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**CONCERNING SIU'S METHOD FOR SOLVING $y'(t) = F(t,$
 $y(g(t)))$**

MURIL LYNN ROBERTSON

CONCERNING SIU'S METHOD FOR SOLVING

$$y'(t) = F(t, y(g(t)))$$

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A procedure is given, which is parameterized by a certain set of real-valued functions, that yields sufficient conditions on each of g and F to guarantee a solution to $y'(t) = F(t, y(g(t)))$.

The following is the main result.

THEOREM 1. *Suppose that f is a real-valued continuous function with connected domain J of real numbers so that*

- (1) $0 \in J$ and $f(0) > 0$
- (2) f is increasing on $J \cap [0, +\infty)$
- (3) f is decreasing on $J \cap (-\infty, 0]$
- (4) $0 < k < 1$, if the range of f is unbounded; and $k = 1$ if the range of f is bounded
- (5) B is a Banach space, $F: J \times B \rightarrow B$, and $N: J \rightarrow \mathbb{R}$
- (6) F is continuous and there is a constant C so that

$$\left| \int_0^x \|F(s, 0)\| ds \right| \leq C f(x), \text{ for all } x \text{ in } J.$$

- (7) $\|F(t, x) - F(t, y)\| \leq N(t)\|x - y\|$, for $t \in J$ and $x, y \in B$
- (8) N is positive and Lebesgue integrable on subintervals of J
- (9) g is any continuous function from J into J so that $g(x) \in f^{-1}[f(0), k|f'^{\text{sign}x}(x)|/N(x)]$ for all $x \in J$. ($f'^{\text{sign}x}$ denotes the right-hand derivative if $x > 0$ and it denotes the left-hand derivative if $x < 0$.)
- (10) $q \in B$.

Then, there is a unique function $y: J \rightarrow B$ so that $y'(t) = F(t, y(g(t)))$, $y(0) = q$ and $\|y(t)\| \leq \text{Constant} \cdot f(t)$, for all t in J .

LEMMA. *If f satisfies conditions (1), (2), and (3) in the statement of Theorem 1, then $\int_0^x f'^{\text{sign}x}(s) ds$ exists in the Lebesgue sense and is less than or equal to $f(x) - f(0)$, for each $x \in J$.*

Proof of Lemma. Suppose $x > 0$. Let $f_n(s) = [f(s + 1/n) - f(s)]n$. Then $f'^+(s) = \lim f_n(s)$ for almost all $s > 0$. Clearly each f_n is summable, because each f_n is continuous. Also, for each n

$$\begin{aligned} \int_0^x f_n(s) ds &= \int_0^x n \left[f\left(s + \frac{1}{n}\right) - f(s) \right] ds \\ &= n \int_{1/n}^{x+1/n} f(s) ds - n \int_0^x f(s) ds \end{aligned}$$

$$\begin{aligned}
&= n \int_x^{x+1/n} f(s) ds - n \int_0^{1/n} f(s) ds \quad (\text{which approaches} \\
&\hspace{15em} f(x) - f(0)) \\
&\leq 2 \cdot \{\text{sup of } f \text{ on } [0, x + 1]\}
\end{aligned}$$

Thus, by Fatou's lemma [see 3, page 39], f'^+ is summable on $[0, x]$ for all $x > 0$ and $x \in J$ and

$$\int_0^x f'^+(s) ds \leq \liminf \int_0^x f'_n(s) ds \leq f(x) - f(0).$$

The proof is similar for $x < 0$.

Proof of Theorem 1. Let $\|\cdot\|$ be the norm of B and define $|\cdot|$ to be the norm defined by $|z| = \sup \{\|z(x)\|/|f(x)| : x \in J\}$ for each z continuous from J into B such that this supremum exists. Let Y denote the Banach space of all such z , with norm $|\cdot|$. For each $z \in Y$ and $x \in J$, let $(Tz)(x) = g + \int_0^x F(s, z(g(s))) ds$. Suppose $z, w \in Y$. Then

$$\begin{aligned}
\|(Tz)(x) - (Tw)(x)\| &= \left\| \int_0^x [F(s, z(g(s))) - F(s, w(g(s)))] ds \right\| \\
&\leq \left| \int_0^x N(s) \|z(g(s)) - w(g(s))\| ds \right| \\
&\leq \left| \int_0^x N(s) f(g(s)) ds \right| |z - w|.
\end{aligned}$$

Thus, $|Tz - Tw| \leq \sup \left\{ \left| \int_0^x N(s) f(g(s)) ds \right| / |f(x)| \right\} |z - w|$. The following shows that T is a contraction.

