LUSIN AREA FUNCTIONS ON LOCAL FIELDS

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We show that over a local field, Lusin area functions and nontangential maximal functions of a regular function are equivalent in the $L^p$ "norm" for $0 < p < \infty$. As a consequence, we have that "nice" singular integral transforms preserve $H^p$-spaces for $0 < p < \infty$.

1. By a local field, we mean a locally compact, nondiscrete, totally disconnected, (complete) field. Various aspects of harmonic analysis on local fields have been studied. A list of references can be found in [4]. We also refer to [4] for notation and preliminaries.

Let $K$ be a fixed local field with the ring of integers $\mathcal{O}$. $\mathcal{O}/\mathfrak{P} \cong GF(q)$ where $\mathfrak{P}$ is the maximal ideal in $\mathcal{O}$ and $q$ is a prime power. For $k \in \mathbb{Z}$, let $\mathfrak{S}^{-k} = \{x \in K : |x| \leq q^k\}$, $(\mathcal{O} = \mathfrak{P}^0)$. $\mathfrak{P}^{-k} = q + \mathfrak{P}^{-k}$ are spheres. The Haar measure on $K$ has been normalized so that $|\mathcal{O}| = \int dx = 1$ and $|\mathfrak{P}^{-k}| = q^k$ for all $k$. The theory of regular functions which are the local field analogue of harmonic functions is studied in [10] and [4]. In particular, distributions on $K$ have been identified with regular functions on $K \times \mathbb{Z}$ and the regularization kernel $R_k(x) = q^{-k}\Phi_{-k}(x)$, where $\Phi_{-k}$ is the characteristic function of $\mathfrak{P}^{-k}$, serves as the Poisson kernel.

Write $(\mathfrak{S}_y^{-1}, k) = \{(x, k) \in K \times \mathbb{Z} : x \in \mathfrak{S}_y^{-1}\}$. For a nonnegative integer $l$ and $z \in K$, let $\Gamma_l(z) = \{(x, k) \in K \times \mathbb{Z} : |x - z| \leq q^{k+l}\} = \bigcup_k (\mathfrak{S}_y^{-1(k+l+1)}, k)$. For a distribution $f$ on $K$ or a regular function $f(x, k)$ on $K \times \mathbb{Z}$, denote $d_kf(x) = f(x, k) - f(x, k + 1)$. The Lusin area function of $f$ with respect to $\Gamma_l$ is given by

$$S^{(l)}f(z) = \left(\sum |d_kf(x)|^p\right)^{1/p}$$

where the sum runs over distinct $(\mathfrak{S}_y^{-k}, k) \subset \Gamma_l(z)$. Write $Sf(z) = S^{(0)}f(z) = \left(\sum |d_kf(x)|^p\right)^{1/p}$. The nontangential maximal function of $f$ with respect to $\Gamma_l$ is given by

$$m^{(l)}f(z) = \sup_{(x, k) \in \Gamma_l(z)} |f(x, k)|.$$ 

Write $f^*(z) = m^{(0)}f(z) = \sup_z |(z, k)|$.

Let us suppose that $f(x, k) \to 0$ as $k \to \infty$ for each $x \in K$. Let $\|f\|_p = \sup_x \|f(\cdot, k)\|_p$ for $0 < p < \infty$. It is shown in [10] that for $1 < p < \infty$,

$$A_p\|f\|_p \leq \|Sf\|_p \leq B_p\|f\|_p$$

with constants $A_p, B_p > 0$. 383
It is easy to see that for $1 < p < \infty$

\begin{equation}
\|f\|_p \leq \|f^*\|_p \leq C_p \|f\|_p \quad \text{with constant } C_p > 0.
\end{equation}

In other words,

\begin{equation}
\|Sf\|_p \approx \|f\|_p \approx \|f^*\|_p \quad \text{for } 1 < p < \infty.
\end{equation}

