}^\perp \). The adjoint is not unique.

This is well borne out with the adjoint actually derived in section IV. The domain of \( L^* \), \( \mathcal{D}^* \), consists of those elements \( z \in X^* \) satisfying

1. for each \( z \) there is an \( r \times 1 \) matrix valued constant \( \phi \) such that

\[
z + K^*[\bar{A}z(0) + \bar{B}z(1)] + K^*_i \phi
\]

is absolutely continuous.
2. \( l^+ z = -(z + K^*[\tilde{A} z(0) + \tilde{B} z(1)] + K^*_i \phi)' + P^* z \) exists a.e. and is in \( X^* \).

3. \( \tilde{C} z(0) + \int_0^1 dH^*(t) z(t) + \tilde{D} z(1) = 0 \), \( \int_0^1 dH^*_i (t) z(t) = 0 \).

The relation \( L^* \) is defined by its graph

\[ L^* = \{(z, l^+ z); z \in D^* \} \subset X^* \times X^* . \]

When \( D \) is not dense in \( X \) because of the absolute continuity of a linear combination of rows of \( K_i \), then \( l^+ z \) is multivalued since \( \phi \) is not unique. Further when \( ly \) is multivalued because of the absolute continuity of linear combination of columns of \( H_i \), then \( D^* \) is orthogonal to those combinations and is not dense in \( X^* \).

Although multivaluedness and nondensity of domains cause problems when the setting is a standard Banach space such as \( X \), the setting of linear relation in \( X \times X \) handles these problems quite nicely.

Further examples illustrating this phenomenon have been presented recently by Coddington [5], [6].

II. Linear relation. (See Arens [1] or Brown [12].) A linear relation \( T \) on \( X \) is a set valued mapping with domain and range in \( X \) whose graph \( G(T) \) is a linear subspace of \( X \times X \).

If \( y \) is in the domain of \( T \), \( D(T) \), and \( Ty \) denotes the image of \( y \) under \( T \), then the graph of \( T \) in \( X \times X \) is given by

\[ G(T) = \{(y, Ty); y \in D(T)\} . \]

(It is clear that a linear operator can be identified with its graph, so that it also can be thought of as a linear relation.) It is easy to see that \( T(0) \) is a subspace of the range of \( T \), \( R(T) \); that \( x, y \in T(y) \) if and only if \( x = y \) mod \( T(0) \); that if \( y_r \in T(y) \), then \( T(y) = y_r + T(0) \); and that

\[ G(T) = \{(y, y_r + T(0)); y \in D(T), y_r \in T(y)\} . \]

The null space of \( T \), \( N(T) \), is given by \( N(T) = \{y: (y, 0) \in G(T)\} \) and is a subspace of \( X \).

\( T \) is closed if \( G(T) \) is closed. The closure of \( T \) is determined by \( \overline{G(T)} \). \( T \) is normally solvable if it is both closed and has closed range. Closure of \( T \) implies the closure of both \( N(T) \) and \( T(0) \).

The purpose of introducing linear relations is to be able to define an adjoint for \( T \). Let \([y, z] = z(y) \) for \( y \in X \), \( z \in X^* \). This can be extended to \( X \times X \) and \( X^* \times X^* \) by setting
when \((y_1, y_2) \in X \times X\) and \((z_1, z_2) \in X^* \times X^*\). Then \(T^*\) is identified with its graph

\[ G(T^*) = \{ (z_1, z_2) : (z_1, z_2) \in X^* \times X^*, [y_2, z_1] - [y_1, z_2] = 0 \} \]

for all \((y_1, y_2) \in G(T)\).

This, of course, agrees with the standard definition when \(T\) is an operator with dense domain. \(T^*\) has a number of properties similar to adjoint operators. We refer the reader to [1] or [12] for further details. We shall use these properties implicitly throughout the remainder of the article.

