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R. T. Rockafellar has proved a formula for the conjugates of convex integral functionals on vector spaces of measurable vector-valued functions. This formula is extended to the case where the spaces of values of the measurable functions is a Souslin locally convex space. Rockafellar's definition of decomposable space of measurable vector-valued functions is extended to other than normed spaces. In the first section technical results on measurable set-valued functions are given.

Introduction. Let $(T, \mathscr{C}, \mu)$ denote a measure space with a positive σ-finite measure μ and $\mathscr{C}$ complete. Let E be a real locally convex space and E' its dual. Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be a vector space of functions from T to E (resp. E'). Hypotheses will be made which ensure that for each $u \in \mathscr{L}$ and each $v \in \mathscr{L}'$, the function $t \mapsto \langle v(t), u(t)\rangle$ is integrable. The pairing $\int \langle v(t), u(t)\rangle \mu(dt)$ will be denoted by $\langle v, u\rangle$.

A function $f: T \times E \rightarrow \bar{\boldsymbol{R}} (= [-\infty, \infty])$ will be called an integrand. Under certain hypotheses $t \mapsto f(t, u(t))$ is measurable for each $u \in \mathscr{L}$. We shall consider the functional on $\mathscr{L}$ defined by $I_f(u) = \int f(t, u(t))\mu(dt)$ (with the convention that the integral is $+\infty$ if the positive part of $f(t, u(t))$ is nonintegrable). Denote by $f^*(t, \cdot)$ the conjugate function of $f(t, \cdot)$ this is

$$f^*(t, x') = \sup \{\langle x', x\rangle - f(t, x) \mid x \in E\} \text{ for } x' \in E' .$$

As above I_{f^*} denotes the functional on $\mathscr{L}'$ defined by $I_{f^*}(v) = \int f^*(t, v(t))\mu(dt)$.

Then roughly speaking the result proved by Rockafellar is the following: if $\mathscr{L}$ contains sufficiently many functions (see below the definition of decomposable spaces) then I_{f^*} is the conjugate functional of I_f with respect to the duality $\mathscr{L}, \mathscr{L}'$.
More precisely, for each $v \in \mathscr{L}'$,

$$I_{f^*}(v) = (I_f)^*(v)(= \sup \{\langle v, u\rangle - I_f(u) \mid u \in \mathscr{L}\})$$

Rockafellar first proved this formula for $E = \boldsymbol{R}^n$ ([11] Th. 2 p. 532), then for separable Banach spaces ([13] Th. 2 p. 225). The inequality $I_{f^*}(v) \geqq (I_f)^*(v)$ is obvious. The converse inequality is proved using measurable selection theorems (and the fact that $\mathscr{L}$ contains sufficiently many functions). Standard measurable selection theorems require Polish spaces. But if E is a separable Banach space its dual may fail to be Polish, and then it is difficult to prove $(I_{f^*})^* = I_{f^{**}}$.

In [5] (Theorem 5) Castaing proved the two formulas $(I_f)^* = I_{f^*}$ and $(I_{f^*})^* = I_{f^{**}}$ for E separable Fréchet but T metrizable compact. In [6] he succeeds to prove the same formulas for abstract T and E separable Banach space, using the fact that E' endowed with $\sigma(E', E)$ is a Lusin space.

Here we shall extend the formulas to a Souslin locally convex space E whose dual E' is also a Souslin space for at least one locally convex topology compatible with duality. That is the case if E is a separable Fréchet space, and many locally convex spaces deduced from separable Fréchet spaces (for example most of the spaces encountered in the theory of distributions) have that property. Some new results on measurable functions and measurable set-valued functions with values in a Souslin space are given in a preliminary section. In particular Lemma 2 extends Bourbaki ([2] Ch. IV § 5). For applications and further references see Ioffe-Levin [7], Ioffe-Tikhomirov [8], Rockafellar [12].

1. Functions and set-valued functions with values in a Souslin space. We shall denote by $\mathscr{B}(E)$ the Borel σ-field of a topological space E. A Souslin space is a Hausdorff topological space S such that there exists a Polish space P and a continuous map h from P onto S.

