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**ON RESTRICTING IRREDUCIBLE CHARACTERS TO
NORMAL SUBGROUPS**

RICHARD LEWIS ROTH

ON RESTRICTING IRREDUCIBLE CHARACTERS TO NORMAL SUBGROUPS

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This paper is about the situation where χ is an irreducible character of a finite group G and K is a normal subgroup. A construction of Serre's relating the characters of G with those of G/K is used to give a new proof of a well-known lemma concerning the case that $\chi|_K$ is irreducible and to generalize this lemma. It is seen that the irreducibility of $\chi|_K$ is equivalent to the property that $(1/|K|) \sum_{x \in hK} |\chi(x)|^2 = 1$ for each coset of G modulo K and also to the property that χ is not a component of $\lambda\chi$ for any irreducible character λ of G/K except for $\lambda = 1$. The subgroup $J_1 = J_1(\chi)$ is defined as the intersection of the kernels of the irreducible characters λ of G/K for which χ is a component of $\lambda\chi$. It is seen that an irreducible component σ of the restriction of χ to K will extend to J_1 , $e_{J_1}(\chi) = e_K(\chi)$ and J_1 is the maximal normal subgroup with these two properties.

Preliminary remarks. \hat{G} denotes the set of irreducible complex characters of G . 1 will often be used for the one-character of the appropriate group (according to context). $\langle \chi, \varphi \rangle_G = (1/|G|) \sum_{g \in G} \chi(g)\bar{\varphi}(g)$, the usual inner product.

We include here a couple of well-known theorems to be referred to later.

THEOREM A. (Clifford) *If $K \triangleleft G$, $\chi \in \hat{G}$, $\sigma \in \hat{K}$ and σ a component of $\chi|_K$ then $\chi|_K = e_K(\chi) \sum_{i=1}^m \sigma^{g_i}$ where $e_K(\chi)$ is a positive integer called the ramification index, $m = [G: I(\sigma)]$ with $I(\sigma)$ being the inertial group for σ and $\{g_1, \dots, g_m\}$ are a set of coset representatives for G modulo $I(\sigma)$. (See for example [1, Theorem 9.10].)*

THEOREM B. *Let $K \triangleleft G$ and χ an irreducible character of G which remains irreducible when restricted to K . Then the characters $\lambda\chi$ are distinct and irreducible as λ varies over the characters of G/K . Further if θ is an irreducible character of G such that $\chi|_K$ is a component of $\theta|_K$, then θ is of the form $\lambda\chi$ as above. (See [3, Lemma 3.1].)*

1. In this section we review a construction due to Serre which bears some resemblance to the familiar process for inducing characters from a subgroup. Theorem 1.1(b) is analogous to the Frobenius reciprocity theorem and was stated by Serre without proof in [8, p. 106].

By a class function on a group G is meant any function from G to the complex numbers which is constant on conjugacy classes. Let K be a normal subgroup of the finite group G . If φ is any class function of G/K , let φ^* denote the corresponding class function on G obtained in the usual way by $\varphi^*(g) = \varphi(gK)$. [Note that it is usually the custom to write φ instead of φ^* and this will be done in the latter part of this paper but here it is useful to make the distinction.] If ψ is a class function on G let ψ_* denote the function on G/K defined by $\psi_*(hK) = (1/|K|) \sum_{x \in hK} \psi(x)$.

THEOREM 1.1. (Serre) (a) ψ_* is a class function on G/K .

(b) $\langle \varphi^*, \psi \rangle_G = \langle \varphi, \psi_* \rangle_{G/K}$ where φ is any class function on G/K .

Proof. (a) If hK and h_1K are conjugate in G/K then $h_1K = g^{-1}hKg$ for some $g \in G$. Hence

$$\begin{aligned} \psi_*(h_1K) &= \frac{1}{|K|} \sum_{x \in h_1K} \psi(x) = \frac{1}{|K|} \sum_{x \in g^{-1}hKg} \psi(x) \\ &= \frac{1}{|K|} \sum_{y \in hK} \psi(g^{-1}yg) = \frac{1}{|K|} \sum_{y \in hK} \psi(y) = \psi_*(hK). \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \langle \varphi, \psi_* \rangle_{G/K} &= \frac{1}{|G/K|} \sum_{hK \in G/K} \varphi(hK) \overline{\psi_*(hK)} \\ &= \frac{|K|}{|G|} \sum_{hK \in G/K} \left[\varphi(hK) \cdot \frac{1}{|K|} \sum_{x \in hK} \overline{\psi(x)} \right] \\ &= \frac{1}{|G|} \sum_{x \in G} \varphi^*(x) \overline{\psi(x)} = \langle \varphi^*, \psi \rangle_G. \end{aligned}$$

The following corollary shows that the construction appears not as promising as Frobenius' induction; nevertheless it has some use as will be seen shortly.