### III. The adjoint of \(L\).

Recall that the expression \(l^+z\) is given by

\[ l^+z = -(z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi') + P^*z. \]

We introduce in addition the expression \(l^{++}z\), given by

\[ l^{++}z = -(z + K^*\phi_1 + K_1^*\phi') + P^*z, \]

where \(\phi\) and \(\phi_1\) are appropriate vector valued constants suitably chosen so the expression within the parentheses is absolutely continuous.

**Theorem 3.1.** (A Green’s formula.) Let \(y, ly \in X\) and let \(z, l^{++}z \in X^*\). Then

\[
\int_0^1 [z^*(ly) - (l^{++}z)^*y]dt
\]

\[
= [\tilde{A}z(0) + \tilde{B}z(1)]^* \left[ Ay(0) + By(1) + \int_0^1 dKy \right]
\]

\[
+ [\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*z]^* \left[ Cy(0) + Dy(1) \right]
\]

\[
+ \phi^* \left[ \int_0^1 dK_1y \right] + \left[ \int_0^1 dH_1^*z \right]^* \phi
\]

\[
+ \left[ \phi_1 - (\tilde{A}z(0) + \tilde{B}z(1)) \right]^* \left[ \int_0^1 dKy \right].
\]

**Proof.** Note that since \(H, H_1, K, K_1\) are regular, then so are \(y\) and \(z\). Thus according to [10; Corollary 2.1] the usual integration by parts formula

\[
\int_0^1 f \cdot dg + \int_0^1 df \cdot g = f \cdot g \bigg|_0^1
\]
holds. If to the terms \( z \) and \( y \) on the left the terms \( K^*\phi + K^t\phi \) and \( H[Cy(0) + Dy(1)] + H^t\psi \) are added and elsewhere subtracted, several integration by parts results in

\[
\int_0^1 [z^*(ly) - (l^+z)^*y]dt = z^*y\big|_0^1 + \phi^*_0 \int_0^1 dKy + \phi^* \int_0^1 dK^t \gamma \\
+ \int_0^1 z^*dH[(y(0) + Dy(1)] + \int_0^1 z^*dH^t \psi.
\]

By using the formulas resulting from multiplying \((AT)\) and its inverse, the term \( z^*y\big|_0^1 \) can be written in terms of end point boundary conditions in \( z \) and \( y \). An appropriate regrouping of terms completes the proof.

We are now in a position to characterize the adjoint linear relation \( L^* \).

**Theorem 3.2.** The domain of the adjoint relation \( L^* \) is \( \mathcal{D}^* \). Further

\[ L^* = \{(z, l^+z) : z \in \mathcal{D}^*\}. \]

**Proof.** If \( z \in \mathcal{D} \), then Green’s formula shows that \((z, l^+z) \in L^*\). Hence

\[ \{(z, l^+z) : z \in \mathcal{D}^*\} \subset L^*. \]

To show the reverse inclusion, let \( y \in \mathcal{D} \cap C_0(0, 1) \), so that

\[ 0 = \int_0^1 dKy = -\int_0^1 K(t)y'(t)dt, \]
\[ 0 = \int_0^1 dK^t \gamma = -\int_0^1 K^t(t)y'(t)dt. \]

Thus \((y, y' + Py) \in L\). If \((z, l^*z) \in \text{dom } L^*\), then

\[ [y, l^*z] - [y' + Py, z] = 0, \]

or

\[ \int_0^1 [(l^*z)^*y - z^*(y' + Py)]dt = 0. \]

If the terms involving \( y \) are integrated by parts, this is equivalent to

\[ \int_0^1 \left\{ z + \int_0^t [(l^*z) - P^*z]d\xi \right\} y'dt = 0. \]
Since $y$ vanishes at 0 and 1, $y'$ is orthogonal constants. From comments above $y'$ is orthogonal to $K(t)^*$. Thus for appropriate $C, \phi, \phi_i$,

$$z + \int_0^1 [(l^+ z) - P^* z] d\xi = C - K(t)^* \phi_i - K_i(t)^* \phi .$$

Hence

$$z + K^* \phi_i + K_i^* \phi = - \int_0^1 [(l^+ z) - P^* z] d\xi + C$$

is absolutely continuous, and

$$l^* z = -(z + K^* \phi_i + K_i^* \phi)' + P^* z = l^+ z .$$

Green's formula now shows for arbitrary $y \in D$

$$0 = \left[ \tilde{C} z(0) + \tilde{D} z(1) + \int_0^1 dH^* z \right] [Cy(0) + Dy(1)]$$

$$+ \left[ \int_0^1 dH^*_i z \right] \psi$$

$$+ [\phi_i - (\tilde{A} z(0) + \tilde{B} z(1))] \left[ \int_0^1 dK y \right] .$$

