Pacific Journal of Mathematics

AUTOMORPHISMS OF COMPACT KLEIN SURFACES WITH BOUNDARY

COY LEWIS MAY

Vol. 59, No. 1 May 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 59, No. 1, 1975

AUTOMORPHISMS OF COMPACT KLEIN SURFACES WITH BOUNDARY

COY L. MAY

A Hurwitz ramification formula for morphisms of compact Klein surfaces is obtained and used to show that a compact Klein surface of genus $g \geqq 2$ with nonempty boundary cannot have more than $12(g-1)$ automorphisms.

0. Introduction. Let X be a compact Klein surface [1], that is, X is a compact surface with boundary together with an equivalence class of dianalytic atlases on X. A homeomorphism $f: X \to X$ of X onto itself that is dianalytic will be called an *automorphism* of X.

A natural task is to seek an upper bound for the order of the automorphism group of X when X is of (algebraic) genus $g \geqq 2$. The corresponding result for Riemann surfaces is well-known; Hurwitz [2] showed that a compact Riemann surface of genus $g \geqq 2$ cannot have more than $84(g-1)$ (orientation preserving) automorphisms. Using this result it is easy to show that the upper bound in the Klein surface case cannot be larger than $84(g-1)$. In fact, Singerman [6] has exhibited a Klein surface without boundary of genus 7 that has $504 = 84(7-1)$ automorphisms.

In this paper then we concentrate on Klein surfaces with boundary. We obtain a Hurwitz ramification formula for morphisms of Klein surfaces and show that a compact Klein surface with boundary of genus $g \geqq 2$ cannot have more than $12(g-1)$ automorphisms. We also show that the bound $12(g-1)$ is the best possible.

1. Let X be a Klein surface. The boundary of X will be denoted ∂X. Let $X^\circ = X \backslash \partial X$. X° will be called the *interior* of X.

Let $p \in X$. Then let $n_p = 1$ if $p \in \partial X$ is a boundary point of X, and let $n_p = 2$ if $p \in X^\circ$ is an interior point of X.

Now we recall the definition of a morphism of Klein surfaces [1, page 17]. Let $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geqq 0\}$, and let $\phi: \mathbb{C} \to \mathbb{C}^+$ be the folding map, so that $\phi(\alpha + \beta i) = \alpha + |\beta| i$.

DEFINITION. Let X, Y be Klein surfaces and $g: X \to Y$ a continuous map. Then g is a *morphism* if $g(\partial X) \subset \partial Y$ and if for every point $p \in X$ there exist dianalytic charts (U, z) and (V, w) at p and $g(p)$ respectively and an analytic function G on $z(U)$ such that the following diagram commutes:

$$\begin{array}{ccccc} U & & \xrightarrow{\quad g \quad} & & V \\ \downarrow z & & & & \downarrow w \\ \mathbb{C}^+ & \xrightarrow{G} & \mathbb{C} & \xrightarrow{\phi} & \mathbb{C}^+ \end{array}$$

Let $g: X \to Y$ be a nonconstant morphism of Klein surfaces. Let $x \in X$. We can find dianalytic charts (U, z) and (V, w) at x and $g(x)$ respectively, such that $z(x) = 0 = w(g(x))$, $g(U) \subset V$, and such that $g|_U$ has the form

$$g|_U = \begin{cases} w^{-1} \circ \phi \circ (\pm z^e) & \text{if} \quad g(x) \in \partial Y \\ w^{-1} \circ (\pm z^e) & \text{if} \quad g(x) \in Y^\circ \end{cases}$$

where e is an integer, $e \geqq 1$ [1, pages 27–30]. The integer e is called the *ramification index* of g at x and will be denoted $e_g(x)$. We say that g is *ramified at* x if $e_g(x) > 1$; otherwise we say that g is *unramified at* x. Also, the *relative degree* of x over $g(x)$, denoted $d_g(x)$, is defined by

$$d_g(x) = \frac{n_x}{n_{g(x)}} .$$

Note that $d_g(x) = 2$ if $x \in X^\circ$ and $g(x) \in \partial Y$; otherwise $d_g(x) = 1$.