Case 1. Suppose $x > 0$. Then if $0 \leq s \leq x$, $f(g(s)) \in [f(0), kf'^+(s)/N(s)]$. Thus, $\left| \int_0^x N(s) f(g(s)) ds \right| = \int_0^x N(s) f(g(s)) ds \leq \int_0^x kf'^+(s) ds \leq k(f(x) - f(0))$, by the lemma.

Case 2. Suppose $x < 0$. Then if

$$x \leq s \leq 0, f(g(s)) \in [f(0), -kf'^-(s)/N(s)].$$

Thus,

$$\left| \int_0^x N(s) f(g(s)) ds \right| = \int_x^0 N(s) f(g(s)) ds \leq sk \int_0^x f'^-(s) ds \leq k(f(x) - f(0)),$$

by the lemma.

Thus, in either case $c = \sup \left\{ \left| \int_0^x N(x) f(g(s)) ds \right| / |f(x)| \right\} \leq \sup \{k(1 - f(0)/f(x))\}$. So, if the range of f is unbounded, $c \leq k$ and if the range of f is bounded by L and $k = 1$, then $c \leq 1 - f(0)/L$. So T

has contraction constant c . The zero function Z is in Y , because $(TZ)(x) = q + \int_0^x F(s, 0)ds$ and $\|(TZ)(x)\|/f(x) \leq [\|q\| + Cf(x)]/f(x) \leq \|q\|/f(0) + C$. Now if $w \in Y$, $\|(Tw)(x)\|/f(x) \leq |Tw - TZ| + |TZ|$. Thus $Tw \in Y$. So, by Banach's contraction mapping principle, T has a unique fixed point in Y . This proves Theorem 1.

REMARKS. (1) Given any g one may find an appropriate N and apply Theorem 1, by requiring $N(x) \leq k|f'^{\text{sign}x}(x)|/f(g(x))$.

(2) At any particular x , there is an f so that $f'^{\text{sign}x}(x) = \infty$ and so that t does not have infinite derivative in a deleted neighborhood of x . For this type f , $g(x)$ could be any number.

In [5], Siu essentially uses the method of Theorem 1 with $f(x) = \exp(|x|/e)$ to obtain:

THEOREM 2 (Siu). *If $|g(x)| \leq |x| + c$, where $0 < c < 1/e$, for all real numbers x , then $y' = y(g)$, $y(0) = q$ has unique solution subject to $\|y(x)\| \leq \text{constant} \cdot \exp(|x|/e)$*

In [4], the author proves:

THEOREM 3. *If $\{I(i)\}$ is a sequence of intervals so that $I(0) = \{0\} \subseteq I(i) \subseteq I(i + 1)$, $I(i) = [a(i), b(i)]$, and $\max\{a(i - 1) - a(i), b(i) - b(i - 1)\} < 1$ for each positive integer i ; then $y' = y(g)$, $y(0) = q$ has unique solution on $\cup\{I(i)\}$, whenever g is continuous and $g(I(i)) \subseteq I(i)$ for each positive integer i .*

The following theorem is comparable to each of Theorem 2 and Theorem 3.