LEMMA 1. *Let S be a Souslin space, P a Polish space and h a continuous map from P onto S. Let Γ be a set-valued function from T to the closed nonempty subsets of S, whose graph belongs to $\mathscr{C} \otimes \mathscr{B}(S)$. Then*

(a) *Γ admits a sequence of selections (u_n), such that, for every t the $u_n(t)$ are dense in $\Gamma(t)$, and such that there exist measurable maps σ_n: $T \to P$ with $u_n = h \circ \sigma_n$.*

(b) *Moreover u_n has the following properties:*

1. *u_n is $(\mathscr{C}, \mathscr{B}(S))$ measurable* (that is $\forall A \in \mathscr{B}(S)$, $u_n^{-1}(A) \in \mathscr{C}$)
2. *u_n is the limit of a sequence of $\mathscr{C}$-measurable functions assuming a finite number of values.*
3. *Moreover if T is a Hausdorff topological space, and μ a Radon measure, then u_n is Lusin μ-measurable.*

Proof. (a) Let G denote the graph of Γ and pr_T denote the map $(t, x) \mapsto t$ from $T \times S$ to T. Put $\phi(t) = h^{-1}(\Gamma(t))$. It is a closed nonempty subset of P. Recall that if U is a subset of P, $\phi^-(U)$ denotes the set $\{t \in T \mid \phi(t) \cap U \neq \varnothing\}$.
Then if $U \in \mathscr{B}(P)$ one has

$$\begin{aligned}\phi^{-}(U) &= \{t \mid \phi(t) \cap U \neq \emptyset\} \\ &= \{t \mid \Gamma(t) \cap h(U) \neq \emptyset\} \\ &= \Gamma^{-}(h(U)) \\ &= \mathrm{pr}_T(G \cap (T \times h(U)) .\end{aligned}$$

It is easy to see that $G \cap (T \times h(U))$ belongs to $\mathscr{C} \otimes \mathscr{B}(h(U))$. But $h(U)$ is a Souslin space (because U is Borel hence Souslin). Thus by the projection theorem (Aumann [1], Sainte Beuve [14]; for completeness we sketch a short proof in the Remark 2 below) $\phi^{-}(U)$ belongs to $\mathscr{C}$. Thus we can apply standard theorems on measurable selections: Castaing ([4] Th. 5.2) and for abstract measurability Valadier ([16] Th. 0.3). The conclusion is: ϕ has a sequence of measurable selections (σ_n) such that for every t the $\sigma_n(t)$ are dense in $\phi(t)$. Put $u_n(t) = h \circ \sigma_n(t)$. Then the $u_n(t)$ are dense in $\Gamma(t)$.

(b) 1. As σ_n is $(\mathscr{C}, \mathscr{B}(P))$ measurable and h is continuous (hence $(\mathscr{B}(P), \mathscr{B}(S))$ measurable), u_n is $(\mathscr{C}, \mathscr{B}(S))$ measurable.

2. The function σ_n is the limit of a sequence of $\mathscr{C}$-measurable functions assuming a finite number of values. Hence u_n has the same property.

3. Finally if T is a Hausdorff topological space and μ a Radon measure (see Bourbaki [3] Schwartz [15] for measures on Hausdorff spaces), it is well known that, as P is Polish, σ_n is Lusin measurable. That is for each compact $K \subset T$ and $\varepsilon > 0$, there exists a compact $K_\varepsilon \subset K$ such that $\mu(K - K_\varepsilon) \leqq \varepsilon$ and σ_n is continuous on K_ε. Obviously u_n has the same property.

REMARKS.

(1) Existence of one measurable selection has been proved by Sainte-Beuve [14] under a weaker hypothesis: Γ is not supposed closed-valued. She extends Aumann's theorem, which was stated for a Lusin space.

(2) We sketch now a short proof of the projection theorem. The statement is the following: if S is Souslin and $G \in \mathscr{C} \otimes \mathscr{B}(S)$, the projection of G onto T belongs to $\mathscr{C}$. When S is compact metrizable, this theorem is well known (Meyer [9], Neveu [10]): G is analytic and its projection is analytic, hence belongs to $\mathscr{C}$ which has been supposed complete.

If S is Polish, S is G_δ (countable intersection of open sets) in a compact metrizable space E. Then it is obvious that $\mathscr{C} \otimes \mathscr{B}(S) \subset \mathscr{C} \otimes \mathscr{B}(E)$ and the projection theorem is true for S Polish. Finally if S is Souslin: let P be a Polish space and $h: P \to S$ continuous and onto. Then if $G \in \mathscr{C} \otimes \mathscr{B}(S)$, $(1_T \times h)^{-1}(G)$ belongs to $\mathscr{C} \otimes \mathscr{B}(P)$ and

$$\mathrm{pr}_T\, G = \mathrm{pr}_T\, [(1_T \times h)^{-1}(G)]\ .$$

Here $1_T \times h$ is the map $(t, x) \mapsto (t, h(x))$ from $T \times P$ to $T \times S$, and pr_T denotes either the projection from $T \times S$ onto T or the projection from $T \times P$ onto T.

