THE INVARIANCE PRINCIPLE FOR WAVE OPERATORS

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The invariance principle for wave operators is proved.
It is shown that the existence of wave operators $W_{\pm}(B, A)$
does not imply the existence of $W_{\pm}(g(B), g(A))$, in general.

1. Introduction. Let $A$ and $B$ be two selfadjoint operators on
a separable Hilbert space $S$ and let $P_A$ and $P_B$ be the orthogonal
projections on the spaces of absolute continuity for $A$ and $B$, respectively. The wave operators $W_{\pm}(B, A)$ are defined by the strong limits

$$W_{\pm}(B, A) = \lim_{t \to \pm \infty} e^{itB} e^{-itA} P_A$$

when they exist (cf. [2, Chapter X]). The invariance principle of
M. S. Birman and T. Kato says: If the wave operators $W_{\pm}(B, A)$
and $W_{\pm}(g(B), g(A))$ exist and $g(\lambda)$ is real-valued and piecewise monotone
increasing, with a certain mild smoothness, then

$$W_{\pm}(g(B), g(A)) = W_{\pm}(B, A).$$

As stated by T. Kato and S. T. Kuroda in [3]: "It would be
nice if the existence of $W_{\pm}(B, A)$ implied the existence of $W_{\pm}(g(B),
g(A))$ and the invariance principle.

However, this has not been shown in general".

For example, the existence of $W_{\pm}(g(B), g(A))$ and the invariance
principle have been proved under the condition that $B - A$ or $(B - \xi)^{-1} - (A - \xi)^{-1} (\xi$ a nonreal number) is a trace-class operator (see for instance [2, Chapter X]).

The aim of this paper is

1. the proof of the invariance principle for wave operators,

2. the proof, that the existence of $W_{\pm}(B, A)$ does not imply the
existence of $W_{\pm}(g(B), g(A))$, in general.

In the present work we restrict our considerations to real-valued
functions $g(\lambda)$ on $(-\infty, \infty)$ with the following properties (cf. [2, p. 543]): The whole interval $(-\infty, \infty)$ can be divided into a countable
number of subintervals $\Delta_n$ with lengths $l_n$ in such a way that $\min l_n > 0$ and in each open subinterval $g(\lambda)$ is differentiable with $g'(\lambda)$
continuous, locally of bounded variation, and positive. A function
with these properties is called an allowable function. Furthermore,
we shall consider only the wave operators $W_+$ because the theorems
and proofs for the wave operators $W_-$ are entirely similar. In §2
we prove

**THEOREM 1** (invariance principle). Let $A$ and $B$ be two selfadjoint
operators on a separable Hilbert space $\mathcal{H}$ and let $g(\lambda)$ be an allowable
function. If $W_+(B, A)$ and $W_+(g(B), g(A))$ exist and if $W_+(B, A)$ is
complete, then $W_+(g(B), g(A)) = W_+(B, A)$.

From Theorem 1 we also see that the existence of $W_+(B, A)$,
$W_+(g(B), g(A))$ and the completeness of $W_+(B, A)$ imply the completeness
of $W_+((g, B), g(A))$. The next two theorems concern the existence of
the wave operator $W_+(g(B), g(A))$. They will be proved in §§2 and
3, respectively.

**THEOREM 2.** Let $A$ and $B$ be two selfadjoint operators on a
separable Hilbert space $\mathcal{H}$ with the absolutely continuous spectrum $\Lambda$
and let $g(\lambda)$ be an allowable function. If the wave operator $W_+(B, A)$
exists, is complete and if $g(\lambda)$ is piecewise linear on $\Lambda$, then
$W_+(g(B), g(A))$ exists.

**THEOREM 3.** Let $A$ be a selfadjoint operator on a separable
Hilbert space $\mathcal{H}$ with the absolutely continuous spectrum $\Lambda \neq 0$.
Let $g(\lambda)$ be an allowable function for which a finite interval $\Delta \subset (-\infty,\infty)$
with $|\Delta \cap \Lambda| \neq 0$ (Lebesgue measure) exists such that on $\Delta g'(\lambda)$
equals continuous strictly monotone function. Then there
is a selfadjoint operator $B$ such that $W_+(B, A)$ exists, is complete,
however, $W_+(g(B), g(A))$ does not exist.

It is easily seen, for instance, that all allowable functions $g(\lambda)$
which are piecewise twice continuously differentiable satisfy the assump-
tions either of Theorem 2 or of Theorem 3 for fixed $\Lambda$.

For the proofs of the theorems we use the following result of
H. Baumgärtel [1, Theorem 3]:

(NS) Let $W$ be a partial isometry with

$$ W^*W = P_A, \quad WW^* = P_B, \quad WAP_AW^* \supseteq BP_B. $$

Then $W_+(B, A)$ exists and $W = W_+(B, A)$ if and only if

$$ W = P_A + C, $$

$$ s\text{-}\lim_{t \to \infty} e^{-itA}Ce^{-itA}P_A = 0. $$
By Theorem 1 we obtain from (NS) that the existence and completeness of $W_+(B, A)$ imply the existence of $W_+(g(B), g(A))$ if and only if for the operator $C$ defined by (1.3) the strong limit

$$s-lim_{t \to \infty} e^{itg(A)}Ce^{-itg(A)}P_A = 0$$

exists. Here it was used that $P_{g(A)} = P_A$ for allowable functions $g(\lambda)$ (see §2).

