P-PRI... OF MAPS INTO AN H-SPACE

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P-PRIMARY DECOMPOSITION OF MAPS INTO AN H-SPACE

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If \( Y \) is a finitely generated homotopy associative \( H \)-space and \( X \) is finite \( CW \) then \([X, Y]\) is a nilpotent group. Using this it is easy to show that for any set of prime integers \( P \), a localization map \( \iota: Y \to Y_P \) induces \( \iota_*[X, Y] \to [X, Y_P] \) with the order of \( \iota_*^{-1}(\alpha) \) prime to \( P \). (e.g. see [2]) Since there is no theory of the localization of algebraic loops the same technique does not apply if \( Y \) is not homotopy associative. The purpose of this paper is to show that the above theorem holds in this situation.

**Theorem A** Let \( X \) be finite \( CW \), \( Y \) be a finitely generated \( H \)-space (or the localization of such a space) and let \( \iota: Y \to Y_P \) be a localization map. Let \( \alpha \in [X, Y_P] \); then the order of \( \iota_*^{-1}(\alpha) \) is prime to \( P \) or is empty. Furthermore there is always a localization map \( L: Y \to Y_P \) such that \( L_*^{-1}(\alpha) \) is not empty.

By [3], \([X, Y]\) is finite if and only if \([X, Y_P]\) is finite and in this situation \( \iota_*: [X, Y] \to [X, Y_P] \) is onto for any \( \iota \). Thus from Theorem A we get the following result.

**Theorem B.** Let \( X \) and \( Y \) be as in A and let \([X, Y]\) be finite. Then \([X, Y] \cong \prod [X, Y_q] \) where \( q \) is a prime integer and the order of \([X, Y_q]\) is a power of \( q \).

The structure of this paper is as follows: in §2 we prove an algebraic lemma which we need and in §3 we prove the main theorem.

With reference to Theorem B it should be noted that \([X, Y]\) is a finite (centrally) nilpotent loop ([5]) which is a product of loops of prime power order. While every finite nilpotent group possesses this property it is known ([1], p. 98) that there exists finite nilpotent loops which are not direct products of loops of prime power order.

2. Recall that an algebraic loop \( G \) is a set with a binary operation with a unit which satisfies the cancellation laws and has left and right inverses.

Consider the following commuting diagram of algebraic loops and homomorphisms.

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1 By space we mean connected simple \( CW \) space.
LEMMA 2.1. Let \( b \in B \) with \( b \in \text{Ker} \ g \). Assume that \( f^{-1}(b) \) is a finite set of order \( n \). Let \( a \in f^{-1}(b) \) and \( a' \) the left inverse for \( a \) (i.e. \( a'a = 1 \)). Then

(1) \[ \text{Ker} \ f = a'f^{-1}(b) = \{a'\alpha \mid \alpha \in f^{-1}(b)\} \]

(2) \[ \text{Ker} \ k \cap f^{-1}(b) \]

is either empty or the order of \( \text{Ker} \ k \cap f^{-1}(b) \) is equal to the order of \( \text{Ker} \ k \cap \text{Ker} \ f \) and divides \( n \).

Proof. (1) Trivially there is a \( 1-1 \) set map \( \Phi : f^{-1}(b) \rightarrow \text{Ker} \ f \) defined by \( \Phi(\alpha) = a'\alpha \) similarly there is a \( 1-1 \) map \( \Psi : \text{Ker} \ f \rightarrow f^{-1}(b) \) defined by \( \Psi(\beta) = a\beta \). Since \( A \) is not associative \( \Phi \) and \( \Psi \) are not necessarily inverses but the existence of \( \Phi \) implies that \( a'f^{-1}(b) \subseteq \text{Ker} \ f \) and \( \Psi \)'s existence implies equality.