DEFINITION. A nonconstant morphism $g: X \to Y$ between two Klein surfaces will be called a *ramified* r*-sheeted covering* of Y if for every point $y \in Y$,

$$\sum_{x \in g^{-1}(y)} e_g(x) \cdot d_g(x) = r .$$

In fact, every nonconstant morphism between two compact Klein sufaces is a ramified r-sheeted covering for some r [1, page 102].

Now let X, Y, and T be Klein surfaces, $g: X \to Y$ and $f: Y \to T$ be nonconstant morphisms. Then $f \circ g: X \to T$ is a nonconstant morphisms [1, page 19]. Also, if g is a ramified r-sheeted covering of Y and f is a ramified m-sheeted covering of T, then it is easily seen that $f \circ g$ is a ramified mr-sheeted covering of T.

Let X be a Klein surface. We will denote the automorphism group of X by $\mathrm{Aut}(X)$. If X is orientable, we will denote the subgroup of orientation preserving automorphisms by $\mathrm{Aut}^+(X)$.

THEOREM 1. *Let X be a compact Klein surface and let $G \subset \mathrm{Aut}(X)$ be a finite group of automorphisms of X. Then the quotient space $\Phi = X/G$ has a unique dianalytic structure such that the canonical map $\pi: X \to \Phi$ is a morphism of Klein surfaces. Moreover, if $|G| = r$,* *then π is a ramified r-sheeted covering of Φ.*

Proof. Alling and Greenleaf have shown that Φ has a unique dianalytic structure such that π is a morphism [1, pages 52–56]. Actually, in the case of a finite group action (they consider the action of a discontinuous group), their proof shows that π is a ramified r-sheeted covering of Φ.

2. Let Y be a compact Klein surface, and let E be the field of all meromorphic functions on Y. E is an algebraic function field in one variable over $\boldsymbol{R}$, and as such has an *algebraic genus* g. We will refer to this nonnegative integer g as the *genus* of the compact Klein surface Y. In case Y is a Riemann surface, g is equal to the topological genus of Y. For more details, see [1].

Henceforth the term Klein surface will be reserved for those Klein surfaces X that are not Riemann surfaces, that is, for those X that are nonorientable or have nonempty boundary or both.

Let X be a compact Klein surface. Let (X_c, π, σ) be the complex double of X, that is, X_c is a compact Riemann surface, $\pi: X_c \rightarrow X$ is an unramified 2-sheeted covering of X, and σ is the unique anti-analytic involution of X_c such that $\pi = \pi \circ \sigma$. For more details, see [1, pages 37–40]. It is well-known that the genus of X is equal to the genus of its complex double X_c. The complex double also has the following important property [1, page 39]:

PROPOSITION 1. *Let M be a compact Riemann surface, X a compact Klein surface, and $f: M \rightarrow X$ a nonconstant morphism. Then there exists a unique analytic map $\rho: M \rightarrow X_c$ such that $\pi \circ \rho = f$.*

We use the complex double to obtain a Hurwitz ramification formula for morphisms of compact Klein surfaces.

THEOREM 2. *Let X and Y be compact Klein surfaces* (that are not Riemann surfaces), *and let $f: X \rightarrow Y$ be a ramified r-sheeted covering of Y. Let g be the genus of X, γ the genus of Y. Then*

$$2g - 2 = r(2\gamma - 2) + \sum_{x \in X} n_x(e_f(x) - 1) .$$

Proof. Let (X_c, π, σ) and (Y_c, ν, τ) denote the complex doubles of X and Y respectively. By Proposition 1, there exists a unique analytic map $\tilde{f}: X_c \rightarrow Y_c$ such that the following diagram commutes:

$$\begin{array}{ccc} X_c & \xrightarrow{\tilde{f}} & Y_c \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\nu} \\ X & \xrightarrow{f} & Y \end{array}$$

$f \circ \pi = \nu \circ \tilde{f}$ is a ramified $2r$-sheeted covering of Y. But $\tilde{f}$ is a nonconstant analytic mapping between compact Riemann surfaces. Thus $\tilde{f}$ is a ramified m-sheeted covering of Y_c for some m [3, page 15]. Since ν is a 2-sheeted covering, clearly $m = r$. Then, since a Klein surface and its complex double have the same genus, the classical Hurwitz ramification formula [3, page 16] gives

$$(2g - 2) = r(2\gamma - 2) + \sum_{p \in X_c} (e_{\tilde{f}}(p) - 1) .$$

Let $p \in X_c$ and note that $e_{\tilde{f}}(p) = e_f(\pi(p))$, since $e_{\tilde{f}}(p) = e_{\nu \circ \tilde{f}}(p) = e_{f \circ \pi}(p) = e_f(\pi(p))$.