Hence we know that the proof of Theorem 3 leads to the construction of an operator $C$ for which $s-lim_{t \to \infty} e^{itA}Ce^{-itA}P_A = 0$ and $s-lim_{t \to \infty} e^{itg(A)}Ce^{-itg(A)}P_A$ does not exist for the function $g(\lambda)$ defined by Theorem 3. To prove Theorem 2 we shall show that the equation

$$s-lim_{t \to \infty} e^{itA}Ce^{-itA}P_A = 0$$

implies $s-lim_{t \to \infty} e^{itg(A)}Ce^{-itg(A)}P_A = 0$ for piecewise linear functions $g(\lambda)$. The invariance principle will be proved by means of

**Lemma 1.** Let $T$ be a nonnegative bounded selfadjoint operator and $g(X)$ an allowable function. If the strong limits

$$s-lim_{t \to \infty} e^{itA}Te^{-itA}P_A = 0, s-lim_{t \to \infty} e^{itg(A)}Te^{-itg(A)}P_A$$

exist, then they are equal.

In §5 we prove Lemma 1 and formulate and prove two other lemmas which concern the behavior of the function $e^{-itg(\lambda)}$ for large $t$.

2. **Proof of Theorem 1.** First we introduce several notations and simple relations which are needed for the proof. As in §1 let $H$ be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$ and $P_H$ the orthogonal projection on the space of absolute continuity. We note that for every allowable function $g(\lambda)$

$$P_{g(H)} = P_H.$$  

(2.1) has been proved in [2, p. 545] for a class of functions slightly more restrictive than the allowable functions. The proof can easily be generalized for all allowable functions. Furthermore, we introduce the notations $\{H\}'$ for the commutant of $H$ and

$$V_H(X) \equiv s-lim_{t \to \infty} e^{itH}Xe^{-itH}P_H,$$

whenever for the bounded operator $X$ the strong limit exists. If $V_H(X)$ for the bounded operator $X$ exists, then we have the unambiguous decomposition (cf. [1])
(2.3) \[ X = X_1 + X_2 , \]
where
(2.4) \[ X_1 = P_H X_1 = X_1 P_H \in \{ H \}' , \]
(2.5) \[ V_H(X_2) = 0 . \]

For continuous functions \( f(\lambda) \) and a selfadjoint operator \( X \) one easily verifies that
(2.6) \[ f(V_H(X)) = V_H(f(X)) . \]

Now we prove Theorem 1. By (NS), we find that \( W_+(B, A) = P_A + C \) with \( V_+(C) = 0 \). Further we also have
(2.7) \[ V_+(C^*) = 0 , \]
since
\[
V_+(C^*) = V_+(C^* + P_A) - P_A = V_+(W_+(B, A)) - P_A \\
= \lim_{t \to \infty} e^{itA} W_+(B, A) e^{-itA} P_A - P_A \\
= \lim_{t \to \infty} W_+(B, A) e^{itB} e^{-itA} P_A - P_A \\
= W_+(B, A) P_A - P_A = 0
\]
with the intertwining relation \( W_+(B, A) e^{itA} = e^{itB} W_+(B, A) \). We define
(2.8) \[ W_1 = W_+(B, A) W_+(g(B), g(A)) . \]
From this definition we obtain that \( V_{\varphi \{ A \}}(W_+(B, A)) \) exists and
(2.9) \[ V_{\varphi \{ A \}}(W_+(B, A)) = W_1 \ , \]
since
\[
V_{\varphi \{ A \}}(W_+(B, A)) = \lim_{t \to \infty} e^{it\varphi(A)} W_+(B, A) e^{-it\varphi(A)} P_A \\
= \lim_{t \to \infty} W_+(B, A) e^{it\varphi(B)} e^{-it\varphi(A)} P_A = W_1 .
\]
By (2.3) and (2.5) then we have
(2.10) \[ W_+(B, A) = W_1 + C_1 \]
with
(2.11) \[ V_{\varphi \{ A \}}(C_1) = 0 \]
(2.12) \[ W_1 = P_A W_1 = W_1 P_A \in \{ g(A) \}' . \]

