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THE INVARIANCE PRINCIPLE FOR WAVE OPERATORS

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**The invariance principle for wave operators is proved.
 It is shown that the existence of wave operators $W_{\pm}(B, A)$
 does not imply the existence of $W_{\pm}(g(B), g(A))$, in general.**

1. Introduction. Let A and B be two selfadjoint operators on a separable Hilbert space \mathcal{H} and let P_A and P_B be the orthogonal projections on the spaces of absolute continuity for A and B , respectively. The wave operators $W_{\pm}(B, A)$ are defined by the strong limits

$$(1.1) \quad W_{\pm}(B, A) \equiv s\text{-}\lim_{t \rightarrow \pm\infty} e^{itB} e^{-itA} P_A$$

when they exist (cf. [2, Chapter X]). The invariance principle of M. S. Birman and T. Kato says: If the wave operators $W_{\pm}(B, A)$ and $W_{\pm}(g(B), g(A))$ exist and $g(\lambda)$ is real-valued and piecewise monotone increasing, with a certain mild smoothness, then

$$(1.2) \quad W_{\pm}(g(B), g(A)) = W_{\pm}(B, A) .$$

As stated by T. Kato and S. T. Kuroda in [3]: "It would be nice if the existence of $W_{\pm}(B, A)$ implied the existence of $W_{\pm}(g(B), g(A))$ and the invariance principle.

However, this has not been shown in general".

For example, the existence of $W_{\pm}(g(B), g(A))$ and the invariance principle have been proved under the condition that $B - A$ or $(B - \xi)^{-1} - (A - \xi)^{-1}$ (ξ a nonreal number) is a trace-class operator (see for instance [2, Chapter X]).

The aim of this paper is

1. the proof of the invariance principle for wave operators,
2. the proof, that the existence of $W_{\pm}(B, A)$ does not imply the existence of $W_{\pm}(g(B), g(A))$, in general.

In the present work we restrict our considerations to real-valued functions $g(\lambda)$ on $(-\infty, \infty)$ with the following properties (cf. [2, p. 543]): The whole interval $(-\infty, \infty)$ can be divided into a countable number of subintervals I_n with lengths l_n in such a way that $\min l_n > 0$ and in each open subinterval $g(\lambda)$ is differentiable with $g'(\lambda)$

continuous, locally of bounded variation, and positive. A function with these properties is called an allowable function. Furthermore, we shall consider only the wave operators W_+ because the theorems and proofs for the wave operators W_- are entirely similar. In §2 we prove

THEOREM 1 (invariance principle). *Let A and B be two selfadjoint operators on a separable Hilbert space \mathcal{H} and let $g(\lambda)$ be an allowable function. If $W_+(B, A)$ and $W_+(g(B), g(A))$ exist and if $W_+(B, A)$ is complete, then $W_+(g(B), g(A)) = W_+(B, A)$.*

From Theorem 1 we also see that the existence of $W_+(B, A)$, $W_+(g(B), g(A))$ and the completeness of $W_+(B, A)$ imply the completeness of $W_+(g(B), g(A))$. The next two theorems concern the existence of the wave operator $W_+(g(B), g(A))$. They will be proved in §§2 and 3, respectively.

THEOREM 2. *Let A and B be two selfadjoint operators on a separable Hilbert space \mathcal{H} with the absolutely continuous spectrum A and let $g(\lambda)$ be an allowable function. If the wave operator $W_+(B, A)$ exists, is complete and if $g(\lambda)$ is piecewise linear on A , then $W_+(g(B), g(A))$ exists.*

THEOREM 3. *Let A be a selfadjoint operator on a separable Hilbert space \mathcal{H} with the absolutely continuous spectrum $A \neq 0$. Let $g(\lambda)$ be an allowable function for which a finite interval $\Delta \subset (-\infty, \infty)$ with $|\Delta \cap A| \neq 0$ (Lebesgue measure) exists such that on Δ $g'(\lambda)$ exists and is a continuous strictly monotone function. Then there is a selfadjoint operator B such that $W_+(B, A)$ exists, is complete, however, $W_+(g(B), g(A))$ does not exist.*

It is easily seen, for instance, that all allowable functions $g(\lambda)$ which are piecewise twice continuously differentiable satisfy the assumptions either of Theorem 2 or of Theorem 3 for fixed A .

For the proofs of the theorems we use the following result of H. Baumgärtel [1, Theorem 3]:

(NS) Let W be a partial isometry with

$$W^*W = P_A, \quad WW^* = P_B, \quad WAP_AW^* \supseteq BP_B.$$

Then $W_+(B, A)$ exists and $W = W_+(B, A)$ if and only if

$$(1.3) \quad W = P_A + C,$$

$$(1.4) \quad s\text{-}\lim_{t \rightarrow \infty} e^{itA} C e^{-itA} P_A = 0.$$

By Theorem 1 we obtain from (NS) that the existence and completeness of $W_+(B, A)$ imply the existence of $W_+(g(B), g(A))$ if and only if for the operator C defined by (1.3) the strong limit

$$(1.5) \quad s\text{-}\lim_{t \rightarrow \infty} e^{itg(A)} C e^{-itg(A)} P_A = 0$$

exists. Here it was used that $P_{g(A)} = P_A$ for allowable functions $g(\lambda)$ (see §2).