Therefore

$$\begin{aligned}(2g - 2) &= r(2\gamma - 2) + \sum_{p \in X_c} (e_f(\pi(p)) - 1) \\ &= r(2\gamma - 2) + \sum_{x \in X} n_x(e_f(x) - 1) .\end{aligned}$$

Finally, we recall how the automorphism group of a compact Klein surface can be obtained from that of its complex double [1, page 79]:

PROPOSITION 2. *Let X be a compact Klein surface with complex double (X_c, π, σ). Then*

$$\mathrm{Aut}\,(X) \cong \{g \in \mathrm{Aut}^+\,(X_c) \mid \sigma \circ g \circ \sigma = g\} .$$

COROLLARY. *If X is a compact Klein surface of genus $g \geqq 2$, then*

$$|\mathrm{Aut}\,(X)| \leqq 84(g - 1) .$$

Thus Aut (X) *is finite group.*

Proof. The genus of X_c is g, so that the corollary follows immediately from the Proposition and Hurwitz's bound for $|\mathrm{Aut}^+(X_c)|$.

3. Applications. Let X be a compact Klein surface of genus g, and let $G \subset \mathrm{Aut}\,(X)$ be a finite group of automorphisms of X of order $|G| = r$. By Theorem 1, the quotient space $\Phi = X/G$ is a compact Klein surface and the canonical map $\pi\colon X \to \Phi$ is a ramified r-sheeted covering of Φ. Let γ denote the genus of Φ.

Let $p \in \Phi$. We will call the set $\pi^{-1}(p)$ the *fiber above* p. If $g \in \mathrm{Aut}(X)$ then $g(\partial X) = \partial X$ and $g(X^\circ) = X^\circ$. Therefore either $\pi^{-1}(p) \subset \partial X$ or $\pi^{-1}(p) \subset X^\circ$. Equivalently, if $x, y \in X$ such that $\pi(x) = \pi(y)$, then $d_\pi(x) = d_\pi(y)$.

Let $S_x = \{g \in G \mid g(x) = x\}$ be the stabilizer subgroup of G of a point $x \in X$. We can find a dianalytic chart (U, z) at x such that $g(U) = U$ for all $g \in S_x$. Let $S'_x = \{g \in S_x \mid z \circ g \circ z^{-1}$ is analytic$\}$. Clearly S'_x is independent of the choice of (U, z). Either $S_x = S'_x$ or S'_x is a subgroup of index 2. $S_x = S'_x$ in case (i) $x \in X^\circ$ and $\pi(x) \in \Phi^\circ$ or (ii) $x \in \partial X$ and $e_\pi(x) = 1$; otherwise $S_x \neq S'_x$. The ramification index $e_\pi(x)$ is the order of S'_x in case $x \in X^\circ$ and $\pi(x) \in \partial\Phi$; otherwise $e_\pi(x)$ is the order of S_x. For more details, see [1, page 52–56]. If $\pi(x) = \pi(y)$, then clearly there are isomorphisms $S_x \cong S_y$ and $S'_x \cong S'_y$, so that $e_\pi(x) = e_\pi(y)$ in any case.

If π is ramified at a point $x \in X$ and $\pi(x) = p$, then we will say that π is *ramified above* p.

Now the quotient map $\pi: X \to \Phi$ is ramified above a finite number of points of Φ, say $a_1, \cdots, a_\omega$. Let k_i denote the ramification index $e_\pi(x)$ of any point x such that $\pi(x) = a_i$. We will write $n_i = n_{a_i}$.