$Cy(0) + Dy(1)$ varies over $C^{2n-m}$, for it not, a linear combination of its rows would vanish, putting an extra constraint on $D$. Likewise it is clear from the definition of $D$ that $\psi$ varies over $C^*$. Finally if a linear combination of rows of $K$ were constant, so then would the same linear combination of components of $\int_0^1 dK y$ be 0. Its coefficient from $\phi_i - (\tilde{A} z(0) + \tilde{B} z(1))$ would be arbitrary. But then the corresponding product within $(z + K^*[\tilde{A} z(0) + \tilde{B} z(1)] + K_i \phi)'$ would vanish. So effectively

$$\tilde{C} z(0) + \tilde{D} z(1) + \int_0^1 dH^*_i z = 0 , \quad \int_0^1 dH^*_i z = 0 ,$$

and

$$\phi_i = \tilde{A} z(0) + \tilde{B} z(1) .$$

Hence $\text{dom} L^* = D^*, l^* z = l^+ z$, and

$$L^* = \{ (z, l^+ z) : z \in D^* \} .$$

This result is identical in form with that derived in [9]. Here, however, because of greatly relaxed assumptions concerning $H, H_i, K, K_i$ linear relations prove to be a very convenient setting.

IV. Self-adjoint differential-boundary relations. In this sec-
tion we restrict our attention to the Hilbert space $X = L^2_{\nu}(0, 1)$ and characterize those linear relations which are self-adjoint. For convenience we replace $L$ and $L^*$ by $M$ and $M^*$, given by

$$M = \{(y, [1/i]y + H[Cy(0) + Dy(1)] + H^*\psi)' + Qy): y \in \mathcal{D}\}$$

and

$$M^* = \{(z, [1/i]z + K^*[Az(0) + Bz(1)] + K^*z)' + Q^*z): z \in \mathcal{D}^*\}$$

where $P = iQ$.

We say that the linear relation $M$ is self-adjoint if $M = M^*$.

Hence we find

**THEOREM 4.1.** The linear relation $M$ is self-adjoint if and only if

1. $Q = Q^*$.
2. $m = n, r = s$.
4. $AA^* = BB^*$.
5. $H[CC^* - DD^*] = 0$ a.e.
6. $Ki = E_iH^*$, where $E_i$ is a nonsingular $r \times r$ matrix.

**Proof.** It is clear that if all these conditions are satisfied, then $M = M^*$.

Conversely if $M = M^*$, then

$$[1/i][y + H[Cy(0) + Dy(1)] + H^\nu] + Qy$$

$$= [1/i][y + K^*[Ay(0) + By(1)] + K^*\phi]' + Q^*y.$$  

If $y \in \mathcal{D}$ vanishes near 0 and 1, is absolutely continuous (so $\psi$ and $\phi$ may be chosen 0), but is otherwise free to vary, then $Q^*y = Qy$ and $Q = Q^*$. From inspection $m = n$ and $r = s$. Otherwise either $\mathcal{D}$ or $\mathcal{D}^*$ would have more boundary constraints than the other. Further

$$Ay(0) + By(1) + \int_0^1 dKy = 0, \quad \int_0^1 dK^*y = 0$$

and

$$\tilde{C}y(0) + \tilde{D}y(1) + \int_0^1 dH^*y = 0, \quad \int_0^1 dH^*y = 0$$

must represent the same boundary conditions. This can only happen if $A = E\tilde{C}$, $B = E\tilde{D}$, $K = EH^*$, for some nonsingular matrix $E$ and $Ki = E_iH^*$ for some nonsingular matrix $E_i$. The equations which result from multiplying $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1} = \begin{pmatrix}
-\bar{A} & -\bar{C} \\
-\bar{B} & -\bar{D}
\end{pmatrix}
\]
show that \( E = [BD^* - AC^*] \), as well as \( AA^* = BB^* \) and \( H[CC^* - DD^*] = 0 \).

