ON RESTRICTING IRREDUCIBLE CHARACTERS TO NORMAL SUBGROUPS

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This paper is about the situation where \( \chi \) is an irreducible character of a finite group \( G \) and \( K \) is a normal subgroup. A construction of Serre's relating the characters of \( G \) with those of \( G/K \) is used to give a new proof of a well-known lemma concerning the case that \( \chi \mid_K \) is irreducible and to generalize this lemma. It is seen that the irreducibility of \( \chi \mid_K \) is equivalent to the property that \( (1/|K|) \sum_{\chi \in \chi(K)} |\chi(g)|^2 = 1 \) for each coset of \( G \) modulo \( K \) and also to the property that \( \chi \) is not a component of \( \lambda \chi \) for any irreducible character \( \lambda \) of \( G/K \) except for \( \lambda = 1 \). The subgroup \( J_1 = J_1(\chi) \) is defined as the intersection of the kernels of the irreducible characters \( \lambda \) of \( G/K \) for which \( \chi \) is a component of \( \lambda \chi \). It is seen that an irreducible component \( \sigma \) of the restriction of \( \chi \) to \( K \) will extend to \( J_1 \), \( e_{J_1}(\gamma) = e_{K}(\gamma) \) and \( J_1 \) is the maximal normal subgroup with these two properties.

Preliminary remarks. \( \hat{G} \) denotes the set of irreducible complex characters of \( G \). 1 will often be used for the one-character of the appropriate group (according to context). \( \langle \chi, \varphi \rangle_\sigma = (1/|G|) \sum_{g \in \sigma} \chi(g)\varphi(g) \), the usual inner product.

We include here a couple of well-known theorems to be referred to later.

THEOREM A. (Clifford) If \( K \triangleleft G \), \( \chi \in \hat{G} \), \( \sigma \in \hat{K} \) and \( \sigma \) a component of \( \chi \mid_K \) then \( \chi \mid_K = e_\chi(\chi) \sum_{i=1}^m \sigma_i \) where \( e_\chi(\chi) \) is a positive integer called the ramification index, \( m = [G:I(\sigma)] \) with \( I(\sigma) \) being the inertial group for \( \sigma \) and \{\( g_1, \ldots, g_m \)\} are a set of coset representatives for \( G \) modulo \( I(\sigma) \). (See for example [1, Theorem 9.10].)

THEOREM B. Let \( K \triangleleft G \) and \( \chi \) an irreducible character of \( G \) which remains irreducible when restricted to \( K \). Then the characters \( \chi \lambda \) are distinct and irreducible as \( \lambda \) varies over the characters of \( G/K \). Further if \( \theta \) is an irreducible character of \( G \) such that \( \chi \mid_K \) is a component of \( \theta \mid_K \), then \( \theta \) is of the form \( \chi \lambda \) as above. (See [3, Lemma 3.1].)

1. In this section we review a construction due to Serre which bears some resemblance to the familiar process for inducing characters from a subgroup. Theorem 1.1(b) is analogous to the Frobenius reciprocity theorem and was stated by Serre without proof in [8, p. 106].
By a class function on a group $G$ is meant any function from $G$ to the complex numbers which is constant on conjugacy classes. Let $K$ be a normal subgroup of the finite group $G$. If $\varphi$ is any class function of $G/K$, let $\varphi^*$ denote the corresponding class function on $G$ obtained in the usual way by $\varphi^*(g) = \varphi(gK)$. [Note that it is usually the custom to write $\varphi$ instead of $\varphi^*$ and this will be done in the latter part of this paper but here it is useful to make the distinction.] If $\psi$ is a class function on $G$ let $\psi_*$ denote the function on $G/K$ defined by $\psi_*(hK) = (1/|K|) \sum_{x \in hK} \psi(x)$.

**Theorem 1.1.** (Serre) (a) $\psi^*$ is a class function on $G/K$.

(b) $\langle \varphi^*, \psi \rangle_G = \langle \varphi, \psi^* \rangle_{G/K}$ where $\varphi$ is any class function on $G/K$.