COROLLARY 1.2. Let $\psi \in \widehat{G}$.

(a) If $K \subseteq \text{Ker } \psi$ and hence ψ may be regarded also as an element of $\widehat{G/K}$, then $\psi_* = \psi$ (under the latter identification).

(b) If $K \not\subseteq \text{Ker } \psi$ then $\psi_* \equiv 0$.

Proof. If $\varphi \in \widehat{G/K}$ then $\langle \varphi, \psi_* \rangle_{G/K} = \langle \varphi^*, \psi \rangle_G = 1$ or 0 depending on whether $\psi = \varphi^*$ or not. Case (b) means that $\psi \neq \varphi^*$ for any $\varphi \in \widehat{G/K}$ and since $\widehat{G/K}$ forms a basis for the class functions on G/K we get that $\psi_* \equiv 0$. If (a) holds, then $\psi = \varphi^*$ for exactly one φ and so $\psi_* = \psi$.

COROLLARY 1.3. If ψ is any class function on G write $\psi =$

$\sum a_i \chi_i + \sum b_j \psi_j$ where $\chi_i, \psi_j \in \widehat{G}$, $K \subseteq \text{Ker } \chi_i$ each i but $K \not\subseteq \text{Ker } \psi_j$ each j . Then $(\psi_*)^* = \sum a_i \chi_i$. Further if ψ is a character then $(\psi_*)^*$ is also a character or the zero function.

In what follows, we omit the upper star and identify characters of G/K with characters of G .

2. We now use the Serre construction to give a proof of a theorem which generalizes both [2, Lemma, p. 178] of Gallagher and [5, Lemma 4.2] of Iwahori and Matsumoto (see corollaries which follow).

THEOREM 2.1. *Let $\chi \in \widehat{G}$. Let $S(\chi)$ denote the set of irreducible characters λ of G such that $\lambda\chi$ contains χ as a component, i.e., $\langle \lambda\chi, \chi \rangle_G = n_\lambda > 0$. Then $(\chi\bar{\chi})_* = \sum_{\lambda \in S(\chi) \cap \widehat{G/K}} n_\lambda \lambda$ i.e., $(1/|K|) \sum_{x \in hK} |\chi(x)|^2 = \sum_{\lambda \in S(\chi) \cap \widehat{G/K}} n_\lambda \lambda(hK)$.*

Proof. $n_\lambda = \langle \lambda\chi, \chi \rangle_G = \langle \lambda, \chi\bar{\chi} \rangle_G$ so that $\chi\bar{\chi} = \sum n_\lambda \lambda$ summed over $\lambda \in S(\chi)$. By Corollary 1.3, $(\chi\bar{\chi})_* = \sum_{\lambda \in S(\chi) \cap \widehat{G/K}} n_\lambda \lambda$.

COROLLARY 2.2. *The one-character always occurs with multiplicity one in $(\chi\bar{\chi})_*$.*

COROLLARY 2.3. (Iwahori-Matsumoto [5, Lemma 4.2]) *If G/K is abelian and $H(\chi)$ is the group of (linear) characters $\lambda \in \widehat{G/K}$ such that $\lambda\chi = \chi$ then $(\chi\bar{\chi})_* = \sum_{\lambda \in H(\chi)} \lambda$.*

Proof. In this case $S(\chi) \cap \widehat{G/K} = H(\chi)$ since if λ is linear and $\langle \lambda\chi, \chi \rangle = n_\lambda > 0$ then $\lambda\chi = \chi$ and $n_\lambda = 1$.

COROLLARY 2.4. (Gallagher [2, Lemma, p. 178]; also Isaacs [4, Lemma 3.4]) *If $\chi|_K$ is irreducible then $(\chi\bar{\chi})_* = 1$.*

It is instructive to give two different short proofs.

Proof 1. By Theorem B in the preliminary remarks the characters $\{\lambda\chi : \lambda \in \widehat{G/N}\}$ are all distinct and irreducible. Thus $S(\chi) \cap \widehat{G/N} = \{1\}$.

Proof 2. $\chi|_K$ irreducible means that

$$1 = \langle \chi, \chi \rangle_K = \frac{1}{|K|} \sum \chi(g)\overline{\chi(g)} = (\chi\bar{\chi})_*(K).$$

Hence $(\chi\bar{\chi})_*$ is a character (Corollary 1.3) of degree 1. By Corollary

2.2, we have $(\chi\bar{\chi})_* = 1$.

