THE INVARIANCE PRINCIPLE FOR WAVE OPERATORS

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The invariance principle for wave operators is proved. It is shown that the existence of wave operators $W_\pm(B, A)$ does not imply the existence of $W_\pm(g(B), g(A))$, in general.

1. **Introduction.** Let A and B be two selfadjoint operators on a separable Hilbert space $\mathscr{H}$ and let P_A and P_B be the orthogonal projections on the spaces of absolute continuity for A and B, respectively. The wave operators $W_\pm(B, A)$ are defined by the strong limits

$$(1.1) \qquad W_\pm(B, A) \equiv \text{s-}\lim_{t\to\pm\infty} e^{itB}e^{-itA}P_A$$

when they exist (cf. [2, Chapter X]). The invariance principle of M. S. Birman and T. Kato says: If the wave operators $W_\pm(B, A)$ and $W_\pm(g(B), g(A))$ exist and $g(\lambda)$ is real-valued and piecewise monotone increasing, with a certain mild smoothness, then

$$(1.2) \qquad W_\pm(g(B), g(A)) = W_\pm(B, A) .$$

As stated by T. Kato and S. T. Kuroda in [3]: "It would be nice if the existence of $W_\pm(B, A)$ implied the existence of $W_\pm(g(B), g(A))$ and the invariance principle.

However, this has not been shown in general".

For example, the existence of $W_\pm(g(B), g(A))$ and the invariance principle have been proved under the condition that $B - A$ or $(B - \xi)^{-1} - (A - \xi)^{-1}(\xi$ a nonreal number) is a trace-class operator (see for instance [2, Chapter X]).

The aim of this paper is

1. the proof of the invariance principle for wave operators,

2. the proof, that the existence of $W_\pm(B, A)$ does not imply the existence of $W_\pm(g(B), g(A))$, in general.

In the present work we restrict our considerations to real-valued functions $g(\lambda)$ on $(-\infty, \infty)$ with the following properties (cf. [2, p. 543]): The whole interval $(-\infty, \infty)$ can be divided into a countable number of subintervals $\varDelta_n$ with lengths l_n in such a way that min $l_n > 0$ and in each open subinterval $g(\lambda)$ is differentiable with $g'(\lambda)$

continuous, locally of bounded variation, and positive. A function with these properties is called an allowable function. Furthermore, we shall consider only the wave operators W_+ because the theorems and proofs for the wave operators W_- are entirely similar. In §2 we prove

THEOREM 1 (invariance principle). *Let A and B be two selfadjoint operators on a separable Hilbert space $\mathscr{H}$ and let $g(\lambda)$ be an allowable function. If $W_+(B, A)$ and $W_+(g(B), g(A))$ exist and if $W_+(B, A)$ is complete, then $W_+(g(B), g(A)) = W_+(B, A)$.*

From Theorem 1 we also see that the existence of $W_+(B, A)$, $W_+(g(B), g(A))$ and the completeness of $W_+(B, A)$ imply the completeness of $W_+((g, B), g(A))$. The next two theorems concern the existence of the wave operator $W_+(g(B), g(A))$. They will be proved in §§2 and 3, respectively.

THEOREM 2. *Let A and B be two selfadjoint operators on a separable Hilbert space $\mathscr{H}$ with the absolutely continous spectrum Λ and let $g(\lambda)$ be an allowable function. If the wave operator $W_+(B, A)$ exists, is complete and if $g(\lambda)$ is piecewise linear on Λ, then $W_+(g(B), g(A))$ exists.*

THEOREM 3. *Let A be a selfadjoint operator on a separable Hilbert space $\mathscr{H}$ with the absolutely continuous spectrum $\Lambda \neq 0$. Let $g(\lambda)$ be an allowable function for which a finite interval $\Delta \subset (-\infty, \infty)$ with $|\Delta \cap \Lambda| \neq 0$ (Lebesgue measure) exists such that on Δ $g'(\lambda)$ exists and is a continuous strictly monotone function. Then there is a selfadjoint operator B such that $W_+(B, A)$ exists, is complete, however, $W_+(g(B), g(A))$ does not exist.*

It is easily seen, for instance, that all allowable functions $g(\lambda)$ which are piecewise twice continuously differentiable satisfy the assumptions either of Theorem 2 or of Theorem 3 for fixed Δ.

For the proofs of the theorems we use the following result of H. Baumgärtel [1, Theorem 3]:

(NS) Let W be a partial isometry with

$$W^*W = P_A, \; WW^* = P_B, \; WAP_A W^* \supseteq BP_B \; .$$

Then $W_+(B, A)$ exists and $W = W_+(B, A)$ if and only if

(1.3) $$W = P_A + C \; ,$$

(1.4) $$\operatorname*{s\text{-}lim}_{t \to \infty} e^{itA} C e^{-itA} P_A = 0 \; .$$

By Theorem 1 we obtain from (NS) that the existence and completeness of $W_+(B, A)$ imply the existence of $W_+(g(B), g(A))$ if and only if for the operator C defined by (1.3) the strong limit

$$(1.5) \qquad \qquad \text{s-}\lim_{t \to \infty} e^{itg(A)} C e^{-itg(A)} P_A = 0$$

exists. Here it was used that $P_{g(A)} = P_A$ for allowable functions $g(\lambda)$ (see §2).