Fix a_i. First suppose that if $\pi(x) = a_i$, then the relative degree $d_\pi(x) = 1$, i.e., $n_x = n_{a_i} = n_i$. Then there are r/k_i points in the fiber $\pi^{-1}(a_i)$, and

$$\sum_{x \in \pi^{-1}(a_i)} n_x(e_\pi(x) - 1) = \frac{r}{k_i} \cdot n_i \cdot (k_i - 1) = rn_i\left(1 - \frac{1}{k_i}\right).$$

Now suppose that if $\pi(x) = a_i$, then $d_\pi(x) = 2$, so that $n_x = 2$, $n_i = 1$. In this case there are $r/2k_i$ points in the fiber $\pi^{-1}(a_i)$, and

$$\sum_{x \in \pi^{-1}(a_i)} n_x(e_\pi(x) - 1) = \frac{r}{2k_i} \cdot 2 \cdot (k_i - 1) = rn_i\left(1 - \frac{1}{k_i}\right).$$

Therefore the Hurwitz ramification formula (Theorem 2) can be rewritten in the following form:

$$(*) \qquad \frac{2g - 2}{r} = 2\gamma - 2 + \sum_{i=1}^{\omega} n_i\left(1 - \frac{1}{k_i}\right).$$

Henceforth we assume that X is of genus $g \geqq 2$. Then, by the corollary to Proposition 2, $\mathrm{Aut}(X)$ is a finite group, so that in our calculations here we *can* let $G = \mathrm{Aut}(X)$. The calculations will be divided into several cases.

A. $\gamma \geqq 1$.

First suppose that $\gamma \geqq 2$. Then, immediately from (*), we have $(2g-2)/r \geqq 2$. Thus $r \leqq g-1$.

Now suppose $\gamma = 1$. Then $\omega \neq 0$, and

$$\frac{2g-2}{r} \geqq n_1\left(1-\frac{1}{k_1}\right) \geqq 1-\frac{1}{k_1} \geqq \frac{1}{2}.$$

Hence $r \leqq 4(g-1)$.

B. $\gamma = 0$, three lemmas.

Recall that there are two compact Klein surfaces of genus zero, the disc D and the real projective plane B. Each has a unique dianalytic structure [1, pages 59–60].

Note that with $\gamma = 0$, (*) implies that $\omega \geqq 2$.

In the following lemmas we will assume that the Klein surface X has nonempty boundary. Then the quotient space Φ has nonempty boundary, and since $\gamma = 0$, Φ is the disc D (with its unique dianalytic structure).

LEMMA 1. *Suppose $\partial X \neq \emptyset$. If π is ramified at a boundary point $x \in \partial X$, then the ramification index $e_\pi(x) = 2$.*

Proof. Let $e = e_\pi(x)$. $\pi(x) \in \partial D$, of course.

We can find dianalytic charts (U, z) and (V, w) at x and $\pi(x)$ respectively, such that $z(x) = 0 = w(\pi(x))$, $\pi(U) \subset V$, and such that

$$\pi|_U = w^{-1} \circ \phi \circ (\pm z^e)$$

$e \geqq 2$, since π is ramified at x. Suppose $e > 2$. $z(U)$ is an open subset of $\mathbb{C}^+$ about the origin. Thus for a small enough real number $t > 0$, both the points $\xi_1 = t$, $\xi_2 = t\exp(2\pi i/e)$ belong to $z(U)$. Then $z^{-1}(\xi_1) \in \partial X$ and $z^{-1}(\xi_2) \in X^\circ$, and clearly $\pi(z^{-1}(\xi_1)) = \pi(z^{-1}(\xi_2))$. But for each point $p \in D$, either $\pi^{-1}(p) \subset \partial X$ or $\pi^{-1}(p) \subset X^\circ$. Thus we have a contradiction. Therefore $e = 2$.

LEMMA 2. *Suppose $\partial X \neq \emptyset$. If π is ramified above a boundary point of D, that is, $a_k \in \partial D$ for some k, then at least two of the fibers $\pi^{-1}(a_i) \subset \partial X$. Further the number of ramified fibers contained in ∂X is even.*

Proof. Suppose $a_k \in \partial D$ for some k.

If $\pi^{-1}(a_k) \subset \partial X$, then let $x \in \partial X$ such that $\pi(x) = a_k$. $e_\pi(x) = 2$ by Lemma 1, and it is easy to see that there is an interior point $q \in X^\circ$ such that $\pi(q) \in \partial D$ (find charts as in the proof of Lemma 1 and look at $\xi = t \exp(\pi i/2)$ for small enought t). Thus regardless of whether $\pi^{-1}(a_k) \subset \partial X$ or $\pi^{-1}(a_k) \subset X^\circ$, there is an interior point $q \in X^\circ$ such that $\pi(q) \in \partial D$.