COROLLARY 2.5 (the converse to Corollary 2.4). *If $(\chi\bar{\chi})_* = 1$ then $\chi|_K$ is irreducible.*

Proof. As in Proof 2 of Corollary 2.4 above, note that $\langle \chi, \chi \rangle_K = (\chi\bar{\chi})_*(K) = 1$.

As a summary it is convenient to make a list of equivalent statements.

THEOREM 2.5. *Let $\chi \in \widehat{G}$, $K \triangleleft G$. The following conditions are equivalent:*

(a) $\chi|_K$ is irreducible.

(b) $(\chi\bar{\chi})_* = 1$.

(c) If $\lambda \in \widehat{G/K}$ and $\langle \lambda\chi, \chi \rangle_G \neq 0$ then $\lambda = 1$.

(d) *The characters in the set $\{\lambda\chi: \lambda \in \widehat{G/K}\}$ are distinct and irreducible.*

Proof. (a) \Rightarrow (b) by Corollaries 2.4 and 2.5. (b) \Leftrightarrow (c) by Theorem 2.1. So (a), (b) and (c) are equivalent. Clearly (d) \Rightarrow (c). (a) \Rightarrow (d) is by Theorem B of the preliminary remarks.

3. In [6] the author considered the effect of the characters $\widehat{G/K}$ on an irreducible character of G in the case that G/K is abelian (see also [5] for a similar treatment). In particular the irreducible characters $H(\chi)$ that “fix” χ (i.e., $\lambda\chi = \chi$) were studied and the intersection of their kernels was singled out as the “dual inertial group” $J(\chi)$. If G/K is non-abelian then its irreducible characters need not be linear, and there are several ways to generalize the above concept. In [7] we called $H(\chi)$ the set of irreducible characters λ such that $\lambda\chi = (\deg \lambda)\chi$. Some properties of $J(\chi)$ were dealt with there where $J(\chi)$ is the intersection of the kernels of set of characters $H(\chi)$. An alternative approach which we look at briefly here is to examine instead $H_1(\chi) = S(\chi) \cap \widehat{G/K}$ = the irreducible characters λ of G/K such that $\lambda\chi$ contains χ as a component. Then let $J_1(\chi) = \bigcap \{\text{Ker } \lambda: \lambda \in H_1(\chi)\}$. It is seen below that $J_1 = J_1(\chi)$ has at least some of the properties of the “dual inertial group” of [6], namely that if, (1) σ is a component of $\chi|_K$ then σ may be extended to $J_1(\chi)$ and (2) $e_{J_1(\chi)} = e_K(\chi)$. Further it is shown (Theorem 3.5) that $J_1(\chi)$ might be characterized as the (unique) maximal normal subgroup between G and K having these two properties. (This latter fact is new even for the case of G/K abelian treated in [6].)

THEOREM 3.1. *Let $\chi \in \widehat{G}$, $K \triangleleft G$, and $J_1 = J_1(\chi)$ be defined as above. Let ψ be an irreducible component of $\chi|_{J_1}$. Then $(\psi\bar{\psi})_* = 1$ on J_1/K and hence $\psi|_K$ is irreducible.*

Proof. $(\chi\bar{\chi})_* = \sum_{\lambda \in H_1(\chi)} n_\lambda \lambda$. So $(\chi\bar{\chi})_*$ restricted to J_1/K consists of a multiple of the one-character. Since ψ is a component of $\chi|_{J_1}$, $\chi\bar{\chi}|_{J_1} = \psi\bar{\psi} + \tau$ where τ is another character of J_1 , and the restriction of $(\chi\bar{\chi})_*$ to J_1/K equals $(\psi\bar{\psi})_* + \tau_*$. Hence $(\psi\bar{\psi})_*$ is a multiple of the one-character, and hence is the one-character by Corollary 2.2.

COROLLARY 3.2. *Let $K \triangleleft G$, $\chi \in \widehat{G}$ and let σ be a component of $\chi|_K$. Then σ may be extended to a character ψ of J_1 and $I(\sigma) \cong J_1(\chi)$ where $I(\sigma)$ denotes the inertial group of σ .*

Proof. Let ψ be a component of $\chi|_K$. By Theorem 3.1 $\psi|_K = \tau$ is an irreducible component of $\chi|_K$. For some $g \in G$, $\sigma = \tau^g$ (by Theorem A in the preliminary remarks) and ψ^g is an extension of σ to J_1 . For simplicity of notation we may assume henceforth that $\psi|_K = \sigma$. Thus if $h \in J_1$ then $\psi^h = \psi$ so $\sigma^h = \sigma$ and hence $h \in I(\sigma)$.