Hence we know that the proof of Theorem 3 leads to the construction of an operator C for which $s\text{-}\lim_{t \rightarrow \infty} e^{itA} C e^{-itA} P_A = 0$ and $s\text{-}\lim_{t \rightarrow \infty} e^{itg(A)} C e^{-itg(A)} P_A$ does not exist for the function $g(\lambda)$ defined by Theorem 3. To prove Theorem 2 we shall show that the equation

$$s\text{-}\lim_{t \rightarrow \infty} e^{itA} C e^{-itA} P_A = 0$$

implies $s\text{-}\lim_{t \rightarrow \infty} e^{itg(A)} C e^{-itg(A)} P_A = 0$ for piecewise linear functions $g(\lambda)$. The invariance principle will be proved by means of

LEMMA 1. *Let T be a nonnegative bounded selfadjoint operator and $g(\lambda)$ an allowable function. If the strong limits*

$$s\text{-}\lim_{t \rightarrow \infty} e^{itA} T e^{-itA} P_A = 0, \quad s\text{-}\lim_{t \rightarrow \infty} e^{itg(A)} T e^{-itg(A)} P_A$$

exist, then they are equal.

In §5 we prove Lemma 1 and formulate and prove two other lemmas which concern the behavior of the function $e^{-itg(\lambda)}$ for large t .

2. *Proof of Theorem 1.* First we introduce several notations and simple relations which are needed for the proof. As in §1 let H be a selfadjoint operator on a separable Hilbert space \mathcal{H} and P_H be the orthogonal projection on the space of absolute continuity. We note that for every allowable function $g(\lambda)$

$$(2.1) \quad P_{g(H)} = P_H.$$

(2.1) has been proved in [2, p. 545] for a class of functions slightly more restrictive than the allowable functions. The proof can easily be generalized for all allowable functions. Furthermore, we introduce the notations $\{H\}'$ for the commutant of H and

$$(2.2) \quad V_H^+(X) \equiv s\text{-}\lim_{t \rightarrow \infty} e^{itH} X e^{-itH} P_H,$$

whenever for the bounded operator X the strong limit exists. If $V_H^+(X)$ for the bounded operator X exists, then we have the unambiguous decomposition (cf. [1])

$$(2.3) \quad X = X_1 + X_2 ,$$

where

$$(2.4) \quad X_1 = P_H X_1 = X_1 P_H \in \{H\}' ,$$

$$(2.5) \quad V_H^+(X_2) = 0 .$$

For continuous functions $f(\lambda)$ and a selfadjoint operator X one easily verifies that

$$(2.6) \quad f(V_H^+(X)) = V_H^+(f(X)) .$$

Now we prove Theorem 1. By (NS), we find that $W_+(B, A) = P_A + C$ with $V_A^+(C) = 0$. Further we also have

$$(2.7) \quad V_A^+(C^*) = 0 ,$$

since

$$\begin{aligned} V_A^+(C^*) &= V_A^+(C^* + P_A) - P_A = V_A^+(W_+^*(B, A)) - P_A \\ &= s\text{-}\lim_{t \rightarrow \infty} e^{itA} W_+^*(B, A) e^{-itA} P_A - P_A \\ &= s\text{-}\lim_{t \rightarrow \infty} W_+^*(B, A) e^{itB} e^{-itA} P_A - P_A \\ &= W_+^*(B, A) W_+(B, A) - P_A = 0 \end{aligned}$$

with the intertwining relation $W_+(B, A) e^{itA} = e^{itB} W_+(B, A)$. We define

$$(2.8) \quad W_1 \equiv W_+^*(B, A) W_+(g(B), g(A)) .$$

From this definition we obtain that $V_{g(A)}^+(W_+^*(B, A))$ exists and

$$(2.9) \quad V_{g(A)}^+(W_+^*(B, A)) = W_1 ,$$

since

$$\begin{aligned} V_{g(A)}^+(W_+^*(B, A)) &= s\text{-}\lim_{t \rightarrow \infty} e^{itg(A)} W_+^*(B, A) e^{-itg(A)} P_A \\ &= s\text{-}\lim_{t \rightarrow \infty} W_+^*(B, A) e^{itg(B)} e^{-itg(A)} P_A = W_1 . \end{aligned}$$