From the Definition (2.8) and the completeness of \( W_+(B, A) \) one
easily verifies that $W_1$ is a partially isometrie with
\begin{equation}
W_1^* W_1 = P_A, \quad W_1 W_1^* = P_1 \leq P_A .
\end{equation}
Further we have
\begin{equation}
V_{\tilde{\xi}(A)}((C^*_1 P_1) = 0 .
\end{equation}
This follows from
\begin{align*}
V_{\tilde{\xi}(A)}((C^*_1 P_1) &= 0 \iff V_{\tilde{\xi}(A)}(W_1^*(B, A) C^*_1 P_1) \\
&= V_{\tilde{\xi}(A)}(W_1^*(B, A) C^*_1 P_1 + C_1 W_1^* + P_1) - P_1 \\
&= V_{\tilde{\xi}(A)}(W_1^*(B, A) (C^*_1 + W_1^*) P_1) - P_1 \\
&= V_{\tilde{\xi}(A)}(W_1^*(B, A) W_1(B, A) P_1) - P_1 = 0
\end{align*}
with $V_{\tilde{\xi}(A)}((C^*_1 W_1^* + P_1) = P_1$ by $W_1^*, P_1 \in \{ g(A) \}'$. Combining (2.10) with $W_1^*(B, A) = P_A + C^*$ and (2.13) we obtain
\begin{align*}
CP_1 C^* &= (W_1 + C_1 - P_A)^* P_1 (W_1 + C_1 - P_A) \\
&= (W_1 - P_1)^* (W_1 - P_1) + (W_1 - P_1)^* C_1 + C_1^* P_1 (W_1 - P_1) \\
&\quad + C_1^* P_1 C_1 = (W_1 - P_1)^* (W_1 - P_1) + C_1
\end{align*}
By (2.11), (2.14) and $(W_1 - P_1), (W_1 - P_1)^* \in \{ g(A) \}'$ we have $V_{\tilde{\xi}(A)}((C_1) = 0$ and therefore,
\begin{equation}
V_{\tilde{\xi}(A)}(CP_1 C^*) = (W_1 - P_1)^* (W_1 - P_1) .
\end{equation}
Furthermore, it follows from (2.7) that
\begin{equation}
V_1(CP_1 C^*) = 0 .
\end{equation}
The operator $CP_1 C^*$ satisfies the assumptions of Lemma 1. Hence, we have $(W_1 - P_1)^* (W_1 - P_1) = 0$ and also $(W_1 - P_1) = 0$. With (2.13) and (2.8), we finally obtain $W_1 = P_A$ and
\begin{align*}
W_+(g(B), g(A)) &= W_+(B, A) .
\end{align*}

3. Proof of Theorem 2. We shall use the same notations as in §2. By Theorem 1 and (NS) it is necessary and sufficient for the existence of $W_+(g(B), g(A))$ that
\begin{equation}
V_{\tilde{\xi}(A)}((C) = 0 .
\end{equation}
Let $\varphi \in P_A \mathcal{H}, \varepsilon > 0$ and $P_A(A)$ be the spectral measure of $A$. Then there is a finite interval $A'$ such that
\begin{equation}
\| \varphi - P_A(A') \varphi \| \leq \frac{\varepsilon}{3\| C\|} .
\end{equation}
From the definition of the functions \( g(\lambda) \) in Theorem 2 we see that there exists a finite number of disjoint intervals \( \Delta_n \subset \mathcal{A} (n = 1, 2, \ldots, N) \) such that

\[
(3.3) \quad g(A) P_d P_d(\Delta_n) = q_n A P_d P_d(\Delta_n) ,
\]

\[
(3.4) \quad \left\| P_d(\Delta') \varphi - \sum_{n=1}^{N} P_d(\Delta_n) \varphi \right\| \leq \frac{\varepsilon}{3 \| C \|}
\]

with \( 0 < q_n < \infty \). By (3.2), (3.3), and (3.4) we find

\[
(3.5) \quad \left\| C e^{-itg(A)} P_d \varphi \right\| \leq \frac{2}{3} \varepsilon + \sum_{n=1}^{N} \left\| C e^{-itg_n \lambda} P_d(\Delta_n) \varphi \right\|.
\]

\( C \) satisfies the relation (see (NS))

\[
\left\| C e^{-itg \lambda} P_d \psi \right\| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty
\]

for every \( \psi \in \mathcal{H} \). Hence, for the functions \( \varphi_n = P_d(\Delta_n) \varphi \) there are numbers \( T_n \) such that

\[
(3.6) \quad \left\| C e^{-itg_n \lambda} \varphi_n \right\| \leq \frac{\varepsilon}{3N} \quad \text{for all} \quad t > T_n .
\]

By (3.5), (3.9) we obtain

\[
\left\| C e^{-itg(A)} P_d \varphi \right\| \leq \varepsilon \quad \text{for} \quad t > T = \max_{n} T_n
\]

and (3.1) is proved.

4. Proof of Theorem 3. For simplicity we shall assume that \( \Delta \subset \mathcal{A} \) and \( \mathcal{A} = [0, 2\pi] \). Let \( u \in P_d(A) P_d \mathcal{H} \) with \( P_d(\Delta') u \neq 0 \) for all \( \Delta' \subset \mathcal{A} \), \( \| \Delta' \| \neq 0 \). A restricted on the subspace \( \mathcal{H}_i = \overline{\text{sp} \{ P_d(\Delta') u, \Delta' \subset \Delta \}} \) Borel set) is an operator with simple absolutely continuous spectrum \( \Delta \). Hence we may identify \( \mathcal{H}_i \) with \( L^2(\Delta) \) and \( \lambda \) restricted on \( \mathcal{H}_i \) with the multiplication operator by \( \lambda \) on \( L^2(\Delta) \) denoted by \( H \).