V. Self-adjoint operators on subspaces of \( \mathbb{L}_n^2[0, 1] \). Let the columns of \( H \) be suitably arranged such that the first \( s \) of them form a maximal independent absolutely continuous collection. Then \( H \) can be partitioned into \( H = (H_c : H_s) \), where \( H_c \) denotes the absolutely continuous columns, and \( H_s \) denotes those singular with respect to Lebesgue measure. In \( X \) let \( \mathcal{H}_1 \) denote the subspace spanned by the columns of \( H_c \).

Likewise, let the rows of \( K \) be suitably arranged so that the first \( r \) form a maximal independent absolutely continuous collection. Then \( K \) can be partitioned into \( K = (K_c : K_s) \), where \( K_c \) denotes the absolutely continuous rows, and \( K_s \) denotes those singular with respect to Lebesgue measure. In \( X^* \) let \( \mathcal{H}_1^* \) denote the subspace spanned by the columns of \( K_c^* \).

Now \( l \) can be rewritten as follows: Let \( l_s \) be defined by
\[
l_s y = (y + H[Cy(0) + Dy(1)] + H_s \psi_s)' + Py .
\]
Then
\[
l y = l_s y + H_c \psi_c
\]
where \( \psi = \begin{pmatrix} \psi_s \\ \psi_c \end{pmatrix} \). The boundary conditions determining \( \mathcal{D} \) can also be more accurately written as
\[
Ay(0) + By(1) + \int_0^1 dK_c y = 0 , \quad \int_0^1 dK_s y = 0 , \quad \int_0^1 K_c y dt = 0 .
\]

Similarly \( l^+ \) can be written by first defining \( l_s^* \) by
\[
l_s^* z = -(z + K^*[\bar{A}z(0) + \bar{B}z(1)] + K_s^* \phi_s)' + P^* z .
\]
Then
\[
l^+ z = l^+_s z + K_s^* \phi_s ,
\]
where \( \phi = \begin{pmatrix} \phi_s \\ \phi_c \end{pmatrix} \). The boundary conditions determining \( \mathcal{D}^* \) can be written as
\[
\bar{C}z(0) + \bar{D}z(1) + \int_0^1 dH_s^* z = 0 , \quad \int_0^1 dH_c^* z = 0 , \quad \int_0^1 H_c^* z dt = 0 .
\]

We now face a rather odd situation. \( \mathcal{D} \) is orthogonal to \( \mathcal{H}_1^* \),
while $l$ is defined on $D$ modulo $\mathcal{H}_1$. That is, for the subspace $\mathcal{H}_1^{*+1}$, is uniquely defined only when set in $\mathcal{H}_1^{*+1}/\mathcal{H}_1^*$.

Likewise $D^*$ is orthogonal to $\mathcal{H}_1$, while $l^+$ is defined on $D^*$ modulo $\mathcal{H}_1^*$. That is, for the subspace $\mathcal{H}_1^*$, $l^+$ is uniquely defined only when set in $\mathcal{H}_1^*/\mathcal{H}_1^*$.

This can be considerably simplified when $X = L^2_s[0, 1]$, $\mathcal{H}_1 = \mathcal{H}_1^*$, and the linear relation $M$ is self-adjoint. The spaces above are all reduced to $\mathcal{H}_1^1$ or its isomorphic copy $L^2_s[0, 1]/\mathcal{H}_1$. We assume without loss of generality that the columns of $H'$ are mutually orthonormal.