By (2.3) and (2.5) then we have

$$(2.10) \quad W_+^*(B, A) = W_1 + C_1$$

with

$$(2.11) \quad V_{g(A)}^+(C_1) = 0$$

$$(2.12) \quad W_1 = P_A W_1 = W_1 P_A \in \{g(A)\}' .$$

From the Definition (2.8) and the completeness of $W_+(B, A)$ one

easily verifies that W_1 is a partially isometric with

$$(2.13) \quad W_1^* W_1 = P_A, \quad W_1 W_1^* = P_1 \leq P_A.$$

Further we have

$$(2.14) \quad V_{g(A)}^+(C_1^* P_1) = 0.$$

This follows from

$$\begin{aligned} V_{g(A)}^+(C_1^* P_1) = 0 &\longleftrightarrow V_{g(A)}^+(W_+^*(B, A) C_1^* P_1) \\ &= V_{g(A)}^+(W_+^*(B, A) C_1^* P_1 + C_1 W_1^* + P_1) - P_1 \\ &= V_{g(A)}^+(W_+^*(B, A) (C_1^* + W_1^*) P_1) - P_1 \\ &= V_{g(A)}^+(W_+^*(B, A) W_+(B, A) P_1) - P_1 = 0 \end{aligned}$$

with $V_{g(A)}^+(C_1 W_1^* + P_1) = P_1$ by $W_1^*, P_1 \in \{g(A)\}'$. Combining (2.10) with $W_+^*(B, A) = P_A + C^*$ and (2.13) we obtain

$$\begin{aligned} CP_1 C^* &= (W_1 + C_1 - P_A)^* P_1 (W_1 + C_1 - P_A) \\ &= (W_1 - P_1)^* (W_1 - P_1) + (W_1 - P_1)^* C_1 + C_1^* P_1 (W_1 - P_1) \\ &\quad + C_1^* P_1 C_1 = (W_1 - P_1)^* (W_1 - P_1) + C_2. \end{aligned}$$

By (2.11), (2.14) and $(W_1 - P_1), (W_1 - P_1)^* \in \{g(A)\}'$ we have $V_{g(A)}^+(C_2) = 0$ and therefore,

$$(2.15) \quad V_{g(A)}^+(CP_1 C^*) = (W_1 - P_1)^* (W_1 - P_1).$$

Furthermore, it follows from (2.7) that

$$(2.16) \quad V_A^+(CP_1 C^*) = 0.$$

The operator $CP_1 C^*$ satisfies the assumptions of Lemma 1. Hence, we have $(W_1 - P_1)^* (W_1 - P_1) = 0$ and also $(W_1 - P_1) = 0$. With (2.13) and (2.8), we finally obtain $W_1 = P_A$ and

$$W_+(g(B), g(A)) = W_+(B, A).$$

3. *Proof of Theorem 2.* We shall use the same notations as in §2. By Theorem 1 and (NS) it is necessary and sufficient for the existence of $W_+(g(B), g(A))$ that

$$(3.1) \quad V_{g(A)}^+(C) = 0.$$

Let $\varphi \in P_A \mathcal{H}$, $\varepsilon > 0$ and $P_A(\mathcal{A})$ be the spectral measure of A . Then there is a finite interval \mathcal{A}' such that

$$(3.2) \quad \|\varphi - P_A(\mathcal{A}')\varphi\| \leq \frac{\varepsilon}{3\|C\|}.$$

From the definition of the functions $g(\lambda)$ in Theorem 2 we see that there exists a finite number of disjoint intervals $\Delta_n \subset \Delta' (n = 1, 2, \dots, N)$ such that

$$(3.3) \quad g(\Delta)P_\Delta P_\Delta(\Delta_n) = q_n A P_\Delta P_\Delta(\Delta_n),$$

$$(3.4) \quad \left\| P_\Delta(\Delta')\varphi - \sum_{n=1}^N P_\Delta(\Delta_n)\varphi \right\| \leq \frac{\varepsilon}{3\|C\|}$$

with $0 < q_n < \infty$. By (3.2), (3.3), and (3.4) we find

$$(3.5) \quad \|C e^{-itg(\Delta)} P_\Delta \varphi\| \leq \frac{2}{3}\varepsilon + \sum_{n=1}^N \|C e^{-itq_n A} P_\Delta(\Delta_n)\varphi\|.$$

C satisfies the relation (see (NS))

$$\|C e^{-itA} P_\Delta \psi\| \longrightarrow 0 \quad \text{as } t \longrightarrow \infty$$

for every $\psi \in \mathcal{H}$. Hence, for the functions $\varphi_n = P_\Delta(\Delta_n)\varphi$ there are numbers T_n such that

$$(3.6) \quad \|C e^{-itq_n A} \varphi_n\| \leq \frac{\varepsilon}{3N} \quad \text{for all } t > T_n.$$

By (3.5), (3.9) we obtain

$$\|C e^{-itg(\Delta)} P_\Delta \varphi\| \leq \varepsilon \quad \text{for } t > T = \max_n T_n$$

and (3.1) is proved.