Hence we know that the proof of Theorem 3 leads to the construction of an operator C for which $\text{s-}\lim_{t \to \infty} e^{itA} C e^{-itA} P_A = 0$ and $\text{s-}\lim_{t \to \infty} e^{itg(A)} C e^{-itg(A)} P_A$ does not exist for the function $g(\lambda)$ defined by Theorem 3. To prove Theorem 2 we shall show that the equation

$$\text{s-}\lim_{t \to \infty} e^{itA} C e^{-itA} P_A = 0$$

implies $\text{s-}\lim_{t \to \infty} e^{itg(A)} C e^{-itg(A)} P_A = 0$ for piecewise linear functions $g(\lambda)$. The invariance principle will be proved by means of

LEMMA 1. *Let T be a nonnegative bounded selfadjoint operator and $g(\lambda)$ an allowable function. If the strong limits*

$$\text{s-}\lim_{t \to \infty} e^{itA} T e^{-itA} P_A = 0, \ \text{s-}\lim_{t \to \infty} e^{itg(A)} T e^{-itg(A)} P_A$$

exist, then they are equal.

In §5 we prove Lemma 1 and formulate and prove two other lemmas which concern the behavior of the function $e^{-itg(\lambda)}$ for large t.

2. *Proof of Theorem 1.* First we introduce several notations and simple relations which are needed for the proof. As in §1 let H be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$ and P_H be the orthogonal projection on the space of absolute continuity. We note that for every allowable function $g(\lambda)$

$$(2.1) \qquad \qquad P_{g(H)} = P_H \, .$$

(2.1) has been proved in [2, p. 545] for a class of functions slightly more restrictive than the allowable functions. The proof can easily be generalized for all allowable functions. Furthermore, we introduce the notations $\{H\}'$ for the commutant of H and

$$(2.2) \qquad \qquad V_H^{\pm}(X) \equiv \text{s-}\lim_{t \to \infty} e^{itH} X e^{-itH} P_H \, ,$$

whenever for the bounded operator X the strong limit exists. If $V_H^{\pm}(X)$ for the bounded operator X exists, then we have the unambiguous decomposition (cf. [1])

(2.3) $$X = X_1 + X_2 \,,$$

where

(2.4) $$X_1 = P_H X_1 = X_1 P_H \in \{H\}' \,,$$

(2.5) $$V_H^+(X_2) = 0 \,.$$

For continuous functions $f(\lambda)$ and a selfadjoint operator X one easily verifies that

(2.6) $$f(V_H^+(X)) = V_H^+(f(X)) \,.$$

Now we prove Theorem 1. By (NS), we find that $W_+(B, A) = P_A + C$ with $V_A^+(C) = 0$. Further we also have

(2.7) $$V_A^+(C^*) = 0 \,,$$

since

$$V_A^+(C^*) = V_A^+(C^* + P_A) - P_A = V_A^+(W_+^*(B, A)) - P_A$$

$$= \text{s-}\lim_{t \to \infty} e^{itA} W_+^*(B, A) e^{-itA} P_A - P_A$$

$$= \text{s-}\lim_{t \to \infty} W_+^*(B, A) e^{itB} e^{-itA} P_A - P_A$$

$$= W_+^*(B, A) W_+(B, A) - P_A = 0$$

with the intertwining relation $W_+(B, A)e^{itA} = e^{itB} W_+(B, A)$. We define

(2.8) $$W_1 \equiv W_+^*(B, A) W_+(g(B), g(A)) \,.$$

From this definition we obtain that $V_{g(A)}^+(W_+^*(B, A))$ exists and

(2.9) $$V_{g(A)}^+(W_+^*(B, A)) = W_1 \,,$$

since

$$V_{g(A)}^+(W_+^*(B, A)) = \text{s-}\lim_{t \to \infty} e^{itg(A)} W_+^*(B, A) e^{-itg(A)} P_A$$

$$= \text{s-}\lim_{t \to \infty} W_+^*(B, A) e^{itg(B)} e^{-itg(A)} P_A = W_1 \,.$$