The restriction of $M$, denoted by $M_1$, which defines an operator from $\mathcal{H}_1^1$ to $\mathcal{H}_1^1$ is uniquely defined by

$$M_1 y = (1/x)L_0 y + (1/i)H_0\psi_0,$$

where, with $\langle \cdot, \cdot \rangle$ denoting the inner product in $L^2_s[0, 1]$,

$$\psi_0 = -\langle l_0 y, H_0' \rangle.$$

Hence

$$M_1 y = (1/i)L_0 y - (1/i)H_0'\langle l_0 y, H_0' \rangle.$$

The relationships between $M$ and $M_1$ can be best illustrated by the following diagram:

**Operator:** $M_1$  
**Space:** $\mathcal{H}_1^1$  
**Linear Space Homomorphism**  
**Isometry**  
**Linear Relation** $M^*$ ($= M$)  
**Linear Space Isomorphism** $L^2_s[0, 1]$  
**Linear Space Isomorphism** $\mathcal{H}_1^1$  
**Linear Space Homomorphism** $\mathcal{H}_1^1$  

It is readily apparent from the diagram that:

**Theorem 6.1.** $M_1$ is a self-adjoint operator on the subspace $\mathcal{H}_1^1$ if and only if $M$ is a self-adjoint linear on $L^2_s[0, 1]$.

We note that the description of $M_1$ is equivalent to that derived by Coddington [4], [5], [6] when Coddington's $n = 1$ and $H_s = 0$, $H$ is absolutely continuous. There certainly exists an extension of the present work to higher order differential-boundary relations which
will duplicate Coddington's results in full generality, although at the present time such work has not been done.

REFERENCES


Received October 16, 1974 and in revised form March 25, 1975.

THE PENNSYLVANIA STATE UNIVERSITY
Shashi Prabha Arya and M. K. Singal, More sum theorems for topological spaces

Goro Azumaya, F. Mbuntum and Kalathoor Varadarajan, On M-projective and M-injective modules

Kong Ming Chong, Spectral inequalities involving the infima and suprema of functions

Alan Hetherington Durfee, The characteristic polynomial of the monodromy

Emilio Gagliardo and Clifford Alfons Kottman, Fixed points for orientation preserving homeomorphisms of the plane which interchange two points

Raymond F. Gittings, Finite-to-one open maps of generalized metric spaces

Andrew M. W. Glass, W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, $a^*$-closures of completely distributive lattice-ordered groups

Matthew Gould, Endomorphism and automorphism structure of direct squares of universal algebras

R. E. Harrell and Les Andrew Karlovitz, On tree structures in Banach spaces

Julien O. Hennefeld, Finding a maximal subalgebra on which the two Arens products agree

William Francis Keigher, Adjunctions and comonads in differential algebra

Robert Bernard Kelman, A Dirichlet-Jordan theorem for dual trigonometric series

Allan Morton Krall, Stieltjes differential-boundary operators. III. Multivalued operators—linear relations

Hui-Hsiung Kuo, On Gross differentiation on Banach spaces

Tom Louton, A theorem on simultaneous observability

Kenneth Mandelberg, Amitsur cohomology for certain extensions of rings of algebraic integers

Coy Lewis May, Automorphisms of compact Klein surfaces with boundary

Peter A. McCoy, Generalized axisymmetric elliptic functions

Muril Lynn Robertson, Concerning Siu’s method for solving $y'(t) = F(t, y(g(t)))$

Richard Lewis Roth, On restricting irreducible characters to normal subgroups

Albert Oscar Shar, $P$-primary decomposition of maps into an $H$-space

Kenneth Barry Stolarsky, The sum of the distances to certain pointsets on the unit circle

Bert Alan Taylor, Components of zero sets of analytic functions in $C^2$ in the unit ball or polydisc

Michel Valadier, Convex integrands on Souslin locally convex spaces

Januario Varela, Fields of automorphisms and derivations of $C^*$-algebras

Arnold Lewis Villone, A class of symmetric differential operators with deficiency indices $(1, 1)$

Manfred Wollenberg, The invariance principle for wave operators