4. Proof of Theorem 3. For simplicity we shall assume that $\Delta \subset \mathbb{A}$ and $\Delta = [0, 2\pi]$. Let $u \in P_\Delta(\Delta)P_\Delta \mathcal{H}$ with $P_\Delta(\Delta')u \neq 0$ for all $\Delta' \subset \Delta$, $|\Delta'| \neq 0$. A restricted on the subspace $\mathcal{H}_1 = \overline{\text{sp}} \{P_\Delta(\Delta')u, \Delta' \subset \Delta \text{ Borel set}\}$ is an operator with simple absolutely continuous spectrum Δ . Hence we may identify \mathcal{H}_1 with $\mathcal{L}^2(\Delta)$ and a restricted on \mathcal{H}_1 with the multiplication operator by λ on $\mathcal{L}^2(\Delta)$ denoted by H .

Therefore it follows that for the proof of Theorem 3 it is sufficient to show that for H such an operator B defined by Theorem 3 exists. At first we construct a projector P such that $V_H^+(P) = 0$ and $V_{\sigma(H)}^+(P)$ do not exist. We consider the function

$$\frac{1}{\sqrt{2\pi}} e^{-itg(\lambda)} = \sum_m \psi_\varepsilon^m \frac{1}{\sqrt{2\pi}} e^{-i\lambda m} \in \mathcal{L}^2(\Delta).$$

By Lemma 2 for $\varepsilon = 1/2$ we can find sequences of natural numbers S_n, N_n with $S_n, N_n \rightarrow \infty$ as $n \rightarrow \infty$ such that

$$(4.1) \quad \sum_{m=N_n}^{N_n+1-1} |\psi_{S_n}^m|^2 = a_n \geq 1 - \varepsilon = \frac{1}{2}.$$

Now we define P by

$$(4.2) \quad P = \sum_{n=1}^{\infty} f_n(\cdot, f_n),$$

$$(4.3) \quad f_n(\lambda) = \sum_{m=N_n}^{N_{n+1}-1} \frac{1}{\sqrt{a_n}} \psi_{S_n}^m \frac{1}{\sqrt{2\pi}} e^{-i\lambda m}.$$

Next we prove $V_H^+(P) = 0$, i.e.,

$$(4.4) \quad \lim_{t \rightarrow \infty} \|Pe^{-itH}\psi\| = 0 \quad \text{for every } \psi \in \mathcal{L}^2(\Delta).$$

As is easily shown, for the proof of (4.4) it is sufficient to consider the sequence $\|Pe^{-inH}\psi_0\|$ with $n \rightarrow \infty$ (n a natural number) and $\psi_0 = 1/\sqrt{2\pi}$.

We have

$$(4.5) \quad \begin{aligned} \|Pe^{-ipH}\psi_0\|^2 &= \sum_{n=1}^{\infty} |(f_n, e^{-ipH}\psi_0)|^2 \\ &= \sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f_n(\lambda) e^{ip\lambda} d\lambda \right|^2 \\ &= \sum_{n=1}^{\infty} \left| \sum_{m=N_n}^{N_{n+1}-1} \psi_{S_n}^m \frac{1}{\sqrt{a_n}} \delta_{m;p} \right|^2 \\ &= \frac{A}{a_{x(p)}} |\psi_{S_{x(p)}}^p|^2, \end{aligned}$$

where $x(p) = r$ if $p \in (N_r, N_{r+1}, \dots, N_{r+1} - 1)$.

It is clear that $x(p) \rightarrow \infty$ as $p \rightarrow \infty$. Since $g(\lambda)$ satisfies the assumptions of Lemma 3 we find $|\psi_{S_{x(p)}}^p|^2 \rightarrow 0$ as $p \rightarrow \infty$ and also $\|Pe^{-ipH}\psi_0\|^2 \rightarrow 0$ as $p \rightarrow \infty$. This proves (4.4). To prove that $V_{g(H)}^+(P)$ does not exist by Lemma 1 it is sufficient to show that there are a $\psi \in \mathcal{L}^2(\Delta)$, a sequence of real numbers $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and an $X > 0$ such that

$$(4.6) \quad \|Pe^{-it_n g(H)}\psi\| > X \quad \text{for all } t_n.$$

We set $\psi = \psi_0 = 1/\sqrt{2\pi}$ and $t_n = S_n$ (see (4.1)). Then by (4.1), (4.2)

$$\begin{aligned} \|Pe^{-iS_n g(H)}\psi_0\|^2 &= \sum_{q=1}^{\infty} |(f_q, e^{-iS_n g(H)}\psi_0)|^2 \\ &\geq |(f_n, e^{-iS_n g(H)}\psi_0)|^2 = \left| \frac{1}{\sqrt{a_n}} \sum_{m=N_n}^{N_{n+1}-1} \psi_{S_n}^m \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\lambda e^{iS_n g(\lambda)} e^{-im\lambda} \right|^2 \\ &= \left| \frac{1}{\sqrt{a_n}} \sum_{m=N_n}^{N_{n+1}-1} |\psi_{S_n}^m|^2 \right|^2 = a_n \geq \frac{1}{2} \quad \text{for all } S_n. \end{aligned}$$

This proves that $U_{g(H)}^+(P)$ does not exist.

