

Pacific Journal of Mathematics

***P*-PRIMARY DECOMPOSITION OF MAPS INTO AN *H*-SPACE**

ALBERT OSCAR SHAR

P-PRIMARY DECOMPOSITION OF MAPS INTO AN H-SPACE

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If Y is a finitely generated homotopy associative H -space¹ and X is finite CW then $[X, Y]$ is a nilpotent group. Using this it is easy to show that for any set of prime integers P , a localization map $\iota: Y \rightarrow Y_P$ induces $\iota_*[X, Y] \rightarrow [X, Y_P]$ with the order of $\iota_*^{-1}(\alpha)$ prime to P . (e.g. see [2]) Since there is no theory of the localization of algebraic loops the same technique does not apply if Y is not homotopy associative. The purpose of this paper is to show that the above theorem holds in this situation.

THEOREM A *Let X be finite CW , Y be a finitely generated H -space (or the localization of such a space) and let $\iota: Y \rightarrow Y_P$ be a localization map. Let $\alpha \in [X, Y_P]$; then the order of $\iota_*^{-1}(\alpha)$ is prime to P or is empty. Furthermore there is always a localization map $L: Y \rightarrow Y_P$ such that $L_*^{-1}(\alpha)$ is not empty.*

By [3], $[X, Y]$ is finite if and only if $[X, Y_P]$ is finite and in this situation $\iota_*: [X, Y] \rightarrow [X, Y_P]$ is onto for any ι . Thus from Theorem A we get the following result.

THEOREM B. *Let X and Y be as in A and let $[X, Y]$ be finite. Then $[X, Y] \cong \prod [X, Y_q]$ where q is a prime integer and the order of $[X, Y_q]$ is a power of q .*

The structure of this paper is as follows: in §2 we prove an algebraic lemma which we need and in §3 we prove the main theorem.

With reference to Theorem B it should be noted that $[X, Y]$ is a finite (centrally) nilpotent loop ([5]) which is a product of loops of prime power order. While every finite nilpotent group possesses this property it is known ([1], p. 98) that there exists finite nilpotent loops which are not direct products of loops of prime power order.

2. Recall that an algebraic loop G is a set with a binary operation with a unit which satisfies the cancellation laws and has left and right inverses.

Consider the following commuting diagram of algebraic loops and homomorphisms.

¹ By space we mean connected simple CW space.

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow k & & \downarrow g \\
 C & \xrightarrow{h} & D
 \end{array}$$

LEMMA 2.1. *Let $b \in B$ with $b \in \text{Ker } g$. Assume that $f^{-1}(b)$ is a finite set of order n . Let $a \in f^{-1}(b)$ and a' the left inverse for a (i.e. $a'a = 1$). Then*

(1) $\text{Ker } f = a'f^{-1}(b) = \{a'\alpha \mid \alpha \in f^{-1}(b)\}$

(2) $\text{Ker } k \cap f^{-1}(b)$

is either empty or the order of $\text{Ker } k \cap f^{-1}(b)$ is equal to the order of $\text{Ker } k \cap \text{Ker } f$ and divides n .

Proof. (1) Trivially there is a 1 – 1 set map $\Phi: f^{-1}(b) \rightarrow \text{Ker } f$ defined by $\Phi(\alpha) = a'\alpha$ similarly there is a 1 – 1 map $\Psi: \text{Ker } f \rightarrow f^{-1}(b)$ defined by $\Psi(\beta) = a\beta$. Since A is not associative Φ and Ψ are not necessarily inverses but the existence of Φ implies that $a'f^{-1}(b) \subseteq \text{Ker } f$ and Ψ 's existence implies equality.

(2) If $\text{Ker } k \cap f^{-1}(b) \neq \emptyset$ we may assume, without loss of generality that $k(a) = 1$. Since $\text{Ker } k \cap \text{Ker } f$ is a normal subloop of $\text{Ker } f$ we have by ([B], p. 92) that the order of $\text{Ker } k \cap \text{Ker } f$ divides n . But $k(a'\alpha) = 1$ if and only if $k(\alpha) = 1$.

3. *Proof of Theorem A.* By 4.1 of [3] there exists a localization $L: Y \rightarrow Y_P$ such that $L_*^{-1}(\alpha) \neq \emptyset$. By 4.2 of [3] or 2.2 of [4] for any localization $l: Y \rightarrow Y_P$, $l_*^{-1}(\alpha)$ is finite. Thus we may assume $l_*^{-1}(\alpha)$ is finite and nonempty. By (1) of 2.1 the order of $l_*^{-1}(\alpha)$ is equal to the order of $\text{Ker } l_*$.