The following notation which was used in [7] will be helpful in proving Theorem 3.3. If $\rho, \chi \in G$ then $\rho*\chi$ is the set of irreducible components of $\rho\chi$. If $T \subseteq \widehat{G}$ then $\rho*T = \bigcup \{\rho*\gamma \mid \gamma \in T\} = T*\rho$. Associativity holds: $(\rho*\gamma)*\tau = \rho*(\gamma*\tau)$ = the irreducible components of $\rho\gamma\tau$. In this notation, $S(\chi) = \chi*\bar{\chi}$ since $\langle \lambda\chi, \chi \rangle = \langle \lambda, \chi\bar{\chi} \rangle$. Theorem 2.5 of [7] states that if $K \triangleleft G$, $\chi \in \widehat{G}$ and ψ an irreducible component of $\chi|_K$ then the set of irreducible components of ψ^σ equals $\bigcup \{\chi*\rho \mid \rho \in \widehat{G/K}\} = \chi*\widehat{G/K}$.

THEOREM 3.3. $e_K(\chi) = e_{J_1}(\chi)$.

Proof. Let ψ be an irreducible component of $\chi|_{J_1}$ and $\psi|_K = \sigma$. then

$$\chi|_K = e_K(\chi) \sum_{i=1}^m \sigma^{g^i} \quad m = [G: I(\sigma)]$$

and

$$\chi|_{J_1} = e_{J_1}(\psi) \sum_{i=1}^n \psi^{h^i} \quad n = [G: I(\psi)]$$

by Clifford's theorem ("Theorem A"). Since $\psi|_K = \sigma$ it is clear that $I(\psi) \subseteq I(\sigma)$. It suffices to prove that $I(\psi) = I(\sigma)$ for then $m = n$ and since $\deg \sigma = \deg \psi$ the above equations show that $e_K(\chi) = e_{J_1}(\chi)$.

Hence let $g \in I(\sigma)$. We must show that $\psi^g = \psi$. Clearly $\psi^g|_K = \psi|_K = \sigma$ hence by Theorem B, $\psi^g = \lambda\psi$ for some $\lambda \in \widehat{J_1/K}$. And $(\lambda\psi)^g = (\psi^g)^g = \psi^g$. Let γ be an irreducible component of λ^g . Then $\gamma|_{J_1} = e_{J_1}(\gamma) \sum \lambda^g$ and $K \subseteq \text{Ker } \gamma$. Thus any irreducible component of $\psi^g = (\lambda\psi)^g$ must be included among the irreducible components of $\gamma\psi^g = (\gamma|_{J_1}\psi)^g$. In particular, $\langle \chi, \psi^g \rangle > 0$ and hence $\langle \chi, \gamma\psi^g \rangle > 0$. By Theorem 2.5 of [7], cited earlier, there exists $\tau \in \widehat{G/J_1}$ such that $\chi \in \gamma^*(\tau^*\chi) = (\gamma^*\tau)^*\chi$. Hence there exists $\delta \in \gamma^*\tau$ such that $\chi \in \delta^*\chi$. Hence $\delta \in S(\chi)$.

Now $\gamma \in \widehat{G/K}$, $\tau \in \widehat{G/J_1} \subseteq G/K$ so $\delta \in \gamma^*\tau \subseteq \widehat{G/K}$; i.e., $\delta \in S(\chi) \cap \widehat{G/K} = H_1(\chi)$. Thus $J_1(\chi) \subseteq \text{Ker } \delta$; i.e., $\delta \in \widehat{G/J_1}$. But $\langle \delta, \gamma\tau \rangle > 0$ means that $\langle \delta\bar{\tau}, \gamma \rangle > 0$ so $\gamma \in \delta^*\bar{\tau} \subseteq \widehat{G/J_1}$ so that $J_1 \subseteq \text{Ker } \gamma$ and hence λ is trivial. Thus $\psi^g = \lambda\psi = \psi$.

COROLLARY 3.4. $I(\psi) = I(\sigma)$.

THEOREM 3.5. *Let G, K, χ, σ be as in the previous theorems. Let N be a normal subgroup of G containing K such that*

(1) $e_N(\chi) = e_K(\chi)$ and

(2) σ extends to an irreducible character θ of N . Then $N \subseteq J_1(\chi)$. Hence $J_1(\chi)$ is the unique normal subgroup which is maximal with respect to having properties (1) and (2).