**Proof.** (a) If $hK$ and $h'K$ are conjugate in $G/K$ then $h'h^{-1}Kg$ for some $g \in G$. Hence

$$
\psi_*(hK) = \frac{1}{|K|} \sum_{x \in hK} \psi(x) = \frac{1}{|K|} \sum_{y \in hKg} \psi(y) = \psi_*(hK).
$$

(b) $\langle \varphi, \psi^* \rangle_{G/K} = \frac{1}{|G/K|} \sum_{hK \in G/K} \varphi(hK) \psi_*(hK)$

$$
= \frac{|K|}{|G|} \sum_{hK \in G/K} \left[ \varphi(hK) \cdot \frac{1}{|K|} \sum_{x \in hK} \overline{\psi(x)} \right]
$$

$$
= \frac{1}{|G|} \sum_{x \in G} \varphi_*(x) \overline{\psi(x)} = \langle \varphi^*, \psi \rangle_G.
$$

The following corollary shows that the construction appears not as promising as Frobenius' induction; nevertheless it has some use as will be seen shortly.

**Corollary 1.2.** Let $\psi \in \hat{G}$.

(a) If $K \subseteq \text{Ker} \psi$ and hence $\psi$ may be regarded also as an element of $\hat{G}/K$, then $\psi_* = \psi$ (under the latter identification).

(b) If $K \nsubseteq \text{Ker} \psi$ then $\psi_* = 0$.

**Proof.** If $\varphi \in \hat{G}/K$ then $\langle \varphi, \psi_* \rangle_{G/K} = \langle \varphi^*, \psi \rangle_G = 1$ or 0 depending on whether $\psi = \varphi^*$ or not. Case (b) means that $\psi \neq \varphi^*$ for any $\varphi \in \hat{G}/K$ and since $\hat{G}/K$ forms a basis for the class functions on $G/K$ we get that $\psi_* = 0$. If (a) holds, then $\psi = \varphi^*$ for exactly one $\varphi$ and so $\psi_* = \psi$.

**Corollary 1.3.** If $\psi$ is any class function on $G$ write $\psi = $
\[ \sum a_i \chi_i + \sum b_j \psi_j \] where \( \chi_i, \psi_j \in \hat{G} \), \( K \subseteq \text{Ker} \chi_i \) each \( i \) but \( K \not\subseteq \text{Ker} \psi_j \) each \( j \). Then \( (\psi_*)^* = \sum a_i \chi_i \). Further if \( \psi \) is a character then \( (\psi_*)^* \) is also a character or the zero function.

In what follows, we omit the upper star and identify characters of \( G/K \) with characters of \( G \).

2. We now use the Serre construction to give a proof of a theorem which generalizes both [2, Lemma, p. 178] of Gallagher and [5, Lemma 4.2] of Iwahori and Matsumoto (see corollaries which follow).

**Theorem 2.1.** Let \( \chi \in \hat{G} \). Let \( S(\chi) \) denote the set of irreducible characters \( \lambda \) of \( G \) such that \( \lambda \chi \) contains \( \chi \) as a component, i.e., \( \langle \chi \chi, \chi \rangle_G = \sum n_\chi > 0 \). Then \( (\chi \chi)_* = \sum_{\lambda \in S(\chi) \cap \hat{G}/K} n_\chi \lambda \) i.e., \( (1/|K|) \sum_{\chi \in \hat{G}/K} |\chi(x)|^2 = \sum_{\chi \in \hat{G}/K} n_\chi \lambda(hK) \).

**Proof.** \( n_\chi = \langle \lambda \chi, \chi \rangle_G = \langle \chi, \lambda \chi \rangle_G \) so that \( \chi \lambda = \sum n_\chi \lambda \) summed over \( \lambda \in S(\chi) \). By Corollary 1.3, \( (\chi \chi)_* = \sum_{\lambda \in S(\chi) \cap \hat{G}/K} n_\chi \lambda \).

**Corollary 2.2.** The one-character always occurs with multiplicity one in \( (\chi \chi)_* \).

**Corollary 2.3.** (Iwahori-Matsumoto [5, Lemma 4.2]) If \( G/K \) is abelian and \( H(\chi) \) is the group of (linear) characters \( \lambda \in \hat{G}/K \) such that \( \lambda \chi = \chi \) then \( (\chi \chi)_* = \sum_{\lambda \in \hat{G}/K} n_\lambda \lambda \).