COROLLARY. *If S is a Souslin space and u: $T \to S$ is a function whose graph belongs to $\mathscr{C} \otimes \mathscr{B}(S)$, then u has the following properties:*

(1) *u is $(\mathscr{C}, \mathscr{B}(S))$ measurable*

(2) *u is the limit of a sequence of $\mathscr{C}$-measurable functions assuming a finite number of values*

(3) *Moreover if T is a Hausdorff topological space, and μ a Radon measure, then u is Lusin μ-measurable.*

Proof. Apply Lemma 1(b) to $\Gamma(t) = \{u(t)\}$.

LEMMA 2. *Let E be a Souslin real locally convex vector space, and u: $T \to E$ a function.*

(a) *Then the four following properties are equivalent:*

(1) *u is $(\mathscr{C}, \mathscr{B}(E))$ measurable*

(2) *u is the limit of a sequence of $\mathscr{C}$-measurable functions assuming a finite number of values*

(3) *u is scalarly measurable* (that is for each $x' \in E'$, $\langle x', u(\cdot)\rangle$ is measurable)

(4) *the graph of u belongs to $\mathscr{C} \otimes \mathscr{B}(E)$.*

(b) *Moreover if T is a Hausdorff topological space, and μ a Radon measure, consider the property*

(5) *u is Lusin μ-measurable.*

Then the five properties (1), $\cdots$, (5) *are equivalent.*

Proof. (a) The corollary to Lemma 1 yields the implications $4 \Rightarrow 1$ and $4 \Rightarrow 2$. $1 \Rightarrow 3$ and $2 \Rightarrow 3$ are obvious. We prove now $3 \Rightarrow 4$. By Lemma 3 below, there exists a sequence (e'_n) in E' which separates points of E. Thus the graph of u is

$$\bigcap_n \{(t, x) \in T \times E \mid \langle e'_n, x\rangle = \langle e'_n, u(t)\rangle\}\ .$$

Hence if u is scalarly measurable, the graph of u belongs to $\mathscr{C} \otimes \mathscr{B}(E)$.

(b) Suppose that T is a Hausdorff topological space and that μ is Radon. Then $5 \Rightarrow 3$ is obvious and $4 \Rightarrow 5$ is the corollary of Lemma 1.

LEMMA 3. *Let S be a Souslin space and $(f_i)_{i \in I}$ a family of real-valued continuous functions which separates points of S* (that is if $x \neq y$, there exists i such that $f_i(x) \neq f_i(y)$), *then there exists a countable subset D of I such that the subfamily $(f_i)_{i \in D}$ separates points of S.*

Proof. The fact that $(f_i)_{i \in I}$ separates points of S is equivalent to

$$S^2 - \Delta_S = \bigcup_{i \in I} (f_i \times f_i)^{-1}(\mathbf{R}^2 - \Delta_R) .$$

In this formula $f_i \times f_i$ denotes the map $(x, y) \mapsto (f_i(x), f_i(y))$, and Δ_E denotes the diagonal in $E \times E$. As S^2 is Souslin, there exists a Polish space Q and a continuous onto map $k: Q \to S^2$. Put

$$U_i = (f_i \times f_i)^{-1}(\mathbf{R}^2 - \Delta_R) .$$

It is an open set. It is well known that there exists a countable subset D of I such that

$$\bigcup_{i \in D} k^{-1}(U_i) = \bigcup_{i \in I} k^{-1}(U_i) .$$

As k is onto, that implies

$$\bigcup_{i \in D} U_i = \bigcup_{i \in I} U_i .$$

Hence the countable subfamily $(f_i)_{i \in D}$ separates points of S.

REMARK. This result has been proved by Schwartz [15] in a more general form.

LEMMA 4. *Let E be a Souslin locally convex space and $u: T \to E$ scalarly measurable. Then the function $(t, x') \mapsto \langle x', u(t) \rangle$ defined on $T \times E'$, is $\mathscr{C} \otimes \mathscr{B}(E')$ measurable.*

Proof. This follows from property (2) of Lemma 2.