Now we define by $U \equiv 1 - 2P$ a unitary operator and we set

$B = UHU$. From the definitions of U , B , and (NS) it immediately follows that $W_+(B, H)$ exists and $U = W_+(B, H)$. Since, however, $V_{g(H)}^+(-2P)$ does not exist, it follows from Theorem 1 and (NS) that also $W_+(g(B), g(H))$ does not exist.

5. *Proofs of the Lemmas.* Proof of Lemma 1: We shall prove Lemma 1 indirectly. Thus we suppose that for a nonnegative bounded selfadjoint operator T and an allowable function $g(\lambda)$ the strong limits

$$(5.1) \quad V_A^+(T) = s\text{-}\lim_{t \rightarrow \infty} e^{itA} T e^{-itA} P_A = 0,$$

$$(5.2) \quad V_{g(A)}^+(T) = s\text{-}\lim_{t \rightarrow \infty} e^{itg(A)} T e^{-itg(A)} P_A = S$$

exist with $S \neq 0$, and from these assumptions we construct a contradiction. It is obvious that S is also a nonnegative bounded selfadjoint operator with $S = SP_A = P_A S \in \{g(A)\}'$ by (2.3) to (2.5). By $S \neq 0$ it exists a $u \in P_A \mathcal{H}$ with $Su \neq 0$. From the definition of the allowable functions $g(\lambda)$ it follows that there is a finite interval $\Delta \subset (-\infty, \infty)$ such that $P_A(\Delta)u \neq 0$, $SP_A(\Delta)u \neq 0$ and $g'(\lambda)$ is continuous, positive and of bounded variation on Δ . For a nonnegative operator S it follows from $Sv \neq 0$ that also $(v, Sv) \neq 0$. Hence we have $(P_A(\Delta)u, SP_A(\Delta)u) \neq 0$ and then $QSQ \neq 0$ where Q is the orthogonal projection on the subspace $\mathcal{H}_1 = \overline{\text{sp}} \{P_A(\Delta')u, \Delta' \subset \Delta\}$. It is $Q \in \{A\}'$ and therefore $Q \in \{g(A)\}'$. By $S \in \{g(A)\}'$ we obtain $QSQ \in \{g(A)\}'$. Since $g(\lambda)$ is strictly increasing on Δ it is clear that $\{Q A Q'\} = \{Q g(A) Q'\}$. From this identity and $QSQ \in \{g(A)\}'$ we finally obtain $QSQ \in \{A\}'$. Furthermore, we have $E(\Delta) \in \{A\}'$, where $E(\Delta)$ is the spectral measure of QSQ . We choose a $\alpha > 0$ such that $E(0, \alpha) < Q$. With $R \equiv (Q - E(0, \alpha)) \in \{A\}'$ and (5.1), (5.2) we find

$$(5.3) \quad V_A^+(RTR) = 0, \quad V_{g(A)}^+(RTR) = RSR \neq 0.$$

RSR is a nonnegative selfadjoint operator with the spectrum $\delta \in 0 \cup [a, b] (0 < a < b < \infty)$. Now we consider continuous functions $f(\lambda)$ which are 1 on $[a, b]$ and 0 in a neighborhood of 0.

By (2.6) and (5.3) we find

$$(5.4) \quad V_A^+(f(RTR)) = 0, \quad V_{g(A)}^+(f(RTR)) = f(RSR) = R.$$

From the independence of the right sides of these $f(\lambda)$ it can easily be shown that (5.4) is also true for the step-function

$$f(\lambda) = \begin{cases} 1 & \text{on } [a_1, b_1] \text{ (} 0 < a_1 < a < b < b_1 < \infty \text{)} \\ 0 & \lambda \notin [a_1, b_1]. \end{cases}$$

Hence we have

$$(5.5) \quad V_A^+(P) = 0, \quad V_{g(A)}^+(P) = R$$

where $P = f(RTR)$ is an orthogonal projection with $P < R$. \mathcal{H}_1 reduces A, P, R and P, R are distinct from 0 only on \mathcal{H}_1 . Thus it is sufficient to consider (5.5) in \mathcal{H}_1 . A restricted on \mathcal{H}_1 is an operator with a simple absolutely continuous spectrum $\sigma \subset \Delta$. Then we may identify \mathcal{H}_1 with $\mathcal{L}^2(\sigma)$ and a restricted on \mathcal{H}_1 with the multiplication operator by λ and regard $\mathcal{H}_1 \cong \mathcal{L}^2(\sigma)$ as a subspace of the large Hilbert space $\mathcal{L}^2(\Delta)$. In $\mathcal{L}^2(\Delta)$ we may identify R with the multiplication operator by $\chi_{\bar{\rho}}(\lambda)$, where $\chi_{\bar{\rho}}(\lambda)$ is the characteristic function on $\bar{\rho} \subset \sigma$ with $|\bar{\rho}| \neq 0$. H denotes the multiplication operator by λ in $\mathcal{L}^2(\Delta)$. Then we obtain from (5.5)

$$(5.6) \quad \lim_{t \rightarrow \infty} \|Pe^{-iHt}\psi\| = 0 \quad \text{for every } \psi \in \mathcal{L}^2(\Delta),$$

$$(5.7) \quad \lim_{t \rightarrow \infty} \|Pe^{-ig(H)t}\psi\| = \|\chi_{\bar{\rho}}\psi\|.$$