Therefore it follows that for the proof of Theorem 3 it is sufficient to show that for \( H \) such an operator \( B \) defined by Theorem 3 exists. At first we construct a projector \( P \) such that \( V^+_H(P) = 0 \) and \( V^+_H(P) \) do not exist. We consider the function

\[
\frac{1}{\sqrt{2\pi}} e^{-itg(\lambda)} = \sum_{m} \psi_{m}^\lambda \frac{1}{\sqrt{2\pi}} e^{-it\lambda m} \in \mathcal{L}^2(\Delta) .
\]

By Lemma 2 for \( \varepsilon = 1/2 \) we can find sequences of natural numbers \( S_n, N_n \) with \( S_n, N_n \rightarrow \infty \) as \( n \rightarrow \infty \) such that

\[
(4.1) \quad \sum_{m=S_n}^{N_n} | \psi_{s}^m |^2 = a_n \geq 1 - \varepsilon = \frac{1}{2} .
\]
Now we define $P$ by
\begin{equation}
P = \sum_{n=1}^{\infty} f_n(\cdot, f_n),
\end{equation}
\begin{equation}
f_n(\lambda) = \sum_{m=\sigma_n}^{N_n+1-1} \frac{1}{\sqrt{a_n}} \psi_{S_n}^m \frac{1}{\sqrt{2\pi}} e^{-im\xi}.
\end{equation}

Next we prove $V^+_H(P) = 0$, i.e.,
\begin{equation}
\lim_{t \to \infty} \|P e^{-itH} \psi\| = 0 \quad \text{for every } \psi \in L^2(\mathcal{D}).
\end{equation}
As is easily shown, for the proof of (4.4) it is sufficient to consider the sequence $\|P e^{-itH} \psi_0\|$ with $n \to \infty$ ($n$ a natural number) and $\psi_0 = 1/\sqrt{2\pi}$.

We have
\begin{equation}
\|P e^{-itH} \psi_0\|^2 = \sum_{n=1}^{\infty} |(f_n, e^{-itH} \psi_0)|^2
= \sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f_n(\lambda) e^{ip\lambda} d\lambda \right|^2
= \sum_{n=1}^{\infty} \left| \sum_{m=\sigma_n}^{N_n+1-1} \psi_{S_n}^m \frac{1}{\sqrt{a_n}} \delta_{m, p} \right|^2
= \frac{A}{a_{x(p)}} |\psi_{S_n(p)}|^2,
\end{equation}
where $x(p) = r$ if $p \in (N_r, N_r + 1, \ldots, N_{r+1} - 1)$.

It is clear that $x(p) \to \infty$ as $p \to \infty$. Since $g(\lambda)$ satisfies the assumptions of Lemma 3 we find $|\psi_{S_n(p)}|^2 \to 0$ as $p \to \infty$ and also $\|P e^{-itH} \psi_0\|^2 \to 0$ as $p \to \infty$. This proves (4.4). To prove that $V^+_H(P)$ does not exist by Lemma 1 it is sufficient to show that there are a $\psi \in L^2(\mathcal{D})$, a sequence of real numbers $t_n \to \infty$ as $n \to \infty$ and an $X > 0$ such that
\begin{equation}
\|P e^{-it_n H} \psi\| > X \quad \text{for all } t_n.
\end{equation}
We set $\psi = \psi_0 = 1/\sqrt{2\pi}$ and $t_n = S_n$ (see (4.1)). Then by (4.1), (4.2)
\begin{align*}
\|P e^{-iS_n H} \psi_0\|^2 &= \sum_{n=1}^{\infty} |(f_n, e^{-iS_n H} \psi_0)|^2 \\
&\leq \|P e^{-iS_n H} \psi_0\|^2 = \left| \frac{1}{\sqrt{a_n}} \sum_{m=N_n}^{N_n+1-1} \psi_{S_n}^m \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\lambda e^{iS_n H \psi_0} e^{-im\xi} \right|^2 \\
&= \left| \frac{1}{\sqrt{a_n}} \sum_{m=N_n}^{N_n+1-1} |\psi_{S_n}^m|^2 \right|^2 = a_n \geq \frac{1}{2} \quad \text{for all } S_n.
\end{align*}

This proves that $U^+_H(P)$ does not exist.
Now we define by $U = 1 - 2P$ a unitary operator and we set
From the definitions of $U$, $B$, and (NS) it immediately follows that $W_{+}(B, H)$ exists and $U = W_{+}(B, H)$. Since, however, $V^+_{g,H}(-2P)$ does not exist, it follows from Theorem 1 and (NS) that also $W_{+}(g(B), g(H))$ does not exist.

5. **Proofs of the Lemmas.** Proof of Lemma 1: We shall prove Lemma 1 indirectly. Thus we suppose that for a nonnegative bounded selfadjoint operator $T$ and an allowable function $g(\lambda)$ the strong limits

\[ V^+_{\lambda}(T) = \operatorname{s-lim}_{t \to \infty} e^{itA} Te^{-itA} P \lambda = 0, \]

\[ V^+_{g(\lambda)}(T) = \operatorname{s-lim}_{t \to \infty} e^{itg(\lambda)A} Te^{-itg(\lambda)A} P \lambda = S \]