Now $\pi(\partial X)$ is a compact and hence closed subset of ∂D. Also, $\partial D \backslash \pi(\partial X) \neq \varnothing$. Topologically ∂D is just a circle, of course. Therefore $\pi(\partial X)$ is a finite union of closed intervals.

It is easy to see that if p is an end-point of one of these closed intervals, then π is ramified above p and $\pi^{-1}(p) \subset \partial X$. The number of such end-points is clearly even and not less than two.

LEMMA 3. *Suppose X is orientable and $\partial X \neq \varnothing$. If $G \subset \mathrm{Aut}^+(X)$, then π is ramified only above interior points of D.*

Proof. Let $x \in X$, and consider the stablizer subgroup S_x and its subgroup S'_x. Since $G \subset \mathrm{Aut}^+(X)$, $S_x = S'_x$, directly from the definition of S'_x. Consequently, if $x \in X^\circ$ then $\pi(x) \in D^\circ$ (π may or may not be ramified at x), and if $x \in \partial X$ then $e_\pi(x) = 1$. Hence π is ramified only above interior points of D.

C. $\gamma = 0$, ramification above Φ° only

Suppose $a_1, \cdots, a_w \in \Phi^\circ$ are interior points of Φ. Then $n_i = 2$ for each i, and by (*)

$$\frac{2g-2}{r} = -2 + 2\sum_{i=1}^{\omega}\left(1 - \frac{1}{k_i}\right)$$

or

$$\frac{g-1}{r} = \omega - 1 - \frac{1}{k_1} - \cdots - \frac{1}{k_\omega}. \tag{1}$$

Again we see that $\omega \geqq 2$.

Suppose $\omega \geqq 3$. Since $k_i \geqq 2$ for each i, by (1)

$$\frac{g-1}{r} \geqq \omega - 1 - \frac{\omega}{2} \geqq \frac{1}{2}.$$

Hence $r \leqq 2(g-1)$.

Suppose $\omega = 2$. $k_1 = k_2 = 2$ is not a possibility, since that would imply $g = 1$. Clearly then

$$\frac{g-1}{r} \geqq 2 - 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Hence $r \leqq 6(g-1)$.

These calculations have already yielded two interesting results:

THEOREM 3. *Let X be a compact Klein surface without boundary* *of genus* $g \geqq 2$. *If* G *is a group of automorphisms of* X *such that* X/G *is the real projective plane* B, *then*

$$|G| \leqq 6(g-1)\ .$$

Proof. $\partial B = \varnothing$, so the the theorem follows from calculations of §C.

THEOREM 4. *Let* X *be a compact orientable Klein surface with boundary of genus* $g \geqq 2$. *Then*

$$|\mathrm{Aut}^+(X)| \leqq 6(g-1)$$

and

$$|\mathrm{Aut}(X)| \leqq 12(g-1)\ .$$

Proof. The first fact follows from the calculations of sections A and C and Lemma 3.

Either $\mathrm{Aut}(X) = \mathrm{Aut}^+(X)$ or $\mathrm{Aut}^+(X)$ is a subgroup of $\mathrm{Aut}(X)$ of index two. Thus the first fact implies the second.

D. $\gamma = 0$, ramification above $\partial\Phi$, $\partial X \neq \varnothing$.

Now we assume that X is a Klein surface *with boundary*. Then the quotient space Φ is the disc D (with its unique dianalytic structure).

We also assume that there is ramification above ∂D. By Lemma 2, at least two of the fibers $\pi^{-1}(a_i) \subset \partial X$. We may suppose that this is the case for a_1 and a_2. Then $k_1 = k_2 = 2$ by Lemma 1. $n_1 = n_2 = 1$, of course, so by (*)

$$\frac{2g-2}{r} = -1 + \sum_{i=3}^{\omega} n_i\left(1 - \frac{1}{k_i}\right)\ . \tag{2}$$

Therefore $\omega \geqq 3$ in this case.

First suppose $\omega \geqq 5$. Then by (2)

$$\frac{2g-2}{r} \geqq -1 + (\omega - 2)\cdot\frac{1}{2} \geqq \frac{1}{2}\ .$$

Thus $r \leqq 4(g-1)$.

Next suppose $\omega = 4$. There are three cases to consider, depending on whether there are 0, 1, or 2 of the points a_3 and a_4 on the boundary of D.