Indeed let $u = \lim u_n$ where the u_n are $\mathscr{C}$-measurable functions assuming a finite number of values. Then $u_n(t) = x_n^p$ if $t \in T_n^p$, and

$$\langle x', u_n(t) \rangle = \langle x', x_n^p \rangle \quad \text{if} \quad t \in T_n^p .$$

Thus $(t, x') \mapsto \langle x', u_n(t) \rangle$ is $\mathscr{C} \otimes \mathscr{B}(E')$ measurable on $T_n^p \times E'$, hence on all $T \times E'$. Finally $\langle x', u(t) \rangle = \lim \langle x', u_n(t) \rangle$ is a $\mathscr{C} \otimes \mathscr{B}(E')$ measurable function of (t, x').

LEMMA 5. *Let E be a Souslin locally convex space and $u: T \to E$ scalarly measurable. Then there exists a sequence (T_n) in $\mathscr{C}$ such that $\overline{u(T_n)}$ is compact, and $T - \bigcup T_n$ is μ-negligible.*

Proof. As μ is σ-finite it is sufficient to prove the result when μ is bounded. By property (1) of Lemma 2 one may consider the measure $\nu = \mu \circ u^{-1}$ on $(E, \mathscr{B}(E))$. As E is Souslin, ν is a Radon measure (Bourbaki [3] Prop. 3 p. 49). Therefore there exists a sequence of compact sets (K_n) in E, such that $\nu(\cup K_n) = \nu(E)$. The sets $T_n = u^{-1}(K_n)$ have the required properties.

2. Decomposable vector spaces of functions. Integrands. From now on E is a Souslin real locally convex vector space and its dual E' is supposed to be Souslin for at least one topology compatible with duality (we remark that this is equivalent to supposing that E' is Souslin for the weak topology $\sigma(E', E)$).

We denote by $\mathscr{L}$ (resp. $\mathscr{L}'$) a vector space of scalarly measurable functions from T to E (resp. E'), and by L (resp. L') the space of equivalence classes for equality almost everywhere. Note that by property (2) of Lemma 2 for each $u \in \mathscr{L}$, and each $v \in \mathscr{L}'$, $t \mapsto \langle v(t), u(t) \rangle$ is measurable. We make the hypothesis that for each $u \in \mathscr{L}$ and each $v \in \mathscr{L}'$, $t \mapsto \langle v(t), u(t) \rangle$ is integrable. We denote by $\langle v, u \rangle$ the number $\int \langle v(t), u(t) \rangle \mu(dt)$. We denote by $\mathscr{M}_E^k$ (resp. $\mathscr{M}_{E'}^k$) the space of scalarly measurable functions from T to E (resp. E') such that $\overline{f(T)}$ is compact (here it is important to choose a Souslin topology on E').

DEFINITION 1. *The space $\mathscr{L}$ is said to be decomposable if $u \in \mathscr{L}$, $f \in \mathscr{M}_E^k$, $A \in \mathscr{C}$ and $\mu(A) < \infty$ imply*

$$\chi_A f + \chi_{T-A} u \in \mathscr{L}$$

(χ_A denotes the characteristic function of A).

REMARK. If E is a separable reflexive Banach space, then E and E' are Polish for the norm topology, hence Souslin for all topologies compatible with duality. Our definition is equivalent to Rockafellar's (where $\mathscr{L}_E^\infty$ is taken in place of $\mathscr{M}_E^k$).

EXAMPLE. Let E be a separable Fréchet space. Then its dual E'_s with the topology $\sigma(E', E)$ is Souslin. We may take $\mathscr{L} = \mathscr{L}_E^1$ and $\mathscr{L}' = \mathscr{L}_{E'_s}^\infty$. Indeed, $\mathscr{L}_E^1$ is obviously decomposable. And $\mathscr{L}_{E'_s}^\infty = \mathscr{M}_{E'_s}^k$, because for a closed subset of E'_s compactness is equivalent to equicontinuity. Thus $\mathscr{L}_{E'_s}^\infty$ is decomposable.

LEMMA 6.

(1) If *$\mathscr{L}$ is decomposable and $v \in \mathscr{L}'$, then $\forall u \in \mathscr{L}$, $\langle v, u \rangle = 0$, implies $v = 0$* a.e.

(2) *If $\mathscr{L}$ and $\mathscr{L}'$ are decomposable, the bilinear map $(u, v) \mapsto \langle v, u\rangle$ defines a separated duality between L and L'.*

Proof. (1) Let $A \in \mathscr{C}$ with $\mu(A) < \infty$. If $x \in E$ and φ is a real valued bounded measurable function, then φx belongs to $\mathscr{M}_E^k$. Hence $\chi_A \varphi x \in \mathscr{L}$. This entails $\int_A \langle v(t), x\rangle \varphi(t)\mu(dt) = 0$, for each x and each φ. Hence $\langle v(\cdot), x\rangle = 0$ almost everywhere on A, hence on T. As E is the dual of E', it contains a sequence (e_n) which separates points of E' (Lemma 3). Therefore $v = 0$ a.e.