(2) If \( \text{Ker} \ k \cap f^{-1}(b) \neq \emptyset \) we may assume, without loss of generality that \( k(a) = 1 \). Since \( \text{Ker} \ k \cap \text{Ker} \ f \) is a normal subloop of \( \text{Ker} \ f \) we have by ([B], p. 92) that the order of \( \text{Ker} \ k \cap \text{Ker} \ f \) divides \( n \). But \( k(a'\alpha) = 1 \) if and only if \( k(\alpha) = 1 \).

3. Proof of Theorem A. By 4.1 of [3] there exists a localization \( L : Y \rightarrow Y_P \) such that \( L^{-1}(\alpha) \neq \emptyset \). By 4.2 of [3] or 2.2 of [4] for any localization \( l : Y \rightarrow Y_P, l^{-1}(\alpha) \) is finite. Thus we may assume \( l^{-1}(\alpha) \) is finite and nonempty. By (1) of 2.1 the order of \( l^{-1}(\alpha) \) is equal to the order of \( \text{Ker} \ l \).

We proceed by induction on the Postnikov systems for \( Y \) and \( Y_P \). Consider the following homotopy commutative diagram:

\[
\begin{array}{ccc}
Y_n & \xrightarrow{l_n} & Y_{P_n} \\
\downarrow p_n & & \downarrow p_{P_n} \\
Y_{n-1} & \xrightarrow{l_{n-1}} & Y_{P_{n-1}} \\
\downarrow p_n & & \downarrow p_{P_n} \\
K(\pi_n(Y), n + 1) & \xrightarrow{f_n} & K(\pi_n(Y_P), n + 1)
\end{array}
\]

where \( f_n \) and \( f_{P_n} \) correspond to the the \( n^{th} \) Postnikov invariants, \( l_n \), \( l_{n-1} \), \( l_e \) are the localization maps induced by \( l : Y \rightarrow Y_P \) and \( p_n \), and \( p_{P_n} \).
are the fibrations induced by \( t^* \) and \( t^*_\rho \) respectively. Note that all the maps in the diagram are \( H \)-maps. Let us assume that the order of \( \text{Ker} l_{n-1}^* \) is prime to \( P \).

By ([5], 2.3) the commuting diagram

\[
\begin{array}{ccc}
[X, Y_{n-1}] & \xrightarrow{l_{n-1}^*} & [X, Y_p] \\
\downarrow{l_n} & & \downarrow{t_p} \\
H^{n+1}(X; \pi_n(Y)) & \xrightarrow{l_{n}^*} & H^{n+1}(X; \pi_n(Y_P))
\end{array}
\]

is a diagram of nilpotent loops and homomorphisms. By 2.1, 2), the subloop \( H \) of \( \text{ker} l_{n-1}^* \) which lifts to \([X, Y_n]\) divides the order of \( \text{ker} l_{n-1}^* \) and hence is prime to \( P \).

Let \( K \) be the subloop of \( H \) which have liftings \( \beta \in [X, Y_n] \) such that \( \beta \in \text{Ker} l_n^* \). Since \( \text{ker} l_{n-1}^* \) is nilpotent ([1], P. 96, 1.1), we have ([1], 93) that the order of \( K \) divides the order of \( H \) and hence is prime to \( P \). But by ([3], 3.3 and 4.1), the set of liftings \( \{ \beta \in [X, Y_n] | \pi_n(\beta) = \alpha, l_n^*(\beta) = 0 \} \) is in 1 – 1 correspondence with a finite group of order prime to \( P \). Thus the order of \( \text{ker} l_n^* \) is again finite of order prime to \( P \). Since the assumption trivially holds at the first stage of the Postnikov decomposition, the result follows.

To prove Theorem B note that by [3] the finiteness of \([X, Y]\) implies that \( l_*: [X, Y] \rightarrow [X, Y_P] \) is onto for any \( I \). Thus \([X, Y_\rho]\) is finite. But \( Y_\rho = \Pi K(Q, n) \), so that

\[
[X, Y_\rho] = [X, \Pi K(Q, n)] = \Pi H^*(X; Q)
\]

which is finite if and only if \([X, Y_\rho] = 0 \).