exist with $S \neq 0$, and from these assumptions we construct a contradiction. It is obvious that $S$ is also a nonnegative bounded selfadjoint operator with $S = SP\lambda = P\lambda SP\lambda \in [g(\lambda)]'$ by (2.3) to (2.5). By $S \neq 0$ it exists a $u \in P\lambda \mathbb{H}$ with $Su \neq 0$. From the definition of the allowable functions $g(\lambda)$ it follows that there is a finite interval $\Delta \subset (- \infty, \infty)$ such that $P\lambda(\Delta)u \neq 0$, $SP\lambda(\Delta)u \neq 0$ and $g(\lambda)$ is continuous, positive and of bounded variation on $\Delta$. For a nonnegative operator $S$ it follows from $Su \neq 0$ that also $(u, Su) \neq 0$. Hence we have $(P\lambda(\Delta)u, SP\lambda(\Delta)u) \neq 0$ and then $QSQ \neq 0$ where $Q$ is the orthogonal projection on the subspace $\mathbb{H}_1 = \overline{\text{sp}\{P\lambda(\Delta)u, \Delta' \subset \Delta\}}$. It is $Q \in \{A\}'$ and therefore $Q \in [g(\lambda)]'$. By $S \in [g(\lambda)]'$ we obtain $QSQ \in [g(\lambda)]'$. Since $g(\lambda)$ is strictly increasing on $\Delta$ it is clear that $(QAQ)' = (Qg(\lambda)Q)'$. From this identity and $QSQ \in [g(\lambda)]'$ we finally obtain $QSQ \in [A]'$. Furthermore, we have $E(\Delta) \in [A]'$, where $E(\Delta)$ is the spectral measure of $QSQ$. We choose a $\alpha > 0$ such that $E(0, \alpha) < Q$. With $R \equiv (Q - E(0, \alpha)) \in \{A\}'$ and (5.1), (5.2) we find

\[ V^+_{\lambda}(RTR) = 0, \quad V^+_{g(\lambda)}(RTR) = RSR \neq 0. \]

$RSR$ is a nonnegative selfadjoint operator with the spectrum $\delta \in 0 \cup [a, b](0 < a < b < \infty)$. Now we consider continuous functions $f(\lambda)$ which are 1 on $[a, b]$ and 0 in a neighborhood of 0.

By (2.6) and (5.3) we find

\[ V^+_{\lambda}(f(RTR)) = 0, \quad V^+_{g(\lambda)}(f(RTR)) = f(RSR) = R. \]

From the independence of the right sides of these $f(\lambda)$ it can easily be shown that (5.4) is also true for the step-function

\[ f(\lambda) = \begin{cases} 
1 & \text{on } [a_i, b_i] \ (0 < a_i < a < b < b_i < \infty) \\
0 & \lambda \in [a_i, b_i] \end{cases}. \]

Hence we have

\[ V^+_{\lambda}(P) = 0, \quad V^+_{g(\lambda)}(P) = R. \]
where $P = f(RTR)$ is an orthogonal projection with $P < R$. $\mathcal{H}_1$ reduces $A, P, R$ and $P, R$ are distinct from 0 only on $\mathcal{H}_1$. Thus it is sufficient to consider (5.5) in $\mathcal{H}_1$. A restricted on $\mathcal{H}_1$ is an operator with a simple absolutely continuous spectrum $\sigma \subset \Delta$. Then we may identify $\mathcal{H}_1$ with $L^2(\sigma)$ and a restricted on $\mathcal{H}_1$ with the multiplication operator by $\lambda$ and regard $\mathcal{H}_1 \equiv L^2(\sigma)$ as a subspace of the large Hilbert space $L^2(\Delta)$. In $L^2(\Delta)$ we may identify $R$ with the multiplication operator by $\chi_\sigma(\lambda)$, where $\chi_\sigma(\lambda)$ is the characteristic function on $\bar{\sigma} \subset \sigma$ with $|\bar{\sigma}| \neq 0$. $H$ denotes the multiplication operator by $\lambda$ in $L^2(\Delta)$. Then we obtain from (5.5)

\[(5.6) \quad \lim_{t \to \infty} \|P e^{-iHt} \psi\| = 0 \quad \text{for every } \psi \in L^2(\Delta),\]

\[(5.7) \quad \lim_{t \to \infty} \|P e^{-i\psi H^{-1} \psi} \| = \|\chi_\sigma \psi\| .\]

For the sake of simplicity, we shall assume that $\Delta = [0, 2\pi]$. We can write $g(\lambda) = \alpha \cdot \tilde{g}(\lambda) + \beta$, where $\tilde{g}(0) = 0$, $\tilde{g}(2\pi) = 2\pi$, and $\alpha, \beta$ are real numbers with $\alpha > 0$. Then we put

\[\varphi_n(\lambda) = 1/\sqrt{2\pi}e^{-i\lambda n}, \psi_n(\lambda) = 1/\sqrt{2\pi}\tilde{g}'(\lambda)e^{-i\lambda n}.\]

It is easy to verify that both $\varphi_n$ and $\psi_n$ form a complete orthonormal family in $L^2(\Delta)$. Furthermore, we have $\|\chi_\sigma \psi_n\|^2 = C > 0$. With these notations we easily obtain from (5.6), (5.7)

\[(5.8) \quad \|P \varphi_n\|^2 = \varepsilon_n \to 0 \quad \text{as } n \to \infty , \quad \|P \psi_n\|^2 = C = \alpha_n \to 0 \quad \text{as } n \to \infty .\]