(2) The second part is obvious from the first.

DEFINITION 2. *A function $f: T \times E \to \bar{R}(= [-\infty, \infty])$ is said to be a normal integrand on $T \times E$ if for every t, $f(t, \cdot)$ is lower semi-continuous and f is $\mathscr{C} \otimes \mathscr{B}(E)$ measurable. It is said to be a convex normal integrand if it is a normal integrand and for every t, $f(t, \cdot)$ is convex.*

In the following lemma epi $f(t, \cdot)$ denotes

$$\{(x, r) \in E \times R \mid r \geqq f(t, x)\} .$$

LEMMA 7. *The function f is a normal integrand iff the set-valued function $t \mapsto$ epi $f(t, \cdot)$ is closed valued and its graph belongs to $\mathscr{C} \otimes \mathscr{B}(E) \times \mathscr{B}(R)$.*

Proof. First note that the closure of epi $f(t, \cdot)$ is equivalent to lower semi-continuity of $f(t, \cdot)$.

(1) Suppose f is a normal integrand. The graph G of $t \mapsto$ epi $f(t, \cdot)$ is given by the formula

$$G = \{(t, x, r) \mid r \geqq f(t, x)\}$$

and hence belongs to $\mathscr{C} \otimes \mathscr{B}(E) \otimes \mathscr{B}(R)$.

(2) Suppose that the graph G belongs to $\mathscr{C} \otimes \mathscr{B}(E) \otimes \mathscr{B}(R)$. Therefore, for each $r \in R$, $\{(t, x) \mid (t, x, r) \in G\}$ belongs to $\mathscr{C} \otimes \mathscr{B}(E)$ (Neveu [10] Prop. III-1-2). But

$$\{(t, x) \mid (t, x, r) \in G\} = \{(t, x) \mid f(t, x) \leqq r\} .$$

Thus f is $\mathscr{C} \otimes \mathscr{B}(E)$ measurable.

REMARK. It is easy to see (using the fact that R has a countable basis of open sets) that $\mathscr{B}(E) \otimes \mathscr{B}(R) = \mathscr{B}(E \times R)$.

LEMMA 8. *If f is a normal integrand on $T \times E$, then the function defined by*

$$f^*(t, x') = \sup \{\langle x', x\rangle - f(t, x) \mid x \in E\}$$

is a convex normal integrand on $T \times E'$.

Proof. By Lemma 7 (and the remark) the set-valued function $t \mapsto \operatorname{epi} f(t, \cdot)$ has a measurable graph. By Lemma 1 there exists a sequence of measurable selections (u_n, r_n) such that for every t the $(u_n(t), r_n(t))$ are dense in $\operatorname{epi} f(t, \cdot)$.
Thus by Lemma 4

$$f^*(t, x') = \sup_n [\langle x', u_n(t)\rangle - r_n(t)]$$

is a measurable function of (t, x').

3. Conjugate integral functionals.

DEFINITION 3. *Let f be a normal integrand on $T \times E$. An integral functional is defined on $\mathscr{L}$ by*
$I_f(u) = \int_T f(t, u(t))\mu(dt)$, *with the convention* $(+\infty) + (-\infty) = +\infty$, *that is the integral is* $+\infty$ *if positive and negative parts of* $f(t, u(t))$ *are nonintegrable.*

THEOREM. *If $\mathscr{L}$ is decomposable, if there exists $u_0 \in \mathscr{L}$ such that $I_f(u_0) < \infty$, then I_{f^*}, is the polar functional of I_f, that is, for every $v \in \mathscr{L}'$*

$$I_{f^*}(v) = \sup \{\langle v, u\rangle - I_f(u) \mid u \in \mathscr{L}\} .$$

If in addition f is convex, $\mathscr{L}'$ decomposable and $I_{f^*}(v_0) < \infty$ for at least one $v_0 \in \mathscr{L}'$, then I_f and I_{f^*} are mutually convex lower semicontinuous polar functional on $\mathscr{L}$ and $\mathscr{L}'$.