We proceed by induction on the Postnikov systems for Y and Y_P . Consider the following homotopy commutative diagram:

$$\begin{array}{ccc}
 Y_n & \xrightarrow{I_n} & Y_{Pn} \\
 \downarrow p_n & & \downarrow p_{Pn} \\
 Y_{n-1} & \xrightarrow{I_{n-1}} & Y_{Pn-1} \\
 \downarrow \mathfrak{f}^n & & \downarrow \mathfrak{f}_P^n \\
 K(\pi_n(Y), n+1) & \xrightarrow{\mathfrak{f}_c} & K(\pi_n(Y_P), n+1)
 \end{array}$$

where \mathfrak{f}^n and \mathfrak{f}_P^n correspond to the the n^{th} Postnikov invariants, I_n, I_{n-1}, I_c are the localization maps induced by $l: Y \rightarrow Y_P$ and p_n , and p_{Pn}

are the fibrations induced by \mathfrak{f}^n and \mathfrak{f}_P^n respectively. Note that all the maps in the diagram are H -maps. Let us assume that the order of $\text{Ker } \mathfrak{I}_{n-1^*}$ is prime to P .

By ([5], 2.3) the commuting diagram

$$\begin{CD} [X, Y_{n-1}] @>\mathfrak{I}_{n-1^*}>> [X, Y_{p^{n-1}}] \\ @VV\mathfrak{f}_nV @VV\mathfrak{f}_{n_P}V \\ H^{n+1}(X; \pi_n(Y)) @>\mathfrak{I}_{c^*}>> H^{n+1}(X; \pi_n(Y_P)) \end{CD}$$

is a diagram of nilpotent loops and homomorphisms. By 2.1, 2), the subloop H of $\text{ker } \mathfrak{I}_{n-1^*}$ which lifts to $[X, Y_n]$ divides the order of $\text{ker } \mathfrak{I}_{n-1^*}$ and hence is prime to P .

Let K be the subloop of H which have liftings $\beta \in [X, Y_n]$ such that $\beta \in \text{Ker } \mathfrak{I}_{n^*}$. Since $\text{ker } \mathfrak{I}_{n-1^*}$ is nilpotent ([1], P. 96, 1.1), we have ([1], 93) that the order of K divides the order of H and hence is prime to P . But by ([3] 3.3 and 4.1), the set of liftings $\{\beta \in [X, Y_n] \mid \mathfrak{p}_{n^*}(\beta) = \alpha, \mathfrak{I}_{n^*}(\beta) = 0\}$ is in 1 - 1 correspondence with a finite group of order prime to P . Thus the order of $\text{ker } \mathfrak{I}_{n^*}$ is again finite of order prime to P . Since the assumption trivially holds at the first stage of the Postnikov decomposition, the result follows.

To prove Theorem B note that by [3] the finiteness of $[X, Y]$ implies that $\mathfrak{I}_*: [X, Y] \rightarrow [X, Y_P]$ is onto for any l . Thus $[X, Y_\phi]$ is finite. But $Y_\phi = \amalg K(Q, n)$, so that

$$[X, Y_\phi] = [X, \amalg K(Q, n)] = \amalg H^n(X; Q)$$

which is finite if and only if $[X, Y_\phi] = 0$.

If q is a prime and \bar{q} its complimentary set of primes then by ([2], [4])

$$\begin{CD} [X, Y] @>>> [X, Y_{\bar{q}}] \\ @VVV @VVV \\ [X, Y_q] @>>> [X, Y_\phi] \end{CD}$$

is a pullback diagram. Therefore

$$\#[X, Y] = \#[X, Y_{\bar{q}}] \cdot \#[X, Y_q] \text{ (where } \#S \text{ is the order of the set } S).$$

Since $\mathfrak{I}_*: [X, Y] \rightarrow [X, X_{\bar{q}}]$ is onto we see, by the proof of A, that there is an integer k such that $\#[\mathfrak{I}_*^{-1}(\alpha)] = q^k$ for all $\alpha \in [X, Y_{\bar{q}}]$.

Thus $\#[X, Y] = q^k \#[X, Y_{\bar{q}}]$ or $[X, Y_q] = q^k$. By [4], and the fact

that $[X, Y_\phi] = 0$ we get $[X, Y] = \Pi[X, Y_q]$.

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Received January 7, 1975 and in revised form March 6, 1975.

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