By (2.3) and (2.5) then we have

(2.10) $$W_+^*(B, A) = W_1 + C_1$$

with

(2.11) $$V_{g(A)}^+(C_1) = 0$$

(2.12) $$W_1 = P_A W_1 = W_1 P_A \in \{g(A)\}' \,.$$

From the Definition (2.8) and the completeness of $W_+(B, A)$ one

easily verifies that W_1 is *a* partially isometrie with

(2.13) $$W_1^* W_1 = P_A, \; W_1 W_1^* = P_1 \leqq P_A \; .$$

Further we have

(2.14) $$V_{g(A)}^+(C_1^* P_1) = 0 \; .$$

This follows from

$$\begin{aligned}
V_{g(A)}^+(C_1^* P_1) = 0 &\longleftarrow V_{g(A)}^+(W_+^*(B, A)C_1^* P_1) \\
&= V_{g(A)}^+(W_+^*(B, A)C_1^* P_1 + C_1 W_1^* + P_1) - P_1 \\
&= V_{g(A)}^+(W_+^*(B, A)(C_1^* + W_1^*)P_1) - P_1 \\
&= V_{g(A)}^+(W_+^*(B, A)W_+(B, A)P_1) - P_1 = 0
\end{aligned}$$

with $V_{g(A)}^+(C_1 W_1^* + P_1) = P_1$ by $W_1^*, \, P_1 \in \{g(A)\}'$. Combining (2.10) with $W_+^*(B, A) = P_A + C^*$ and (2.13) we obtain

$$\begin{aligned}
CP_1C^* &= (W_1 + C_1 - P_A)^* P_1(W_1 + C_1 - P_A) \\
&= (W_1 - P_1)^*(W_1 - P_1) + (W_1 - P_1)^* C_1 + C_1^* P_1(W_1 - P_1) \\
&\quad + C_1^* P_1 C_1 = (W_1 - P_1)^*(W_1 - P_1) + C_2 \; .
\end{aligned}$$

By (2.11), (2.14) and $(W_1 - P_1), \, (W_1 - P_1)^* \in \{g(A)\}'$ we have $V_{g(A)}^+(C_2) = 0$ and therefore,

(2.15) $$V_{g(A)}^+(CP_1C^*) = (W_1 - P_1)^*(W_1 - P_1) \; .$$

Furthermore, it follows from (2.7) that

(2.16) $$V_A^+(CP_1C^*) = 0 \; .$$

The operator CP_1C^* satisfies the assumptions of Lemma 1. Hence, we have $(W_1 - P_1)^*(W_1 - P_1) = 0$ and also $(W_1 - P_1) = 0$. With (2.13) and (2.8), we finally obtain $W_1 = P_A$ and

$$W_+(g(B), g(A)) = W_+(B, A) \; .$$

3. *Proof of Theorem 2.* We shall use the same notations as in §2. By Theorem 1 and (NS) it is necessary and sufficient for the existence of $W_+(g(B), g(A))$ that

(3.1) $$V_{g(A)}^+(C) = 0 \; .$$

Let $\varphi \in P_A \mathscr{H}, \, \varepsilon > 0$ and $P_A(\varDelta)$ be the spectral measure of A. Then there is a finite interval $\varDelta'$ such that

(3.2) $$\|\varphi - P_A(\varDelta')\varphi\| \leqq \frac{\varepsilon}{3\|C\|} \; .$$

From the definition of the functions $g(\lambda)$ in Theorem 2 we see that there exists a finite number of disjoint intervals $\varDelta_n \subset \varDelta'(n = 1, 2, \cdots, N)$ such that

$$(3.3) \qquad g(A)P_A P_A(\varDelta_n) = q_n A P_A P_A(\varDelta_n) \,,$$

$$(3.4) \qquad \left\| P_A(\varDelta')\varphi - \sum_{n=1}^{N} P_A(\varDelta_n)\varphi \right\| \leqq \frac{\varepsilon}{3\|C\|}$$

with $0 < q_n < \infty$. By (3.2), (3.3), and (3.4) we find

$$(3.5) \qquad \| C e^{-itg(A)} P_A \varphi \| \leqq \frac{2}{3}\varepsilon + \sum_{n=1}^{N} \| C e^{-itq_n A} P_A(\varDelta_n)\varphi \| \,.$$

C satisfies the relation (see (NS))

$$\| C e^{-itA} P_A \psi \| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty$$

for every $\psi \in \mathscr{H}$. Hence, for the functions $\varphi_n = P_A(\varDelta_n)\varphi$ there are numbers T_n such that

$$(3.6) \qquad \| C e^{-itq_n A} \varphi_n \| \leqq \frac{\varepsilon}{3N} \quad \text{for all} \quad t > T_n \,.$$

By (3.5), (3.9) we obtain

$$\| C e^{-itg(A)} P_A \varphi \| \leqq \varepsilon \quad \text{for} \quad t > T = \max_n T_n$$

and (3.1) is proved.