We set $\psi_n(\lambda) = \sum_m a_m^* \varphi_m(\lambda)$. Now we consider the functions $\psi_n(\lambda)$ for which

\[(5.10) \quad \left\| \psi_n \right\|^2 = \frac{1}{\sqrt{2\pi}} \tilde{g}'(\lambda) e^{-i\lambda n} \leq \frac{C}{2} \]

with fixed $N > 0$. For the functions $\psi_n(\lambda) = (\tilde{g}'(\lambda))^{-1} \psi_n(\lambda)$ by Lemma 2 we have

\[\left\| \psi_n - \sum_{m=[q_1]}^{N} a_m^\ast \varphi_m \right\| \leq \varepsilon ,\]

where $q_1$, $q_2$ are positive real numbers independent of $s$, $\varepsilon$ and $p$ is a natural number independent of $s$. It is clear that then also

\[(5.11) \quad \left\| \psi_n - \sum_{m=[q_1]}^{N} a_m^\ast \varphi_m \right\|^2 \leq \frac{C}{2} ,\]

with an appropriate $p'$ independent of $s$. An elementary computation
shows that for all \( \psi_s \) with
\[
s \in \left( \left[ \frac{1 + p'}{q_1}, \frac{1 + p'}{q_1} \right], \left[ \frac{1 + p'}{q_1}, \frac{N - p'}{q_2} \right] \right)
\]
the inequality (5.10) is true.

Hence there are natural numbers \( N, s_i \) and an \( \alpha > 0 \) such that for every fixed \( N \) with \( N > N_i \) and \( s \in (s_i, s_i + 1, \ldots, [\alpha N] + s_i) \) \( \psi_s \) satisfies (5.10). Now we consider the sum
\[
S_N = \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N] + s_i} |(\varphi_n, \psi_s)|^2
\]
and introduce the orthogonal projection \( \bar{P} = 1 - P \). Then
\[
S_N = \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N] + s_i} |(p\varphi_n, \psi_s) + (\varphi_n, \bar{P}\psi_s)|^2
\]
\[
\leq \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N] + s_i} \left| (p\varphi_n, \psi_s) \right|^2 + \left| (\varphi_n, \bar{P}\psi_s) \right|^2 + 2\left| (p\varphi_n, \psi_s) \cdot (\varphi_n, \bar{P}\psi_s) \right|
\]
\[
\leq \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N] + s_i} \left| (p\varphi_n, \psi_s) \right|^2 + \left| (\varphi_n, \bar{P}\psi_s) \right|^2
\]
\[
+ 2\sqrt{\left( \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N] + s_i} \left| (p\varphi_n, \psi_s) \right|^2 \right) \left( \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N] + s_i} \left| (\varphi_n, \bar{P}\psi_s) \right|^2 \right)}
\]
(5.12)
\[
\leq \sum_{n=1}^{N} ||p\varphi_n||^2 + \sum_{s=s_1}^{[\alpha N] + s_i} ||\bar{P}\psi_s||^2
\]
\[
+ 2\sqrt{\left( \sum_{n=1}^{N} ||p\varphi_n||^2 \right) \left( \sum_{s=s_1}^{[\alpha N] + s_i} ||\bar{P}\psi_s||^2 \right)}
\].

By (5.9) we have \( ||\bar{P}\psi_s||^2 = 1 - C - \alpha_s \) and with (5.8), (5.12)

\[
S_N \leq \sum_{n=1}^{N} \varepsilon_n + [\alpha N](1 - C) \sum_{s=s_1}^{[\alpha N] + s_i} \alpha_s
\]
(5.13)
\[
+ 2\sqrt{\left( \sum_{n=1}^{N} \varepsilon_n \right) \left( 1 - C - \sum_{s=s_1}^{[\alpha N] + s_i} \alpha_s \right)}
\].

On the other hand, by (5.10) we find

\[
S_N \geq [\alpha N] \left( 1 - \frac{C}{2} \right)
\]
(5.14)

Combining (5.13), (5.14) we obtain
\[
\frac{C}{2} \leq \frac{1}{[\alpha N]} \left( \sum_{n=1}^{N} \varepsilon_n - \sum_{s=s_1}^{[\alpha N] + s_i} \alpha_s \right) + 2\sqrt{\frac{1}{[\alpha N]} \sum_{n=1}^{N} \varepsilon_n} \sqrt{1 - C} \frac{1}{[\alpha N]} \sum_{s=s_1}^{[\alpha N] + s_i} \alpha_s
\].

Since \( \varepsilon_n, \alpha_s \) are zero sequences, also
\[
\gamma_N = \frac{1}{[\alpha N]} \sum_{n=1}^{N} \varepsilon_n , \quad \delta_N = \frac{1}{[\alpha N]} \sum_{s=s_1}^{[\alpha N] + s_i} \alpha_s
\]
are zero sequences. Hence for sufficiently large $N$ the last inequality which we have got from the assumption $S \neq 0$ is not true, which proves Lemma 1.