**Proof.** In this case \( S(\chi) \cap \hat{G}/K = H(\chi) \) since if \( \lambda \) is linear and \( \langle \lambda \chi, \chi \rangle_G = n_\chi > 0 \) then \( \lambda \chi = \chi \) and \( n_\chi = 1 \).

**Corollary 2.4.** (Gallagher [2, Lemma, p. 178]; also Isaacs [4, Lemma 3.4]) If \( \chi \mid_K \) irreducible then \( (\chi \chi)_* = 1 \).

It is instructive to give two different short proofs.

**Proof 1.** By Theorem B in the preliminary remarks the characters \( \{\lambda \chi : \lambda \in \hat{G}/N\} \) are all distinct and irreducible. Thus \( S(\chi) \cap \hat{G}/N = \{1\} \).

**Proof 2.** \( \chi \mid_K \) irreducible means that

\[
1 = \langle \chi, \chi \rangle_K = \frac{1}{|K|} \sum \chi(g) \overline{\chi(g)} = (\chi \chi)_*(K).
\]

Hence \( (\chi \chi)_* \) is a character (Corollary 1.3) of degree 1. By Corollary
2.2, we have \((\chi \overline{\chi})_* = 1\).

**Corollary 2.5** (the converse to Corollary 2.4). If \((\chi \overline{\chi})_* = 1\) then \(\chi \mid_K\) is irreducible.

**Proof.** As in Proof 2 of Corollary 2.4 above, note that \(\langle \chi, \chi \rangle_K = (\chi \overline{\chi})_* (K) = 1\).

As a summary it is convenient to make a list of equivalent statements.

**Theorem 2.5.** Let \(\chi \in \hat{G}, K \lhd G\). The following conditions are equivalent:

(a) \(\chi \mid_K\) is irreducible.
(b) \((\chi \overline{\chi})_* = 1\).
(c) If \(\lambda \in \hat{G}/\hat{K}\) and \(\langle \lambda \chi, \chi \rangle_K \neq 0\) then \(\lambda = 1\).
(d) The characters in the set \(\{\lambda \chi: \lambda \in \hat{G}/\hat{K}\}\) are distinct and irreducible.

**Proof.** (a) \(\iff\) (b) by Corollaries 2.4 and 2.5. (b) \(\implies\) (c) by Theorem 2.1. So (a), (b) and (c) are equivalent. Clearly (d) \(\implies\) (c). (a) \(\iff\) (d) is by Theorem B of the preliminary remarks.

3. In [6] the author considered the effect of the characters \(\hat{G}/\hat{K}\) on an irreducible character of \(G\) in the case that \(G/\hat{K}\) is abelian (see also [5] for a similar treatment). In particular the irreducible characters \(H(\chi)\) that "fix" \(\chi\) (i.e., \(\lambda \chi = \chi\)) were studied and the intersection of their kernels was singled out as the "dual inertial group" \(J(\chi)\). If \(G/\hat{K}\) is non-abelian then its irreducible characters need not be linear, and there are several ways to generalize the above concept. In [7] we called \(H(\chi)\) the set of irreducible characters \(\lambda\) such that \(\lambda \chi = (\deg \lambda) \chi\). Some properties of \(J(\chi)\) were dealt with there where \(J(\chi)\) is the intersection of the kernels of set of characters \(H(\chi)\). An alternative approach which we look at briefly here is to examine instead \(H_1(\chi) = S(\chi) \cap \hat{G}/\hat{K}\) = the irreducible characters \(\lambda\) of \(G/\hat{K}\) such that \(\lambda \chi\) contains \(\chi\) as a component. Then let \(J_1(\chi) = \bigcap \{\ker \lambda: \lambda \in H_1(\chi)\}\). It is seen below that \(J_1 = J_1(\chi)\) has at least some of the properties of the "dual inertial group" of [6], namely that if, (1) \(\sigma\) is a component of \(\chi \mid_K\) then \(\sigma\) may be extended to \(J_1(\chi)\) and (2) \(e_\chi(\sigma) = e_\chi(\sigma)\). Further it is shown (Theorem 3.5) that \(J_1(\chi)\) might be characterized as the (unique) maximal normal subgroup between \(G\) and \(K\) having these two properties. (This latter fact is new even for the case of \(G/\hat{K}\) abelian treated in [6].)
THEOREM 3.1. Let $\chi \in \hat{G}$, $K \triangleleft G$, and $J_1 = J_1(\chi)$ be defined as above. Let $\psi$ be an irreducible component of $\chi|_{J_1}$. Then $(\psi \bar{\psi})_* = 1$ on $J_1/K$ and hence $\psi|_K$ is irreducible.