4. *Proof of Theorem* 3. For simplicity we shall assume that $\varDelta \subset \varLambda$ and $\varDelta = [0, 2\pi]$. Let $u \in P_A(\varDelta)P_A\mathscr{H}$ with $P_A(\varDelta')u \neq 0$ for all $\varDelta' \subset \varDelta, |\varDelta'| \neq 0$. A restricted on the subspace $\mathscr{H}_1 = \overline{\text{sp}}\{P_A(\varDelta')u, \varDelta' \subset \varDelta$ Borel set$\}$ is an operator with simple absolutely continuous spectrum $\varDelta$. Hence we may identify $\mathscr{H}_1$ with $\mathscr{L}^2(\varDelta)$ and a restricted on $\mathscr{H}_1$ with the multiplication operator by λ on $\mathscr{L}^2(\varDelta)$ denoted by H.

Therefore it follows that for the proof of Theorem 3 it is sufficient to show that for H such an operator B defined by Theorem 3 exists. At first we construct a projector P such that $V_H^+(P) = 0$ and $V_{g(H)}^+(P)$ do not exist. We consider the function

$$\frac{1}{\sqrt{2\pi}} e^{-itg(\lambda)} = \sum_m \psi_t^m \frac{1}{\sqrt{2\pi}} e^{-i\lambda m} \in \mathscr{L}^2(\varDelta) \,.$$

By Lemma 2 for $\varepsilon = 1/2$ we can find sequences of natural numbers S_n, N_n with $S_n, N_n \to \infty$ as $n \to \infty$ such that

$$(4.1) \qquad \sum_{m=N_n}^{N_{n+1}-1} |\psi_{S_n}^m|^2 = a_n \geqq 1 - \varepsilon = \frac{1}{2} \,.$$

Now we define P by

(4.2)
$$P = \sum_{n=1}^{\infty} f_n(\cdot, f_n) ,$$

(4.3)
$$f_n(\lambda) = \sum_{m=N_n}^{N_{n+1}-1} \frac{1}{\sqrt{a_n}} \psi_{S_n}^m \frac{1}{\sqrt{2\pi}} e^{-i\lambda m} .$$

Next we prove $V_H^+(P) = 0$, i.e.,

(4.4)
$$\lim_{t\to\infty} \| P e^{-itH} \psi \| = 0 \quad \text{for every} \quad \psi \in \mathscr{L}^2(\Delta) .$$

As is easily shown, for the proof of (4.4) it is sufficient to consider the sequence $\| P e^{-inH} \psi_0 \|$ with $n \to \infty$ (n a natural number) and $\psi_0 = 1/\sqrt{2\pi}$.

We have

(4.5)
$$\| P e^{-ipH} \psi_0 \|^2 = \sum_{n=1}^{\infty} |(f_n, e^{-ipH} \psi_0)|^2$$
$$= \sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f_n(\lambda) e^{ip\lambda} d\lambda \right|^2$$
$$= \sum_{n=1}^{\infty} \left| \sum_{m=N_n}^{N_{n+1}-1} \psi_{S_n}^m \frac{1}{\sqrt{a_n}} \delta_{m;p} \right|^2$$
$$= \frac{\Lambda}{a_{x(p)}} |\psi_{Sx(p)}^p|^2 ,$$

where $x(p) = r$ if $p \in (N_r, N_r + 1, \cdots, N_{r+1} - 1)$.

It is clear that $x(p) \to \infty$ as $p \to \infty$. Since $g(\lambda)$ satisfies the assumptions of Lemma 3 we find $|\psi_{Sx(p)}^p|^2 \to 0$ as $p \to \infty$ and also $\| P e^{-ipH} \psi_0 \|^2 \to 0$ as $p \to \infty$. This proves (4.4). To prove that $V_{g(H)}^+(P)$ does not exist by Lemma 1 it is sufficient to show that there are a $\psi \in \mathscr{L}^2(\Delta)$, a sequence of real numbers $t_n \to \infty$ as $n \to \infty$ and an $X > 0$ such that

(4.6)
$$\| P e^{-it_n \cdot g(H)} \psi \| > X \quad \text{for all} \quad t_n .$$

We set $\psi = \psi_0 = 1/\sqrt{2\pi}$ and $t_n = S_n$ (see (4.1)). Then by (4.1), (4.2)

$$\| P e^{-iS_n \cdot g(H)} \psi_0 \|^2 = \sum_{q=1}^{\infty} |(f_q, e^{-iS_n \cdot g(H)} \psi_0)|^2$$
$$\geq |(f_n, e^{-iS_n \cdot g(H)} \psi_0)|^2 = \left| \frac{1}{\sqrt{a_n}} \sum_{m=N_n}^{N_{n+1}-1} \psi_{S_n}^m \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\lambda e^{iS_n \cdot g(\lambda)} e^{-im\lambda} \right|^2$$
$$= \left| \frac{1}{\sqrt{a_n}} \sum_{m=N_n}^{N_{n+1}-1} |\psi_{S_n}^m|^2 \right|^2 = a_n \geq \frac{1}{2} \quad \text{for all} \quad S_n .$$

This proves that $U_{g(H)}^+(P)$ does not exist.