If \( q \) is a prime and \( \bar{q} \) its complimentary set of primes then by ([2], [4])

\[
\begin{array}{ccc}
[X, Y] & \longrightarrow & [X, Y_\bar{q}] \\
\downarrow & & \downarrow \\
[X, Y_\rho] & \longrightarrow & [X, Y_\rho]
\end{array}
\]

is a pullback diagram. Therefore

\[
\#[X, Y] = \#[X, Y_\bar{q}] \cdot \#[X, Y_\rho] \quad \text{(where \#S is the order of the set S).}
\]

Since \( l_*: [X, Y] \rightarrow [X, Y_\bar{q}] \) is onto we see, by the proof of A, that there is an integer \( k \) such that \( \#[l_*^{-1}(\alpha)] = q^k \) for all \( \alpha \in [X, Y_\bar{q}] \).

Thus \( \#[X, Y] = q^k \#[X, Y_\bar{q}] \) or \([X, Y_\rho] = q^k \). By [4], and the fact
that \([X, Y_\phi] = 0\) we get \([X, Y] = II[X, Y_\phi]\).

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Goro Azumaya, F. Mbuntum and Kalathoor Varadarajan, On M-projective and M-injective modules .................................................. 9
Kong Ming Chong, Spectral inequalities involving the infima and suprema of functions .................................................. 17
Alan Hetherington Durfee, The characteristic polynomial of the monodromy .... 21
Emilio Gagliardo and Clifford Alfons Kottman, Fixed points for orientation preserving homeomorphisms of the plane which interchange two points .... 27
Raymond F. Gittings, Finite-to-one open maps of generalized metric spaces .... 33
Andrew M. W. Glass, W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, $a^*$-closures of completely distributive lattice-ordered groups .................. 43
Matthew Gould, Endomorphism and automorphism structure of direct squares of universal algebras ................................................. 69
R. E. Harrell and Les Andrew Karlovitz, On tree structures in Banach spaces .... 85
Julien O. Hennefeld, Finding a maximal subalgebra on which the two Arens products agree .................................................. 93
Robert J. Keigher, Adjunctions and comonads in differential algebra .... 99
Robert Bernard Kelman, A Dirichlet-Jordan theorem for dual trigonometric series .................................................. 113
Allan Morton Krall, Stieltjes differential-boundary operators. III. Multivalued operators–linear relations .......................... 125
Hui-Hsiung Kuo, On Gross differentiation on Banach spaces .................. 135
Tom Louton, A theorem on simultaneous observability .......................... 147
Kenneth Mandelberg, Amitsur cohomology for certain extensions of rings of algebraic integers .................................................. 161
Coy Lewis May, Automorphisms of compact Klein surfaces with boundary .... 199
Peter A. McCoy, Generalized axisymmetric elliptic functions .................. 211
Muril Lynn Robertson, Concerning Siu's method for solving $y'(t) = F(t, y(g(t)))$ .................................................. 223
Richard Lewis Roth, On restricting irreducible characters to normal subgroups .................................................. 229
Albert Oscar Shar, $P$-primary decomposition of maps into an $H$-space .......... 237
Kenneth Barry Stolarsky, The sum of the distances to certain pointsets on the unit circle .................................................. 241
Bert Alan Taylor, Components of zero sets of analytic functions in $C^2$ in the unit ball or polydisc .................................................. 253
Michel Valadier, Convex integrands on Souslin locally convex spaces .......... 267
Januario Varela, Fields of automorphisms and derivations of $C^*$-algebras .......... 277
Arnold Lewis Villone, A class of symmetric differential operators with deficiency indices $(1, 1)$ .................................................. 295
Manfred Wollenberg, The invariance principle for wave operators .......... 303