Proof. $(\chi \bar{\chi})_* = \sum_{\lambda \in H(I(\chi))} n_\lambda \lambda$. So $(\chi \bar{\chi})_*$ restricted to $J_1/K$ consists of a multiple of the one-character. Since $\psi$ is a component of $\chi|_{J_1}$, $\chi \bar{\chi}|_{J_1} = \psi \bar{\psi} + \tau$ where $\tau$ is another character of $J_1$, and the restriction of $(\chi \bar{\chi})_*$ to $J_1/K$ equals $(\psi \bar{\psi})_* + \tau_*$. Hence $(\psi \bar{\psi})_*$ is a multiple of the one-character, and hence is the one-character by Corollary 2.2.

COROLLARY 3.2. Let $K \triangleleft G$, $\chi \in \hat{G}$ and let $\sigma$ be a component of $\chi|_K$. Then $\sigma$ may be extended to a character $\psi$ of $J_1$ and $I(\sigma) \subseteq J_1(\chi)$ where $I(\sigma)$ denotes the inertial group of $\sigma$.

Proof. Let $\psi$ be a component of $\chi|_K$. By Theorem 3.1 $\psi|_K = \tau$ is an irreducible component of $\chi|_K$. For some $g \in G$, $\sigma = \tau^g$ (by Theorem A in the preliminary remarks) and $\psi^g$ is an extension of $\sigma$ to $J_1$. For simplicity of notation we may assume henceforth that $\psi|_K = \sigma$. Thus if $h \in J_1$ then $\psi^h = \psi$ so $\sigma^h = \sigma$ and hence $h \in I(\sigma)$.

The following notation which was used in [7] will be helpful in proving Theorem 3.3. If $\rho, \chi \in G$ then $\rho * \chi$ is the set of irreducible components of $\rho \chi$. If $T \subseteq \hat{G}$ then $\rho^* T = \bigcup \{\rho^* \gamma \mid \gamma \in T\} = T^* \rho$. Associativity holds: $(\rho^* \gamma)^* \tau = \rho^* (\gamma^* \tau)$ is the irreducible components of $\rho^* \gamma \tau$. In this notation, $S(\chi) = \chi^* \bar{\chi}$ since $\langle \lambda \chi, \chi \rangle = \langle \lambda, \chi \bar{\chi} \rangle$. Theorem 2.5 of [7] states that if $K \triangleleft G$, $\chi \in \hat{G}$ and $\psi$ an irreducible component of $\chi|_K$ then the set of irreducible components of $\psi^\sigma$ equals $\bigcup \{\chi^* \rho \mid \rho \in \hat{G}/K\} = \chi^* G/K$.

THEOREM 3.3. $e_K(\chi) = e_{J_1}(\chi)$.