Now we define by $U \equiv 1 - 2P$ a unitary operator and we set

$B = UHU$. From the definitions of U, B, and (NS) it immediately follows that $W_+(B, H)$ exists and $U = W_+(B, H)$. Since, however, $V_{g(H)}^+(-2P)$ does not exist, it follows from Theorem 1 and (NS) that also $W_+(g(B), g(H))$ does not exist.

5. *Proofs of the Lemmas.* Proof of Lemma 1: We shall prove Lemma 1 indirectly. Thus we suppose that for a nonnegative bounded selfadjoint operator T and an allowable function $g(\lambda)$ the strong limits

(5.1) $$V_A^+(T) = \text{s-}\lim_{t\to\infty} e^{itA} T e^{-itA} P_A = 0 \ ,$$

(5.2) $$V_{g(A)}^+(T) = \text{s-}\lim_{t\to\infty} e^{itg(A)} T e^{-itg(A)} P_A = S$$

exist with $S \neq 0$, and from these assumptions we construct a contradiction. It is obvious that S is also a nonnegative bounded selfadjoint operator with $S = SP_A = P_A S \in \{g(A)\}'$ by (2.3) to (2.5). By $S \neq 0$ it exists a $u \in P_A \mathscr{H}$ with $Su \neq 0$. From the definition of the allowable functions $g(\lambda)$ it follows that there is a finite interval $\varDelta \subset (-\infty, \infty)$ such that $P_A(\varDelta)u \neq 0$, $SP_A(\varDelta)u \neq 0$ and $g'(\lambda)$ is continuous, positive and of bounded variation on $\varDelta$. For a nonnegative operator S it follows from $Sv \neq 0$ that also $(v, Sv) \neq 0$. Hence we have $(P_A(\varDelta)u, SP_A(\varDelta)u) \neq 0$ and then $QSQ \neq 0$ where Q is the orthogonal projection on the subspace $\mathscr{H}_1 = \overline{\text{sp}} \{P_A(\varDelta')u, \varDelta' \subset \varDelta\}$. It is $Q \in \{A\}'$ and therefore $Q \in \{g(A)\}'$. By $S \in \{g(A)\}'$ we obtain $QSQ \in \{g(A)\}'$. Since $g(\lambda)$ is strictly increasing on $\varDelta$ it is clear that $\{QAQ\}' = \{Qg(A)Q\}'$. From this identity and $QSQ \in \{g(A)\}'$ we finally obtain $QSQ \in \{A\}'$. Furthermore, we have $E(\varDelta) \in \{A\}'$, where $E(\varDelta)$ is the spectral measure of QSQ. We choose a $\alpha > 0$ such that $E(0, \alpha) < Q$. With $R \equiv (Q - E(0, \alpha)) \in \{A\}'$ and (5.1), (5.2) we find

(5.3) $$V_A^+(RTR) = 0, \ V_{g(A)}^+(RTR) = RSR \neq 0 \ .$$

RSR is a nonnegative selfadjoint operator with the spectrum $\delta \in 0 \cup [a, b](0 < a < b < \infty)$. Now we consider continuous functions $f(\lambda)$ which are 1 on $[a, b]$ and 0 in a neighborhood of 0.
        By (2.6) and (5.3) we find

(5.4) $$V_A^+(f(RTR)) = 0, \ V_{g(A)}^+(f(RTR)) = f(RSR) = R \ .$$

From the independence of the right sides of these $f(\lambda)$ it can easily be shown that (5.4) is also true for the step-function

$$f(\lambda) = \begin{cases} 1 \ \text{on } [a_1, b_1] \ (0 < a_1 < a < b < b_1 < \infty) \\ 0 \ \ \lambda \notin [a_1, b_1] \ . \end{cases}$$

Hence we have

(5.5) $$V_A^+(P) = 0, \ V_{g(A)}^+(P) = R$$

where $P = f(RTR)$ is an orthogonal projection with $P < R$. $\mathscr{H}_1$ reduces A, P, R and P, R are distinct from 0 only on $\mathscr{H}_1$. Thus it is sufficient to consider (5.5) in $\mathscr{H}_1$. A restricted on $\mathscr{H}_1$ is an operator with a simple absolutely continuous spectrum $\sigma \subset \varDelta$. Then we may identify $\mathscr{H}_1$ with $\mathscr{L}^2(\sigma)$ and a restricted on $\mathscr{H}_1$ with the multiplication operator by λ and regard $\mathscr{H}_1 \cong \mathscr{L}^2(\sigma)$ as a subspace of the large Hilbert space $\mathscr{L}^2(\varDelta)$. In $\mathscr{L}^2(\varDelta)$ we may identify R with the multiplication operator by $\chi_{\bar{\rho}}(\lambda)$, where $\chi_{\bar{\rho}}(\lambda)$ is the characteristic function on $\bar{\rho} \subset \sigma$ with $|\bar{\rho}| \neq 0$. H denotes the multiplication operator by λ in $\mathscr{L}^2(\varDelta)$. Then we obtain from (5.5)