If $a_3, a_4 \in D^\circ$, then

$$\frac{2g-2}{r} \geqq -1 + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = 1\ ,$$

and $r \leqq 2(g-1)$.

If one of the two points, say a_3, is a boundary point and $a_4 \in D^\circ$, then

$$\frac{2g-2}{r} \geqq -1 + \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{1}{2},$$

and $r \leqq 4(g-1)$.

If $a_3, a_4 \in \partial D$, then note that $k_3 = k_4 = 2$ is not a possibility. Clearly then

$$\frac{2g-2}{r} \geqq -1 + \frac{1}{2} + \frac{2}{3} = \frac{1}{6},$$

and $r \leqq 12(g-1)$.

Finally, suppose $\omega = 3$. Then from (2) we see that $n_3 = 2$, i.e., $a_3 \in D^\circ$. Then

$$\frac{2g-2}{r} = 1 - \frac{2}{k_3}.$$

Hence $k_3 \geqq 3$ and $r \leqq 6(g-1)$ in this case.

A review of the calculations of §§A, C, and D gives our main result:

THEOREM 5. *Suppose X is a compact Klein surface with boundary of genus $g \geqq 2$. Then*

$$|\mathrm{Aut}(X)| \leqq 12(g-1).$$

4. Sharpness of the bounds. Here we consider three compact Klein surfaces of low genus and determine their automorphism groups directly.

EXAMPLE 1. Let Y be a sphere with 3 holes, with the holes placed around the equator, centered around the vertices of an inscribed equilateral triangle. Y is an orientable Klein surface of genus 2. Y has a group (isomorphic to the dihedral group D_3) of orientation-preserving automorphisms of order 6. Reflection in the plane of the equator is an orientation-reversing automorphism. Y therefore has $12 = 12(2-1)$ automorphisms. The automorphism group is just $C_2 \times D_3$, where C_2 denotes the cyclic group of order 2.

EXAMPLE 2. Let X be a sphere with 6 holes, with the holes centered around the vertices of an inscribed regular octahedron. X is an orientable Klein surface of genus 5. X has a group of automorphisms isomorphic to the complete symmetry group (including reflections) of the regular octahedron, which is $C_2 \times S_4$. Thus X has $48 = 12(5-1)$ automorphisms.

EXAMPLE 3. Let X be the Klein surface of Example 2, and let $\tau: X \to X$ denote the antipodal map. The quotient space $W = X/\tau$ is a real projective plane with 3 holes, a nonorientable Klein surface of genus 3. By considering the action of $C_2 \times S_4$ on X, it is easy to see that there is a group of automorphisms of W isomorphic to S_4.

Thus the bounds obtained in Theorems 4 and 5 are best possible. The bound $12(g-1)$ is attained for both orientable and nonorientable surfaces. Theorem 3 was obtained incidentally in our proof of Theorem 4. We do not know if the bound of Theorem 3 is the best possible.

In a forthcoming article [5] we study those finite groups that act as a group of $12(g-1)$ automorphisms of a compact Klein surface of genus $g \geqq 2$ with nonempty boundary. There we exhibit several infinite families of values of g for which there is a compact Klein surface with boundary of genus g that has $12(g-1)$ automorphisms.

5. Nevertheless it is possible to improve the bound $12(g-1)$ for a large number of topological types of Klein surfaces. Our main tool is a theorem of Maskit.

Let X be a compact orientable Klein surface with boundary. By the *analytic genus* p of X we mean the topological genus of the compact surface X^* obtained by attaching a disc to each boundary component of X. The relationship between p and the (algebraic) genus g of X is given by

$$g = 2p + k - 1 ,$$

where k is the number of boundary components of X.

THEOREM 6. *Let X be a compact orientable Klein surface of genus g with k boundary components. If*

$$\frac{6(g-1)}{7} < k \leqq g - 3 ,$$

then

$$|\mathrm{Aut}(X)| \leqq 84(g-k-1) < 12(g-1) .$$

Proof. Let p be the analytic genus of X. Maskit has shown that there exists a compact Riemann surface X^* of genus p and an analytic embedding of X into X^* such that, under this embedding, every orientation-preserving automorphism of X is the restriction of an orientation-preserving automorphism of X^* [4, page 718]. Thus $|\mathrm{Aut}^+(X)| \leqq |\mathrm{Aut}^+(X^*)|$.