Proof. Let $\psi$ be an irreducible component of $\chi|_{J_1}$ and $\psi|_K = \sigma$. then

$$\chi|_K = e_K(\chi) \sum_{i=1}^m \sigma^i \quad m = [G: I(\sigma)]$$

and

$$\chi|_{J_1} = e_{J_1}(\psi) \sum_{i=1}^n \psi^h \quad n = [G: I(\psi)]$$

by Clifford's theorem ("Theorem A"). Since $\psi|_K = \sigma$ it is clear that $I(\psi) \subseteq I(\sigma)$. It suffices to prove that $I(\psi|_K) = I(\sigma)$ for then $m = n$ and since $\deg \sigma = \deg \psi$ the above equations show that $e_K(\chi) = e_{J_1}(\chi)$. 
Hence let \( g \in I(\sigma) \). We must show that \( \psi^g = \psi \). Clearly \( \psi^g |_K = \psi |_K = \sigma \) hence by Theorem B, \( \psi^g = \lambda \psi \) for some \( \lambda \in \hat{J}/\hat{K} \). And \((\lambda \psi)^g = (\psi^g)^g = \psi^g \). Let \( \gamma \) be an irreducible component of \( \lambda^g \). Then \( \gamma |_{J_1} = e_J(\gamma) \sum \lambda^g \) and \( K \subseteq \text{Ker } \gamma \). Thus any irreducible component of \( \psi^g = (\lambda \psi)^g \) must be included among the irreducible components of \( \gamma \psi^g = (\gamma |_{J_1} \psi)^g \). In particular, \( \langle \chi, \psi^g \rangle > 0 \) and hence \( \langle \chi, \gamma \psi^g \rangle > 0 \).

By Theorem 2.5 of [7], cited earlier, there exists \( \tau \in \hat{G}/\hat{J}_1 \) such that \( \chi \in \gamma^*(\tau^* \chi) = (\gamma^* \tau)^* \chi \). Hence there exists \( \delta \in \gamma^* \tau \) such that \( \chi \in \delta^* \chi \). Thus \( \psi^g = \lambda \psi = \psi \).

**COROLLARY 3.4.** \( I(\psi) = I(\sigma) \).

**THEOREM 3.5.** Let \( G, K, \chi, \sigma \) be as in the previous theorems. Let \( N \) be a normal subgroup of \( G \) containing \( K \) such that

1. \( e_N(\chi) = e_K(\chi) \) and
2. \( \sigma \) extends to an irreducible character \( \theta \) of \( N \). Then \( N \subseteq J_1(\chi) \). Hence \( J_1(\chi) \) is the unique normal subgroup which is maximal with respect to having properties (1) and (2).

**Proof.** Since \( \theta |_K = \sigma \) is irreducible, \( \theta^h |_K = \sigma^h \) is irreducible for each \( h \in G \). Using Theorem A and writing \( e = e_K(\chi) = e_N(\chi) \) we have:

\[
\chi |_N = e \sum \theta^h \quad \text{\{set of distinct conjugates of } \theta \text{\},}
\]

\[
\chi |_N = e \sum \lambda^h \quad \text{\{set of distinct conjugates of } \lambda \text{\}.}
\]

Since \( \chi |_K = (\chi |_N) |_K \) we see that different conjugates \( \theta^h \) of \( \theta \) must restrict to different conjugates of \( \sigma \). Hence if \( \theta \neq \theta^h \) then \( \sigma = \theta |_K \neq \theta^h |_K = \sigma^h \).

Now let \( \gamma \in H_1(\chi) \). We will show that \( N \subseteq \text{Ker } \gamma \) and hence \( N \subseteq J_1(\chi) = \bigcap (\text{Ker } \gamma : \gamma \in H_1(\chi)) \).

\( \gamma \in H_1(\chi) \) means \( \chi \in \gamma^* \chi \) and hence \( \gamma |_N \chi |_N \) contains \( \theta \) as a component. Hence there exists \( \lambda \) an irreducible component of \( \gamma |_N \), and \( \theta^\sigma \) a conjugate of \( \theta \) such that \( \theta \in \lambda^* \theta^\sigma \). So \((\lambda \theta^\sigma) |_N \) contains \( \sigma \) as a component and yet \( (\lambda \theta^\sigma) |_N = (\deg \lambda) \sigma^\sigma \). Thus \( \sigma = \sigma^\sigma \) and by the initial discussion this means that \( \theta = \theta^\sigma \) and that \( \theta \in \lambda^* \theta \). By Theorem B, \( \lambda \) must be trivial. Since any component of \( \gamma |_N \) is a conjugate of \( \lambda \), we have that \( N \subseteq \text{Ker } \gamma \).
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