(5.6) $$\lim_{t \to \infty} || P e^{-iHt} \psi || = 0 \quad \text{for every} \quad \psi \in \mathscr{L}^2(\varDelta) ,$$

(5.7) $$\lim_{t \to \infty} || P e^{-ig(H)t} \psi || = || \chi_{\bar{\rho}} \psi || .$$

For the sake of simplicity, we shall assume that $\varDelta = [0, 2\pi]$. We can write $g(\lambda) = \alpha \cdot \bar{g}(\lambda) + \beta$, where $\bar{g}(0) = 0$, $\bar{g}(2\pi) = 2\pi$, and α, β are real numbers with $\alpha > 0$. Then we put

$$\varphi_n(\lambda) \equiv \frac{1}{\sqrt{2\pi}} e^{-i\lambda n}, \quad \psi_n(\lambda) \equiv \frac{1}{\sqrt{2\pi}} \sqrt{\bar{g}'(\lambda)} e^{-in\bar{g}(\lambda)} .$$

It is easy to verify that both φ_n and ψ_n form a complete orthonormal family in $\mathscr{L}^2(\varDelta)$. Furthermore, we have $|| \chi_{\bar{\rho}} \psi_n ||^2 = C > 0$. With these notations we easily obtain from (5, 6), (5.7)

(5.8) $$|| P\varphi_n ||^2 = \varepsilon_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

(5.9) $$|| P\psi_n ||^2 - C = \alpha_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

We set $\psi_s(\lambda) = \sum_m a_s^m \varphi_m(\lambda)$. Now we consider the functions $\psi_s(\lambda)$ for which

(5.10) $$\left\| \psi_s - \sum_{n=1}^{N} a_s^m \varphi_m \right\|^2 \leq \frac{C}{2}$$

with fixed $N > 0$. For the functions $\bar{\psi}_s(\lambda) = (\sqrt{\bar{g}'(\lambda)})^{-1} \psi_s(\lambda)$ by Lemma 2 we have

$$\left\| \bar{\psi}_s - \sum_{m=[sq_1]-p}^{[sq_2]-p} \bar{a}_s^m \varphi_m \right\| \leq \varepsilon$$

where q_1, q_2 are positive real numbers independent of s, ε and p is a natural number independent of s. It is clear that then also

(5.11) $$\left\| \psi_s - \sum_{m=[sq_1]-p'}^{[sq_2]+p'} a_s^m \varphi_m \right\|^2 \leq \frac{C}{2} ,$$

with an appropriate p' independent of s. An elementary computation

shows that for all ψ_s with

$$s \in \left(\left[\frac{1 + p'}{q_1} \right], \left[\frac{1 + p'}{q_1} \right] + 1, \cdots, \left[\frac{N - p'}{q_2} \right] \right)$$

the inequality (5.10) is true.

Hence there are natural numbers N_1, s_1 and an $\alpha > 0$ such that for every fixed N with $N > N_1$ and $s \in (s_1, s_1 + 1, \cdots, [\alpha N] + s_1)$ ψ_s satisfies (5.10). Now we consider the sum

$$S_N = \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} |(\varphi_n, \psi_s)|^2$$

and introduce the orthogonal projection $\bar{P} = 1 - P$. Then

$$S_N = \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} |(p\varphi_n, \psi_s) + (\varphi_n, \bar{p}\psi_s)|^2$$

$$\leqq \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} \{ |(p\varphi_n, \psi_s)|^2 + |(\varphi_n, \bar{p}\psi_s)|^2 + 2|(p\varphi_n, \psi_s)| \cdot |(\varphi_n, \bar{p}\psi_s)| \}$$

$$\leqq \sum_{n=1}^{N} \sum_{s=s_1}^{[\beta N]+s_1} \{ |(p\varphi_n, \psi_s)|^2 + |(\varphi_n, \bar{p}\psi_s)|^2 \}$$

(5.12)
$$+ 2\sqrt{\left(\sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} |(p\varphi_{n_1}\psi_s)|^2 \right) \left(\sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} |(\varphi_{n_1}\bar{p}\psi_s)|^2 \right)}$$

$$\leqq \sum_{n=1}^{N} \|P\varphi_n\|^2 + \sum_{s=s_1}^{[\alpha N]+s_1} \|\bar{P}\psi_s\|^2$$

$$+ 2\sqrt{\left(\sum_{n=1}^{N} \|P\varphi_n\|^2 \right) \left(\sum_{s=s_1}^{[\alpha N]+s_1} \|\bar{P}\psi_s\|^2 \right)} .$$