Proof. Since $\theta|_K = \sigma$ is irreducible, $\theta^h|_K = \sigma^h$ is irreducible for each $h \in G$. Using Theorem A and writing $e = e_K(\chi) = e_N(\chi)$ we have:

$$\begin{aligned} \chi_K &= e \sum \sigma^h & \{\sigma^h\} &= \text{set of distinct conjugates of } \sigma, \\ \chi|_N &= e \sum \theta^g & \{\theta^g\} &= \text{set of distinct conjugates of } \theta. \end{aligned}$$

Since $\chi|_K = (\chi|_N)|_K$ we see that different conjugates θ^h of θ must restrict to different conjugates of σ . Hence if $\theta \neq \theta^h$ then $\sigma = \theta|_K \neq \theta^h|_K = \sigma^h$.

Now let $\gamma \in H_1(\chi)$. We will show that $N \subseteq \text{Ker } \gamma$ and hence $N \subseteq J_1(\chi) = \bigcap \{\text{Ker } \gamma : \gamma \in H_1(\chi)\}$.

$\gamma \in H_1(\chi)$ means $\chi \in \gamma^*\chi$ and hence $\gamma|_N \chi|_N$ contains θ as a component. Hence there exists λ an irreducible component of $\gamma|_N$, and θ^g a conjugate of θ such that $\theta \in \lambda^*\theta^g$. So $(\lambda\theta^g)|_N$ contains σ as a component and yet $(\lambda\theta^g)|_N = (\text{deg } \lambda)\sigma^g$. Thus $\sigma = \sigma^g$ and by the initial discussion this means that $\theta = \theta^g$ and that $\theta \in \lambda^*\theta$. By Theorem B, λ must be trivial. Since any component of $\gamma|_N$ is a conjugate of λ , we have that $N \subseteq \text{Ker } \gamma$.

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Shashi Prabha Arya and M. K. Singal, <i>More sum theorems for topological spaces</i>	1
Goro Azumaya, F. Mbuntum and Kalathoor Varadarajan, <i>On M-projective and M-injective modules</i>	9
Kong Ming Chong, <i>Spectral inequalities involving the infima and suprema of functions</i>	17
Alan Hetherington Durfee, <i>The characteristic polynomial of the monodromy</i>	21
Emilio Gagliardo and Clifford Alfons Kottman, <i>Fixed points for orientation preserving homeomorphisms of the plane which interchange two points</i>	27
Raymond F. Gittings, <i>Finite-to-one open maps of generalized metric spaces</i>	33
Andrew M. W. Glass, W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, <i>a^*-closures of completely distributive lattice-ordered groups</i>	43
Matthew Gould, <i>Endomorphism and automorphism structure of direct squares of universal algebras</i>	69
R. E. Harrell and Les Andrew Karlovitz, <i>On tree structures in Banach spaces</i>	85
Julien O. Hennefeld, <i>Finding a maximal subalgebra on which the two Arens products agree</i>	93
William Francis Keigher, <i>Adjunctions and comonads in differential algebra</i>	99
Robert Bernard Kelman, <i>A Dirichlet-Jordan theorem for dual trigonometric series</i>	113
Allan Morton Krall, <i>Stieltjes differential-boundary operators. III. Multivalued operators—linear relations</i>	125
Hui-Hsiung Kuo, <i>On Gross differentiation on Banach spaces</i>	135
Tom Louton, <i>A theorem on simultaneous observability</i>	147
Kenneth Mandelberg, <i>Amitsur cohomology for certain extensions of rings of algebraic integers</i>	161
Coy Lewis May, <i>Automorphisms of compact Klein surfaces with boundary</i>	199
Peter A. McCoy, <i>Generalized axisymmetric elliptic functions</i>	211
Muril Lynn Robertson, <i>Concerning Siu's method for solving $y'(t) = F(t, y(g(t)))$</i>	223
Richard Lewis Roth, <i>On restricting irreducible characters to normal subgroups</i>	229
Albert Oscar Shar, <i>P-primary decomposition of maps into an H-space</i>	237
Kenneth Barry Stolarsky, <i>The sum of the distances to certain pointsets on the unit circle</i>	241
Bert Alan Taylor, <i>Components of zero sets of analytic functions in C^2 in the unit ball or polydisc</i>	253
Michel Valadier, <i>Convex integrands on Souslin locally convex spaces</i>	267
Januario Varela, <i>Fields of automorphisms and derivations of C^*-algebras</i>	277
Arnold Lewis Villone, <i>A class of symmetric differential operators with deficiency indices $(1, 1)$</i>	295
Manfred Wollenberg, <i>The invariance principle for wave operators</i>	303