THEOREM 4. *Suppose the hypothesis of Theorem 3 holds and k is in $(0, 1)$ such that $\max\{a(i - 1) - a(i), b(i) - b(i - 1)\} < k$ for each positive integer i . Then, for each positive integer i , there exists $\delta(i) > 0$ such that if g is a continuous function from $\cup\{I(i)\}$ into $\cup\{I(i)\}$ such that $g(I(i)) \subseteq [a(i) - \delta(i), b(i) + \delta(i)]$, then $y'(t) = F(t, y(g(t)))$, $y(0) = q$ has a solution on $\cup\{I(i)\}$ for any F such that $N = 1$, where F and N satisfy the conditions listed for them in the hypothesis of Theorem 1.*

Proof of Theorem 4. Let f be a positive continuous piecewise linear function with domain $\cup\{I(i)\}$ such that f has slope $M(i)$ or $(b(i - 1), b(i))$ and slope $-M(i)$ on $(a(i), a(i - 1))$ where the sequence $\{M(i)\}$ is chosen such that for each nonnegative integer n ,

$M(n + 1)$

$$\begin{aligned}
 > \max \left\{ \left[f(0) + \sum_{i=1}^n M(i)(a(i - 1) - a(i)) \right] / [k - (a(n) - a(n + 1))] , \right. \\
 \left. \left[f(0) + \sum_{i=1}^n M(i)(b(i) - b(i - 1)) \right] / [k - (b(n + 1) - b(n))] \right\} .
 \end{aligned}$$

Let

$$\begin{aligned}
 \delta(n + 1) = \min \left\{ a(n + 1) - a(n + 2), b(n + 2) - b(n + 1) , \right. \\
 \left. \left[kM(n + 1) - \left[f(0) + \sum_{i=1}^{n+1} M(i)(a(i - 1) - a(i)) \right] \right] / M(n + 2) , \right. \\
 \left. \left[kM(n + 1) - \left[f(0) + \sum_{i=1}^{n+1} M(i)(b(i) - b(i - 1)) \right] \right] / M(n + 2) \right\} .
 \end{aligned}$$

It follows that the hypotheses of Theorem 1 hold.

REMARK. The solution in Theorem 4 is unique in the Banach space Y of Theorem 1, which depends on f .

The following is a straightforward application of Theorem 1.

THEOREM 5. Suppose

(1) $-1 < -k < a < 0 < b < k < 1$

(2) $m \geq \max \{1/(k + a), 1/(k - b)\}$

(3) $nk \geq \max \{(1 - ma)/(1 + mb), (1 + mb)(1 - ma)\}$

(4)
$$l(x) = \begin{cases} x - (\log nk)/n & , \quad x \leq a \\
 a - (\log km/(1 - ma))/n & , \quad a < x < b \\
 -x + a + b - (\log kn(mb + 1)/(1 - ma))/n & , \quad b \leq x \end{cases}$$

and

$$u(x) = \begin{cases} -x + a + b + (\log nk/(1 - ma)/(1 + mb))/n & , \quad x \leq a \\
 b + (\log km/(1 + mb))/n & , \quad a < x < b \\
 x + (\log nk)/n & , \quad b \leq x \end{cases}$$

(5) $l \leq g \leq u$ and g is continuous .

Then, there is a unique solution to $y'(t) = F(t, y(g(t)))$, $y(0) = g$ where F and N satisfy the hypotheses of Theorem 1, $N = 1$ and, $\|y(x)\| \leq \text{constant} \cdot f(x)$, for all real numbers x , where

$$f(x) = \begin{cases} (1 - ma) \exp(-n(x - a)) & , \quad x \leq a \\
 1 - mx & , \quad a \leq x \leq 0 \\
 1 + mx & , \quad 0 \leq x \leq b \\
 (mb + 1) \exp(n(x - b)) & , \quad b \leq x \end{cases}$$

REMARK. If $a = -b$, then $u = -l$.

The following is a generalization of a theorem by D. R. Anderson [1].

COROLLARY TO THEOREM 5. Suppose $0 < \beta < 1$, $\varepsilon > 0$, and

$$|g(x)| \leq \begin{cases} -x + \frac{1}{e} - \varepsilon & x \leq -\beta \\ \beta + \left(\log \frac{1}{\beta}\right) / e - \varepsilon & -\beta < x < \beta \\ x + \frac{1}{e} - \varepsilon & \beta \leq x. \end{cases}$$

Then, there is a solution to $y'(t) = F(t, y(g(t)))$, $y(0) = q$ for $N = 1$ any y subject to $\|y(x)\| \leq \text{constant} \cdot f(x)$ for an appropriate f .

Proof. Straightforward.

REMARK. As β approaches 0, g is allowed to become indefinitely large at 0.

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