By (5.9) we have $\|\bar{P}\psi_s\|^2 = 1 - C - \alpha_s$ and with (5.8), (5.12)

(5.13)
$$S_N \leqq \sum_{n=1}^{N} \varepsilon_n + [\alpha N](1 - C) - \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s$$
$$+ 2\sqrt{\left(\sum_{n=1}^{N} \varepsilon_n \right) \left(1 - C - \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s \right)} .$$

On the other hand, by (5.10) we find

(5.14)
$$S_N \geqq [\alpha N] \left(1 - \frac{C}{2} \right) .$$

Combining (5.13), (5.14) we obtain

$$\frac{C}{2} \leqq \frac{1}{[\alpha N]} \left\{ \sum_{n=1}^{N} \varepsilon_n - \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s \right\} + 2\sqrt{\frac{1}{[\alpha N]} \sum_{n=1}^{N} \varepsilon_n} \cdot \sqrt{1 - C - \frac{1}{[\alpha N]} \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s} .$$

Since ε_n, α_n are zero sequences, also

$$\gamma_N = \frac{1}{[\alpha N]} \sum_{n=1}^{N} \varepsilon_n , \quad \delta_N = \frac{1}{[\alpha N]} \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s$$

are zero sequences. Hence for sufficiently large N the last inequality which we have got from the assumption $S \neq 0$ is not true, which proves Lemma 1.

LEMMA 2.   *Let* $g(\lambda)$ *be a real-valued function and* $g'(\lambda)$ *a continuous positive function on* $\varDelta = [0, 2\pi]$. *Then the functions* $\bar{\psi}_s(\lambda) \in \mathscr{L}^2(\varDelta)$ *defined by*

$$(5.15) \qquad \bar{\psi}_s(\lambda) \equiv \frac{1}{\sqrt{2\pi}} e^{-isg(\lambda)} = \sum_{m=-\infty}^{\infty} \bar{a}_s^m \frac{1}{\sqrt{2\pi}} e^{-i\lambda m}$$

possess the following properties: For every ε *with* $0 < \varepsilon < 1$ *and every natural number* $s > 0$ *there exist two real positive numbers* q_1, q_2 *independent of* s, ε *and a natural number* p *independent of* s *such that*

$$(5.16) \qquad \sum_{m=[sq_1]-p}^{[sq_2]+p} |\bar{a}_s^m|^2 \geqq 1 - \varepsilon \ .$$

Proof.   Let $a_1 = \min_{\lambda \in \varDelta} g'(\lambda)$, $a_2 = \max_{\lambda \in \varDelta} g'(\lambda)$. Let s be a fixed natural number. We consider integral numbers m with $m > s \cdot a_2$ or $s \cdot a_1 > m$. For these m we have $|s \cdot g'(\lambda) - m| > 0$ and we can write

$$\bar{a}_s^m = \frac{1}{2\pi} \int_0^{2\pi} d\lambda e^{-is \cdot g(\lambda)} e^{im\lambda} = \frac{1}{2\pi} \int_0^{2\pi} d\lambda \frac{1}{-i(sg'(\lambda) - m)} \left(\frac{d}{d\lambda} e^{-i(sg(\lambda) - m \cdot \lambda)} \right)$$

Integrating by parts and an elementary computation shows that

$$
\begin{aligned}
(5.17) \qquad |\bar{a}_s^m| &\leqq \frac{1}{2\pi} \left\| \left[\frac{1}{-i(sg'(\lambda) - m)} e^{-i(sg(\lambda) - m\lambda)} \right]_0^{2\pi} \right. \\
&\qquad \left. + \int_0^{2\pi} e^{-i(sg(\lambda) - m\lambda)} d\left(\frac{1}{i(g'(\lambda)s - m)} \right) \right\| \\
&\leqq \frac{1}{2\pi} \left\{ \frac{2}{|s\alpha - m|} + \int_0^{2\pi} \left| d\left(\frac{1}{sg'(\lambda) - m} \right) \right| \right\} \\
&\leqq \frac{1}{2\pi} \left\{ \frac{2}{|s\alpha - m|} + \frac{M \cdot s}{|s\alpha - m|^2} \right\} \\
&= \frac{1}{2\pi |s\alpha - m|} \left\{ 2 + \frac{M}{\left| \alpha - \dfrac{m}{s} \right|} \right\} \ ,
\end{aligned}
$$

where M is the total variation of $g'(\lambda)$ on $\varDelta$ and $\alpha = a_1$ if $m < s \cdot a_1$ or $\alpha = a_2$ if $m > s \cdot a_2$. Let p' be a positive integral number, then by (5.17) we have