From [4], we have that, for all nonnegative $l$ and $h$,

\begin{equation}
\{x \in K: S^{(l)}f(x) < \infty\} = \left\{x \in K: \lim_{k \to -\infty} f(x, k) \text{ exists}\right\}
\end{equation}

\begin{equation}
\approx \{x \in K: m^{(h)}f(x) < \infty\};
\end{equation}

i.e., the above sets are equal except possibly for a set of measure 0. Our main objective is to show that

\begin{equation}
\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p \quad \text{for } 0 < p < \infty.
\end{equation}

As a consequence, we show that “nice” singular integral transforms preserve $H^p$-space ($0 < p < \infty$) which is the space of distributions whose maximal function are in $L^p$. The last result is the main contribution of [5].

The euclidean version of the main theorem can be found in [2] (see also [7]); its martingale version about $Sf$ and $f^*$ is proved in [1]. Our work has been motivated by these results. In Appendix we shall discuss briefly how our argument can be applied to certain martingales.

**Remark 1.** The equivalence in $L^p$ “norm” is interpreted in the obvious way, i.e., if one side is finite, so is the other and is bounded by a constant multiple of the former one. The restriction that $f(x, k) \to 0$ as $k \to \infty$ is needed only for the first inequality of (1) and $\|m^{(h)}f\|_p \leq A_p \|S^{(l)}f\|_p$.

**Remark 2.** A trivial modification gives us the same result for $K^*$, the $n$-dimensional vector space over $K$. The “$\Phi$-inequalities” of Burkholder-Gundy [1][2] for $S^{(l)}$ and $m^{(h)}$ could also be proved.

2. We first show that $\|f^*\|_p \approx \|m^{(h)}f\|_p$ for $0 < p < \infty$.

**Lemma 1.** For $\lambda > 0,$

\begin{equation}
|\{x \in K: f^*(x) > \lambda\}| \leq \left|\{z \in K: m^{(h)}f(z) > \lambda\}\right| \leq \left|\{x \in K: f^*(x) > \lambda\}\right|.
\end{equation}

**Proof.** $|\{f^* > \lambda\}| \leq \left|\{m^{(h)}f > \lambda\}\right|$ is obvious since $f^* \leq m^{(h)}f$.

Suppose $m^{(h)}f(z) > \lambda$. Then there exists $(x, k) \in \Gamma^*(z)$ such that $|f(x, k)| > \lambda$. Hence $\mathcal{P}^{-h} \subset \{f^* > \lambda\}$ and $z \in \mathcal{P}^{-h}^{(k+1)}$. Therefore
THEOREM 1. \( \| f^* \|_p \leq \| m^{(1)} f \|_p \leq \| f^* \|_p \) for \( 0 < p < \infty \).

Proof. This follows from Lemma 1 and the following identity:

\[
(5) \quad \| g \|_p^p = p \int_0^\infty \lambda^{p-1} |g(\lambda)| d\lambda, \quad 0 < p < \infty.
\]

Now let us break up the proof of \( \| S^{(1)} f \|_p \approx \| m^{(1)} f \|_p \) into several lemmas:

**Lemma 2.** \( \| S^{(1)} f \|_p = q \| S f \|_p = q \| f \|_p^2 \).

Proof. Easy and known. (See Lemma 2.8(c) of [4].)

**Lemma 3.** \( \| f^* \|_p \leq A_p \| S f \|_p \) for \( 0 < p < 2 \).

Proof. By (5), it suffices to show the following estimate:

\[
(6) \quad |\{ f^* > \lambda \}| \leq A \lambda^{-2} \int_0^\lambda t \{ S f > t \} dt \text{ for } \lambda > 0.
\]

For a fixed \( \lambda > 0 \), let

\[
\sigma(x) = \sup \{ n: S_n f(x) > \lambda \text{ for some } z \in F_z^{(n+1)} \}
\]

where \( S_n f(z) = (\sum_{k \leq n} |d_k f(z)|^2)^{1/2} \). (Convention: sup \( \emptyset = -\infty \).)