Now $2p = g - k + 1 \geqq 4$, so that $p \geqq 2$ and we may apply Hurwitz's bound for $|\mathrm{Aut}^+(X^*)|$. Hence $|\mathrm{Aut}(X)| \leqq 2 \cdot 84(p-1) = 84(g-k-1)$.

Note that $84(g-k-1) < 12(g-1)$ if and only if $6(g-1) < 7k$.

If $g < 16$, there are no integer values of k such that $6(g-1)/7 < k \leqq g-3$. The improved bound of Theorem 6 does apply to orientable Klein surfaces of genus 16 with 13 boundary components.

For large values of g and suitable values of k, Theorem 6 gives a much better bound than Theorem 5. In fact, if $(g-k)$ is held fixed (that is, the analytic genus remains constant), Theorem 6 gives a uniform bound for the size of the automorphism group. On the other hand, there are orientable Klein surfaces with boundary of each genus $g \geqq 2$ to which Theorem 6 does not apply.

Finally, we obtain a similar result for nonorientable Klein surfaces with boundary.

THEOREM 7. *Let X be a compact nonorientable Klein surface of genus g with k boundary components. If*

$$\frac{6(g-1)}{7} < k \leqq g - 2\,,$$

then

$$|\mathrm{Aut}(X)| \leqq 84(g-k-1) < 12(g-1)\,.$$

Proof. Let (X_0, ν, τ) denote the orienting double of X, that is, X_0 is a compact orientable Klein surface with $2k$ boundary components, $\nu: X_0 \to X$ is an unramified 2-sheeted covering of X, and τ is the unique antianalytic involution of X_0 such that $\nu \circ \tau = \nu$. Further the genus g' of X_0 is $g' = 2g - 1$. For more details, see [1, pages 42–43].

Suppose $f: X \to X$ is an automorphism of X. Then there exists a unique orientation-preserving automorphism $\tilde{f}$ of X_0 such that

$$\begin{array}{ccc} X_0 & \xrightarrow{\tilde{f}} & X_0 \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu} \\ X & \xrightarrow{f} & X \end{array}$$

commutes [1, page 42]. Hence $|\mathrm{Aut}(X)| \leqq |\mathrm{Aut}^+(X_0)|$.

Let p' be the analytic genus of X_0.

$$p' = \frac{(2g-1) - 2k + 1}{2} = g - k \geqq 2\,.$$

Then, using Maskit's theorem as in the proof of Theorem 6, we have that

$$|\mathrm{Aut}(X)| \leqq |\mathrm{Aut}^+(X_0)| \leqq 84(p' - 1) = 84(g - k - 1) .$$

As before, $84(g - k - 1) < 12(g - 1)$ if and only if $6(g - 1) < 7k$.

Note that the improved upper bound in Theorem 7 is the same as in Theorem 6. The bound is applicable to a larger range of values of g and k in the nonorientable case, however.

The lowest genus to which Theorem 7 applies is the case of nonorientable Klein surfaces of genus 9 with 7 boundary components.

I am indebted to Robert Speiser not only for the suggestion that an approach similar to that of Hurwitz [2] might yield results about Klein surfaces but also for several helpful discussions. I would also like to thank Newcomb Greenleaf and Richard Preston for their advice. Finally, thanks are due the referee for bringing [4] to our attention and for making several helpful suggestions.

REFERENCES

1. N. Alling and N. Greenleaf, *Foundations of the Theory of Klein Surfaces*, Berlin: Springer-Verlag, 1971.

2. A. Hurwitz, *Über algebraiche Gebilde mit eindentigen Transformationen in sich*, Math. Ann., **41** (1879), 428–471.

3. I. Kra, *Automorphic Forms and Kleinian Groups*, Reading, Mass.: W. A. Benjamin, Inc., 1972.

4. B. Maskit. *The conformal Group of a Plane Domain*, Amer. J. Math., **90** (1968), 718–722.

5. C. May, *Large Automorphism Groups of Compact Klein Surfaces with Boundary*, (to appear).

6. D. Singerman, *Automorphisms of Compact Non-orientable Riemann Surfaces*, Glasgow Math J., **12** (1971), 50–59.

Received October 30, 1974 and in revised form February 6, 1975.

UNIVERSITY OF TEXAS

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 59, No. 1 May, 1975