**Lemma 2.** Let $g(\lambda)$ be a real-valued function and $g'(\lambda)$ a continuous positive function on $\mathcal{J} = [0, 2\pi]$. Then the functions $\tilde{\psi}_s(\lambda) \in L^2(\mathcal{J})$ defined by

$$
\tilde{\psi}_s(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-is\lambda} = \sum_{m=-\infty}^{\infty} \tilde{a}_m^s \frac{1}{\sqrt{2\pi}} e^{-im\lambda}
$$

possess the following properties: For every $\varepsilon$ with $0 < \varepsilon < 1$ and every natural number $s > 0$ there exist two real positive numbers $q_1, q_2$ independent of $s, \varepsilon$ and a natural number $p$ independent of $s$ such that

$$
\sum_{m=\lfloor s\varepsilon \rfloor - p}^{\lfloor s\varepsilon \rfloor + p} |\tilde{a}_m^s|^2 \geq 1 - \varepsilon.
$$

**Proof.** Let $a_1 = \min_{\lambda \in \mathcal{J}} g'(\lambda), a_2 = \max_{\lambda \in \mathcal{J}} g'(\lambda)$. Let $s$ be a fixed natural number. We consider integral numbers $m$ with $m > s \cdot a_2$ or $s \cdot a_1 > m$. For these $m$ we have $|s \cdot g'(\lambda) - m| > 0$ and we can write

$$
\tilde{a}_m^s = \frac{1}{2\pi} \int_0^{2\pi} d\lambda e^{-is\lambda} g'(\lambda) e^{im\lambda} = \frac{1}{2\pi} \int_0^{2\pi} d\lambda \frac{1}{-i(sg'(\lambda) - m)} \left( \frac{d}{d\lambda} e^{-is\lambda} \right)
$$

Integrating by parts and an elementary computation shows that

$$
|\tilde{a}_m^s| \leq \frac{1}{2\pi} \left[ \frac{1}{-i(sg'(\lambda) - m)} \right]_0^{2\pi} + \int_0^{2\pi} e^{-is\lambda} g'(\lambda) d\left( \frac{1}{i(g'(\lambda)s - m)} \right)\left( \frac{1}{s\alpha - m} \right) + \frac{2}{\alpha - m^2}+

\left( \frac{M}{s\alpha - m} \right) + \frac{2}{\alpha - m^2}+\frac{M}{s\alpha - m^2}+\frac{M}{s\alpha - m^2},
$$

where $M$ is the total variation of $g'(\lambda)$ on $\mathcal{J}$ and $\alpha = a_1$ if $m < s \cdot a_1$, or $\alpha = a_2$ if $m > s \cdot a_2$. Let $p'$ be a positive integral number, then by (5.17) we have
\[ \sum_{m = \lceil s/2a_1 \rceil - p' - 1}^{\infty} |\bar{\alpha}_m^s|^2 \leq \frac{1}{(2\pi)^2} \left( 2 + \frac{M}{a_2} \right)^2 \sum_{m = \lceil s/2a_1 \rceil - p' - 1}^{\infty} \frac{1}{|m - [sa_2] - 1|^2} \]

and entirely analogous

\[ \sum_{m = \lceil s/2a_1 \rceil - p' - 1}^{\infty} |\bar{\alpha}_m^s|^2 \leq \frac{1}{(2\pi)^2} \left( 2 + \frac{2M}{a_1} \right)^2 \sum_{m = \lceil s/2a_1 \rceil - p' - 1}^{\infty} \frac{1}{|sa_1 - m|^2} \]

where \([\alpha]\) is the smallest integer \(r > a - 1\).

For sufficiently large \(p'\) from (5.18), (5.19) we find

\[ \sum_{m = \lceil s/2a_2 \rceil + p' + 1}^{\infty} |\bar{\alpha}_m^s|^2 + \sum_{m = \lceil s/2a_1 \rceil - p' - 1}^{\infty} |\bar{\alpha}_m^s|^2 < \varepsilon \]

for all positive integral numbers \(s\) and every \(\varepsilon > 0\). With \(q_1 = (1/2)a_1\), \(q_2 = 2a_2\) and by \(|\gamma_s|^2 = \sum_m |\bar{\alpha}_m^s|^2 = 1\) we finally obtain (5.16).

**Lemma 3.** Let \(g(\lambda), \bar{\gamma}_s(\lambda)\) be defined as in Lemma 2 and let \(g'(\lambda)\) be continuous, strictly monotone on \(A\). Then the functions \(\bar{\gamma}_s(\lambda)\) possess the following properties: For every \(\varepsilon\) with \(0 < \varepsilon < 1\) there exists an \(N\) such that for all integral numbers \(m, s\) with \(s > N\)

\[ |\bar{\alpha}_m^s| < \varepsilon \]

Proof. From the continuity and strict monotony of the positive function \(g'(\lambda)\) on \(A\) it follows that for every real number \(x\) and \(\varepsilon > 0\) there is an interval \(A_x \subseteq A\) of the length \(l_x \leq \varepsilon\) such that

\[ \alpha(\varepsilon) = \min (\min_{x \in A_x} |g'(\lambda) + x|) \]

exists and \(\alpha(\varepsilon) > 0\). Hence with \(x = -m/s\) we have

\[ |\bar{\alpha}_m^s| = \frac{1}{2\pi} \int_{0}^{2\pi} d\lambda e^{-i\lambda(x - \frac{1}{2})} e^{im\lambda} \leq \frac{\varepsilon}{2} + \frac{1}{2\pi} \left| \int_{A_x} d\lambda e^{-i\lambda(x - \frac{1}{2})} \right| \]