For the sake of simplicity, we shall assume that $\Delta = [0, 2\pi]$. We can write $g(\lambda) = \alpha \cdot \bar{g}(\lambda) + \beta$, where $\bar{g}(0) = 0, \bar{g}(2\pi) = 2\pi$, and α, β are real numbers with $\alpha > 0$. Then we put

$$\varphi_n(\lambda) \equiv \frac{1}{\sqrt{2\pi}}e^{-i\lambda n}, \quad \psi_n(\lambda) \equiv \frac{1}{\sqrt{2\pi}}\sqrt{\bar{g}'(\lambda)}e^{-i n \bar{g}(\lambda)}.$$

It is easy to verify that both φ_n and ψ_n form a complete orthonormal family in $\mathcal{L}^2(\Delta)$. Furthermore, we have $\|\chi_{\bar{\rho}}\psi_n\|^2 = C > 0$. With these notations we easily obtain from (5, 6), (5.7)

$$(5.8) \quad \|P\varphi_n\|^2 = \varepsilon_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

$$(5.9) \quad \|P\psi_n\|^2 - C = \alpha_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

We set $\psi_s(\lambda) = \sum_m a_s^m \varphi_m(\lambda)$. Now we consider the functions $\bar{\psi}_s(\lambda)$ for which

$$(5.10) \quad \left\| \bar{\psi}_s - \sum_{m=1}^N a_s^m \varphi_m \right\|^2 \leq \frac{C}{2}$$

with fixed $N > 0$. For the functions $\bar{\psi}_s(\lambda) = (\sqrt{\bar{g}'(\lambda)})^{-1}\psi_s(\lambda)$ by Lemma 2 we have

$$\left\| \bar{\psi}_s - \sum_{m=[sq_1]-p}^{[sq_2]+p} \bar{a}_s^m \varphi_m \right\| \leq \varepsilon$$

where q_1, q_2 are positive real numbers independent of s, ε and p is a natural number independent of s . It is clear that then also

$$(5.11) \quad \left\| \psi_s - \sum_{m=[sq_1]-p'}^{[sq_2]+p'} a_s^m \varphi_m \right\|^2 \leq \frac{C}{2},$$

with an appropriate p' independent of s . An elementary computation

shows that for all ψ_s with

$$s \in \left(\left[\frac{1+p'}{q_1} \right], \left[\frac{1+p'}{q_1} \right] + 1, \dots, \left[\frac{N-p'}{q_2} \right] \right)$$

the inequality (5.10) is true.

Hence there are natural numbers N_1, s_1 and an $\alpha > 0$ such that for every fixed N with $N > N_1$ and $s \in (s_1, s_1 + 1, \dots, [\alpha N] + s_1)$ ψ_s satisfies (5.10). Now we consider the sum

$$S_N = \sum_{n=1}^N \sum_{s=s_1}^{[\alpha N]+s_1} |(\varphi_n, \psi_s)|^2$$

and introduce the orthogonal projection $\bar{P} = 1 - P$. Then

$$\begin{aligned} (5.12) \quad S_N &= \sum_{n=1}^N \sum_{s=s_1}^{[\alpha N]+s_1} |(p\varphi_n, \psi_s) + (\varphi_n, \bar{p}\psi_s)|^2 \\ &\leq \sum_{n=1}^N \sum_{s=s_1}^{[\alpha N]+s_1} \{ |(p\varphi_n, \psi_s)|^2 + |(\varphi_n, \bar{p}\psi_s)|^2 + 2|(p\varphi_n, \psi_s)| \cdot |(\varphi_n, \bar{p}\psi_s)| \} \\ &\leq \sum_{n=1}^N \sum_{s=s_1}^{[\alpha N]+s_1} \{ |(p\varphi_n, \psi_s)|^2 + |(\varphi_n, \bar{p}\psi_s)|^2 \} \\ &\quad + 2\sqrt{\left(\sum_{n=1}^N \sum_{s=s_1}^{[\alpha N]+s_1} |(p\varphi_n, \psi_s)|^2 \right) \left(\sum_{n=1}^N \sum_{s=s_1}^{[\alpha N]+s_1} |(\varphi_n, \bar{p}\psi_s)|^2 \right)} \\ &\leq \sum_{n=1}^N \|P\varphi_n\|^2 + \sum_{s=s_1}^{[\alpha N]+s_1} \|\bar{P}\psi_s\|^2 \\ &\quad + 2\sqrt{\left(\sum_{n=1}^N \|P\varphi_n\|^2 \right) \left(\sum_{s=s_1}^{[\alpha N]+s_1} \|\bar{P}\psi_s\|^2 \right)}. \end{aligned}$$