$$(5.18) \quad \sum_{m=[s2a_2]+p'+1}^{\infty} |\bar{a}_s^m|^2 \leqq \frac{1}{(2\pi)^2}\left(2 + \frac{M}{a_2}\right)^2 \sum_{m=[s2a_2]+p'+1}^{\infty} \frac{1}{|m - [sa_2] - 1|^2}$$

$$< \frac{1}{(2\pi)^2}\left(2 + \frac{M}{a_2}\right)^2 \sum_{n=p'}^{\infty} \frac{1}{n^2}$$

and entirely analogous

$$(5.19) \quad \sum_{m=[s(1/2)a_1]-p'-1}^{-\infty} |\bar{a}_s^m|^2 \leqq \frac{1}{(2\pi)^2}\left(2 + \frac{2M}{a_1}\right)^2 \sum_{m=[s(1/2)a_1]-p'-1}^{-\infty} \frac{1}{|sa_1 - m|^2}$$

$$< \frac{1}{(2\pi)^2}\left(2 + \frac{2M}{a_1}\right)^2 \sum_{n=-p'}^{\infty} \frac{1}{n^2} \,,$$

where $[a]$ is the smallest integer $r > a - 1$.

For sufficiently large p' from (5.18), (5.19) we find

$$\sum_{m=[s2a_2]+p'+1}^{\infty} |\bar{a}_s^m|^2 + \sum_{m=[s(1/2)a_1]-p'-1}^{-\infty} |\bar{a}_s^m|^2 < \varepsilon$$

for all positive integral numbers s and every $\varepsilon > 0$. With $q_1 = (1/2)a_1$, $q_2 = 2 \cdot a_2$ and by $|\psi_s|^2 = \sum_m |\bar{a}_s^m|^2 = 1$ we finally obtain (5.16).

LEMMA 3. *Let* $g(\lambda)$, $\bar{\psi}_s(\lambda)$ *be defined as in Lemma 2 and let* $g'(\lambda)$ *be continuous, strictly monotone on* Δ. *Then the functions* $\bar{\psi}_s(\lambda)$ *possess the following properties: For every* ε *with* $0 < \varepsilon < 1$ *there exists an* N *such that for all integral numbers* m, s *with* $s > N$

$$(5.20) \quad\quad\quad\quad\quad\quad |\bar{a}_s^m| < \varepsilon .$$

Proof. From the continuity and strict monotony of the positive function $g'(\lambda)$ on Δ it follows that for every real number x and $\varepsilon > 0$ there is an interval $\Delta_x \subseteqq \Delta$ of the length $l_x \leqq \varepsilon \cdot \pi$ such that

$$(5.21) \quad\quad\quad \alpha(\varepsilon) \equiv \min_{x \in R_1}\left(\min_{\lambda \in \Delta - \Delta_x} |g'(\lambda) + x|\right)$$

exists and $\alpha(\varepsilon) > 0$. Hence with $x = -m/s$ we have

$$(5.22) \quad\quad |\bar{a}_s^m| = \left|\frac{1}{2\pi}\int_0^{2\pi} d\lambda\, e^{-isg(\lambda)} e^{im\lambda}\right|$$

$$\leqq \frac{\varepsilon}{2} + \frac{1}{2\pi}\left|\int_{\Delta-\Delta_x} d\lambda\, e^{-is(g(\lambda)-x\lambda)}\right| .$$

The domain of integration $(\Delta - \Delta_x)$ consists of one or two intervals in dependence on x and ε. Let $\Delta' = [a, b] \subseteqq \Delta$ be such an interval. Then

$$\left|\int_a^b d\lambda\, e^{-is(g(\lambda)-x\lambda)}\right| = \left|\int_a^b \frac{d\lambda}{-is(g'(\lambda) - x)}\left(\frac{d}{d\lambda}e^{-is(g(\lambda)-x\lambda)}\right)\right|$$

$$\leqq \left| \left[\frac{1}{-is(g'(\lambda) - x)} e^{-is(g(\lambda) - x\lambda)} \right]_a^b + \int_a^b e^{-is(g(\lambda) - x\lambda)} d\left(\frac{1}{is(g'(\lambda) - x)} \right) \right|$$

$$(5.23) \quad \leqq \frac{2}{s \cdot \alpha} + \int_a^b \left| d\left(\frac{1}{s(g'(\lambda) - x)} \right) \right|$$

$$\leqq \frac{2}{s \cdot \alpha} + \frac{M}{s \cdot \alpha}$$

where α is defined by (5.21) and $M = |g'(b) - g'(a)|$. From (5.22) and (5.23) we have

$$|a_s^m| \leqq \frac{\varepsilon}{2} + \frac{1}{\pi s} \cdot \frac{2 + M}{\alpha} .$$

If we put $N = (2/\varepsilon \cdot \pi) \cdot \dfrac{2 + M}{\alpha}$, then this implies (5.20).

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