The domain of integration \((A - A_x)\) consists of one or two intervals in dependence on \(x\) and \(\varepsilon\). Let \(A' = [a, b] \subseteq A\) be such an interval. Then

\[ \left| \int_{A'} d\lambda e^{-i\lambda(x - \frac{1}{2})} \right| = \left| \int_{A'} \frac{d\lambda}{-i\lambda(x - \frac{1}{2})} \left( \frac{d}{d\lambda} e^{-i\lambda(x - \frac{1}{2})} \right) \right| \]
\[ \leq \left| \int_{s}^{1} e^{-i\theta(g'(\lambda) - x)} \right| \]

\[ \leq \frac{2}{s \cdot \alpha} + \int_{s}^{1} \left| \frac{1}{\theta(g'(\lambda) - x)} \right| \]

\[ \leq \frac{2}{s \cdot \alpha} + \frac{M}{s \cdot \alpha} \]

where \( \alpha \) is defined by (5.21) and \( M = |g'(b) - g'(a)| \). From (5.22) and (5.23) we have

\[ |a_{\epsilon}| \leq \epsilon + \frac{1}{\pi \epsilon} \cdot \frac{2 + M}{\alpha} . \]

If we put \( N = \frac{2 + M}{\alpha} \), then this implies (5.20).

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REFERENCES


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AKADEMIE DER WISSENSCHAFTEN DER DDR
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shashi Prabha Arya and M. K. Singal, <em>More sum theorems for topological spaces</em></td>
<td>1</td>
</tr>
<tr>
<td>Kong Ming Chong, <em>Spectral inequalities involving the infima and suprema of functions</em></td>
<td>17</td>
</tr>
<tr>
<td>Alan Hetherington Durfee, <em>The characteristic polynomial of the monodromy</em></td>
<td>21</td>
</tr>
<tr>
<td>Emilio Gagliardo and Clifford Alfons Kottman, <em>Fixed points for orientation preserving homeomorphisms of the plane which interchange two points</em></td>
<td>27</td>
</tr>
<tr>
<td>Raymond F. Gittings, <em>Finite-to-one open maps of generalized metric spaces</em></td>
<td>33</td>
</tr>
<tr>
<td>Andrew M. W. Glass, W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, <em>a</em>-closures of completely distributive lattice-ordered groups</td>
<td>43</td>
</tr>
<tr>
<td>Matthew Gould, <em>Endomorphism and automorphism structure of direct squares of universal algebras</em></td>
<td>69</td>
</tr>
<tr>
<td>R. E. Harrell and Les Andrew Karlovitz, <em>On tree structures in Banach spaces</em></td>
<td>85</td>
</tr>
<tr>
<td>Julien O. Hennefeld, <em>Finding a maximal subalgebra on which the two Arens products agree</em></td>
<td>93</td>
</tr>
<tr>
<td>William Francis Keigher, <em>Adjunctions and comonads in differential algebra</em></td>
<td>99</td>
</tr>
<tr>
<td>Robert Bernard Kelman, <em>A Dirichlet-Jordan theorem for dual trigonometric series</em></td>
<td>113</td>
</tr>
<tr>
<td>Allan Morton Krall, <em>Stieltjes differential-boundary operators. III. Multivalued operators–linear relations</em></td>
<td>125</td>
</tr>
<tr>
<td>Hui-Hsiung Kuo, <em>On Gross differentiation on Banach spaces</em></td>
<td>135</td>
</tr>
<tr>
<td>Tom Louton, <em>A theorem on simultaneous observability</em></td>
<td>147</td>
</tr>
<tr>
<td>Kenneth Mandelberg, <em>Amitsur cohomology for certain extensions of rings of algebraic integers</em></td>
<td>161</td>
</tr>
<tr>
<td>Coy Lewis May, <em>Automorphisms of compact Klein surfaces with boundary</em></td>
<td>199</td>
</tr>
<tr>
<td>Peter A. McCoy, <em>Generalized axisymmetric elliptic functions</em></td>
<td>211</td>
</tr>
<tr>
<td>Muril Lynn Robertson, <em>Concerning Siu's method for solving ( y'(t) = F(t, y(g(t))) )</em></td>
<td>223</td>
</tr>
<tr>
<td>Richard Lewis Roth, <em>On restricting irreducible characters to normal subgroups</em></td>
<td>229</td>
</tr>
<tr>
<td>Albert Oscar Shar, <em>P-primary decomposition of maps into an H-space</em></td>
<td>237</td>
</tr>
<tr>
<td>Kenneth Barry Stolarsky, <em>The sum of the distances to certain pointsets on the unit circle</em></td>
<td>241</td>
</tr>
<tr>
<td>Bert Alan Taylor, <em>Components of zero sets of analytic functions in ( C^2 ) in the unit ball or polydisc</em></td>
<td>253</td>
</tr>
<tr>
<td>Michel Valadier, <em>Convex integrands on Souslin locally convex spaces</em></td>
<td>267</td>
</tr>
<tr>
<td>Januario Varela, <em>Fields of automorphisms and derivations of ( C^</em>-)algebras*</td>
<td>277</td>
</tr>
<tr>
<td>Arnold Lewis Villone, <em>A class of symmetric differential operators with deficiency indices ((1, 1))</em></td>
<td>295</td>
</tr>
<tr>
<td>Manfred Wollenberg, <em>The invariance principle for wave operators</em></td>
<td>303</td>
</tr>
</tbody>
</table>