By (5.9) we have $\|\bar{P}\psi_s\|^2 = 1 - C - \alpha_s$ and with (5.8), (5.12)

$$(5.13) \quad \begin{aligned} S_N &\leq \sum_{n=1}^N \varepsilon_n + [\alpha N](1 - C) - \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s \\ &\quad + 2\sqrt{\left(\sum_{n=1}^N \varepsilon_n \right) \left(1 - C - \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s \right)}. \end{aligned}$$

On the other hand, by (5.10) we find

$$(5.14) \quad S_N \geq [\alpha N] \left(1 - \frac{C}{2} \right).$$

Combining (5.13), (5.14) we obtain

$$\frac{C}{2} \leq \frac{1}{[\alpha N]} \left\{ \sum_{n=1}^N \varepsilon_n - \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s \right\} + 2\sqrt{\frac{1}{[\alpha N]} \sum_{n=1}^N \varepsilon_n} \cdot \sqrt{1 - C - \frac{1}{[\alpha N]} \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s}.$$

Since ε_n, α_n are zero sequences, also

$$\gamma_N = \frac{1}{[\alpha N]} \sum_{n=1}^N \varepsilon_n, \quad \delta_N = \frac{1}{[\alpha N]} \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s$$

are zero sequences. Hence for sufficiently large N the last inequality which we have got from the assumption $S \neq 0$ is not true, which proves Lemma 1.

LEMMA 2. Let $g(\lambda)$ be a real-valued function and $g'(\lambda)$ a continuous positive function on $\mathcal{I} = [0, 2\pi]$. Then the functions $\bar{\psi}_s(\lambda) \in \mathcal{L}^2(\mathcal{I})$ defined by

$$(5.15) \quad \bar{\psi}_s(\lambda) \equiv \frac{1}{\sqrt{2\pi}} e^{-isg(\lambda)} = \sum_{m=-\infty}^{\infty} \bar{a}_s^m \frac{1}{\sqrt{2\pi}} e^{-i\lambda m}$$

possess the following properties: For every ε with $0 < \varepsilon < 1$ and every natural number $s > 0$ there exist two real positive numbers q_1, q_2 independent of s, ε and a natural number p independent of s such that

$$(5.16) \quad \sum_{m=[sq_1]-p}^{[sq_2]+p} |\bar{a}_s^m|^2 \geq 1 - \varepsilon.$$

Proof. Let $a_1 = \min_{\lambda \in \mathcal{I}} g'(\lambda)$, $a_2 = \max_{\lambda \in \mathcal{I}} g'(\lambda)$. Let s be a fixed natural number. We consider integral numbers m with $m > s \cdot a_2$ or $s \cdot a_1 > m$. For these m we have $|s \cdot g'(\lambda) - m| > 0$ and we can write

$$\bar{a}_s^m = \frac{1}{2\pi} \int_0^{2\pi} d\lambda e^{-is \cdot g(\lambda)} e^{im\lambda} = \frac{1}{2\pi} \int_0^{2\pi} d\lambda \frac{1}{-i(sg'(\lambda) - m)} \left(\frac{d}{d\lambda} e^{-i(sg(\lambda) - m \cdot \lambda)} \right)$$

Integrating by parts and an elementary computation shows that

$$(5.17) \quad \begin{aligned} |\bar{a}_s^m| &\leq \frac{1}{2\pi} \left| \left[\frac{1}{-i(sg'(\lambda) - m)} e^{-i(sg(\lambda) - m\lambda)} \right]_0^{2\pi} \right. \\ &\quad \left. + \int_0^{2\pi} e^{-i(sg(\lambda) - m\lambda)} d \left(\frac{1}{i(g'(\lambda)s - m)} \right) \right| \\ &\leq \frac{1}{2\pi} \left\{ \frac{2}{|s\alpha - m|} + \int_0^{2\pi} \left| d \left(\frac{1}{sg'(\lambda) - m} \right) \right| \right\} \\ &\leq \frac{1}{2\pi} \left\{ \frac{2}{|s\alpha - m|} + \frac{M \cdot s}{|s\alpha - m|^2} \right\} \\ &= \frac{1}{2\pi |s\alpha - m|} \left\{ 2 + \frac{M}{\left| \alpha - \frac{m}{s} \right|} \right\}, \end{aligned}$$

where M is the total variation of $g'(\lambda)$ on \mathcal{I} and $\alpha = a_1$ if $m < s \cdot a_1$ or $\alpha = a_2$ if $m > s \cdot a_2$. Let p' be a positive integral number, then by (5.17) we have