Proof. The proof follows Rockafellar [13].
We can rewrite the formula

$$\int \sup \{\langle v(t), x\rangle - f(t, x) \mid x \in E\}\mu(dt)$$
$$= \sup \left\{\int \langle v(t), u(t)\rangle\mu(dt) - \int f(t, u(t))\mu(dt) \mid u \in \mathscr{L}\right\} .$$

(If we rewrite the second member

$$\sup \left\{\int [\langle v(t), u(t)\rangle - f(t, u(t))]\mu(dt) \mid u \in \mathscr{L}\right\}$$

we have to use the opposite convention, that $-\infty$ prevails over $+\infty$.) Thus the inequality $\geqq$ is obvious. To prove $\leqq$, let $\beta \in \boldsymbol{R}$ such that $\beta < I_{f^*}(v)$ and let us find u such that $\langle v, u\rangle - I_f(u) \geqq \beta$.

(1) As $I_f(u_0) < \infty$ there exists $\alpha_0 \in \mathscr{L}^1$ such that $\langle v(t), u_0(t)\rangle - f(t, u_0(t)) \geqq \alpha_0(t)$ a.e. (for example one can take $\alpha_0(t) = \langle v(t), u_0(t)\rangle - f^+(t, u_0(t)))$. Remark that $f^*(t, v(t)) \geqq \alpha_0(t)$.

(2) Now we prove that there exists $\alpha_1 \in \mathscr{L}^1$ such that $\int \alpha_1(t)\mu(dt) > \beta$ and $\alpha_1(t) < f^*(t, v(t))$ a.e. Indeed let $h \in \mathscr{L}^1$ have strictly positive finite values (we recall that μ is σ-finite). If $I_{f^*}(v) < \infty$ put $\alpha_1(t) = f^*(t, v(t)) - \varepsilon h(t)$ with $\varepsilon > 0$ sufficiently small.

If $I_{f^*}(v) = +\infty$ put

$$\xi_n(t) = \begin{cases} \inf\left(nh(t), \dfrac{1}{2} f^*(t, v(t))\right) & \text{if} \quad f^*(t, v(t)) > 0 \\ f^*(t, v(t)) - h(t) & \text{if} \quad f^*(t, v(t)) \leqq 0 \,. \end{cases}$$

Then $\xi_n \in \mathscr{L}^1$, (ξ_n) is increasing, and $\xi_n(t) \to (1/2)f^*(t, v(t))$ if $f^*(t, v((t) > 0$. By the monotone convergence theorem $\int \xi_n \mu \to \infty$. Choose n large enough such that $\int \xi_n \mu > \beta$ and put $\alpha_1 = \xi_n$. In each of the three cases $f^*(t, v(t)) = +\infty$, finite > 0 or $\leqq 0$, one has $\alpha_1(t) < f^*(t, v(t))$.

(3) Let $\Gamma(t) = \{x \in E \mid \langle v(t), x\rangle - f(t, x) \geqq \alpha_1(t)\}$.

It is a closed almost everywhere nonempty set. The graph of Γ is

$$\{(t, x) \mid \langle v(t), x\rangle - f(t, x) \geqq \alpha_1(t)\}$$

and therefore belongs to $\mathscr{C} \otimes \mathscr{B}(E)$. Then (Lemma 1) Γ has a measurable selection u_1. By Lemma 5 there exists an increasing sequence (T_n) in $\mathscr{C}$ such that

- $\mu(T_n) < \infty$
- $T - \bigcup T_n$ is negligible
- $\overline{u_1(T_n)}$ is compact .

For n large enough one has

$$\int_{T_n} \alpha_1 \mu + \int_{T-T_n} \alpha_0 \mu \geqq \beta \,.$$

Put

$$u(t) = \begin{cases} u_1(t) & \text{if} \quad t \in T_n \\ u_0(t) & \text{if} \quad t \in T - T_n \,. \end{cases}$$

Then $u \in \mathscr{L}$ because $\mathscr{L}$ is decomposable.

On T_n one has

$$\langle v(t), u(t)\rangle - f(t, u(t)) \geqq \alpha_1(t)$$

and on $T - T_n$

$$\langle v(t), u(t)\rangle - f(t, u(t)) \geqq \alpha_0(t) .$$

Hence

$$\int_T \langle v(t), u(t)\rangle \mu(dt) - \int_T f(t, u(t))\mu(dt) \geqq \int_T \alpha_1\mu + \int_{T-T_n} \alpha_0\mu \geqq \beta .$$

(Note that $f^+(t, u(t))$ is integrable so that $\int f(t, u(t))\mu(dt)$ is not $+\infty$.) That proves the inequality $\leqq$. The remainder of the theorem is obvious.

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