$$\begin{aligned}
 (5.18) \quad \sum_{m=[s2a_2]+p'+1}^{\infty} |\bar{\alpha}_s^m|^2 &\leq \frac{1}{(2\pi)^2} \left(2 + \frac{M}{a_2}\right)^2 \sum_{m=[s2a_2]+p'+1}^{\infty} \frac{1}{|m - [sa_2] - 1|^2} \\
 &< \frac{1}{(2\pi)^2} \left(2 + \frac{M}{a_2}\right)^2 \sum_{n=p'}^{\infty} \frac{1}{n^2}
 \end{aligned}$$

and entirely analogous

$$\begin{aligned}
 (5.19) \quad \sum_{m=[s(1/2)a_1]-p'-1}^{-\infty} |\bar{\alpha}_s^m|^2 &\leq \frac{1}{(2\pi)^2} \left(2 + \frac{2M}{a_1}\right)^2 \sum_{m=[s(1/2)a_1]-p'-1}^{-\infty} \frac{1}{|sa_1 - m|^2} \\
 &< \frac{1}{(2\pi)^2} \left(2 + \frac{2M}{a_1}\right)^2 \sum_{n=-p'}^{\infty} \frac{1}{n^2},
 \end{aligned}$$

where $[a]$ is the smallest integer $r > a - 1$.

For sufficiently large p' from (5.18), (5.19) we find

$$\sum_{m=[s2a_2]+p'+1}^{\infty} |\bar{\alpha}_s^m|^2 + \sum_{m=[s(1/2)a_1]-p'-1}^{-\infty} |\bar{\alpha}_s^m|^2 < \varepsilon$$

for all positive integral numbers s and every $\varepsilon > 0$. With $q_1 = (1/2)a_1$, $q_2 = 2 \cdot a_2$ and by $|\psi_s|^2 = \sum_m |\bar{\alpha}_s^m|^2 = 1$ we finally obtain (5.16).

LEMMA 3. *Let $g(\lambda)$, $\bar{\psi}_s(\lambda)$ be defined as in Lemma 2 and let $g'(\lambda)$ be continuous, strictly monotone on Δ . Then the functions $\bar{\psi}_s(\lambda)$ possess the following properties: For every ε with $0 < \varepsilon < 1$ there exists an N such that for all integral numbers m, s with $s > N$*

$$(5.20) \quad |\bar{\alpha}_s^m| < \varepsilon.$$

Proof. From the continuity and strict monotony of the positive function $g'(\lambda)$ on Δ it follows that for every real number x and $\varepsilon > 0$ there is an interval $\Delta_x \subseteq \Delta$ of the length $l_x \leq \varepsilon \cdot \pi$ such that

$$(5.21) \quad \alpha(\varepsilon) \equiv \min_{x \in R_1} (\min_{\lambda \in \Delta - \Delta_x} |g'(\lambda) + x|)$$

exists and $\alpha(\varepsilon) > 0$. Hence with $x = -m/s$ we have

$$\begin{aligned}
 (5.22) \quad |\bar{\alpha}_s^m| &= \left| \frac{1}{2\pi} \int_0^{2\pi} d\lambda e^{-isg(\lambda)} e^{im\lambda} \right| \\
 &\leq \frac{\varepsilon}{2} + \frac{1}{2\pi} \left| \int_{\Delta - \Delta_x} d\lambda e^{-is(g(\lambda) - x\lambda)} \right|.
 \end{aligned}$$

The domain of integration $(\Delta - \Delta_x)$ consists of one or two intervals in dependence on x and ε . Let $\Delta' = [a, b] \subseteq \Delta$ be such an interval. Then

$$\left| \int_a^b d\lambda e^{-is(g(\lambda) - x\lambda)} \right| = \left| \int_a^b \frac{d\lambda}{-is(g'(\lambda) - x)} \left(\frac{d}{d\lambda} e^{-is(g(\lambda) - x\lambda)} \right) \right|$$

$$\begin{aligned}
 & \leq \left| \left[\frac{1}{-is(g'(\lambda) - x)} e^{-is(g(\lambda) - x\lambda)} \right]_a^b + \int_a^b e^{-is(g(\lambda) - x\lambda)} d\left(\frac{1}{is(g'(\lambda) - x)} \right) \right| \\
 (5.23) \quad & \leq \frac{2}{s \cdot \alpha} + \int_a^b \left| d\left(\frac{1}{s(g'(\lambda) - x)} \right) \right| \\
 & \leq \frac{2}{s \cdot \alpha} + \frac{M}{s \cdot \alpha}
 \end{aligned}$$

where α is defined by (5.21) and $M = |g'(b) - g'(a)|$. From (5.22) and (5.23) we have

$$|\alpha_s^m| \leq \frac{\varepsilon}{2} + \frac{1}{\pi s} \cdot \frac{2 + M}{\alpha}.$$

If we put $N = (2/\varepsilon \cdot \pi) \cdot \frac{2 + M}{\alpha}$, then this implies (5.20).

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