For \( x \in K \) with \( \sigma(x) = n \), let

\[
g(x, k) = \begin{cases} f(x, k) & \text{if } k \geq n + 1, \\ f(x, n + 1) & \text{if } k \leq n. \end{cases}
\]

Hence \( S g(x) \leq \lambda \) and \( S g(x) \leq S f(x) \) for all \( x \). Moreover, for \( x \in \{ \sigma = -\infty \} \subset \{ S f \leq \lambda \} \), we have \( g^*(x) = f^*(x) \) and \( S g(x) = S f(x) \). On the other hand, suppose \( \sigma(x) = n > -\infty \). Then there exists \( z \in F_z^{-(n+1)} \) such that \( S_n f(z) > \lambda \). Thus \( F_z^{-(n+1)} \subset \{ z: S f(x) > \lambda \} \) with \( x \in F_z^{-(n+1)} \). Therefore we have

\[
|\{ x: \sigma(x) > -\infty \}| \leq q |\{ z: S f(x) > \lambda \}|.
\]

Now

\[
|\{ f^* > \lambda, \sigma > -\infty \}| \leq q |\{ S f > \lambda \}|
\]

\[
\leq 2q \lambda^{-2} \int_0^\lambda t \{ S f > t \} dt
\]

and, by Lemma 2 and (5),
\[ |\{f^* > \lambda, \sigma = -\infty\}| \leq |\{g^* > \lambda\}| \leq 2\lambda^{-2}||g||^2 \]
\[ = 2\lambda^{-2}||S g||^2 = 4\lambda^{-2}\int_0^\infty t|\{S g > t\}| dt \]
\[ = 4\lambda^{-2}\int_0^1 t|\{S g > t\}| dt \]
\[ \leq 4\lambda^{-2}\int_0^1 t|\{S f > t\}| dt . \]

Thus
\[ |\{f^* > \lambda\}| \leq |\{f^* > \lambda, \sigma > -\infty\}| + |\{f^* > \lambda, \sigma = -\infty\}| \]
\[ \leq (2q + 4)\lambda^{-2}\int_0^1 t|\{S f > t\}| dt . \]

This establishes (6) and Lemma 3.

**Lemma 4.** For \( l > 0 \) and \( 0 < p < 2 \),
\[ ||S^{(l)} f||_p \leq B_p||m^{(l)} f||_p . \]

**Proof.** Again, it suffices to show that for \( l > 0 \) and \( \lambda > 0 \),
\[ |\{S^{(l)} f > \lambda\}| \leq B\lambda^{-2}\int_0^1 t|\{m^{(l)} f > t\}| dt . \]

Let \( \mu(z) = \sup\{n : |f(x, n)| > \lambda \text{ for some } x \in \mathcal{F}_z^{-(a+1)}\} \). For \( z \in K \) with \( \mu(z) = n \), we have \( \mu(x) = n \) for all \( x \in \mathcal{F}_z^{-(a+1)} \); and let
\[ g(z, k) = \begin{cases} f(x, k) & \text{if } k \geq n + 1, \\ f(x, n + 1) & \text{if } k \leq n. \end{cases} \]

Hence \( \{ \mu = -\infty \} = \{ m^{(l)} f \leq \lambda \} \) and for \( \mu(z) = -\infty \), we have \( g(x, k) = f(x, k) \) if \( x \in \mathcal{F}_z^{-(a+1)} \) or \( (x, k) \in \Gamma_1(z) \). Thus on \( \{ z: \mu(z) = -\infty \} \), \( S^{(l)} g(z) = S^{(l)} f(z) \) and \( m^{(l)} g(z) = m^{(l)} f(z) \leq \lambda \). Now
\[ |\{S^{(l)} f > \lambda, \mu > -\infty\}| \leq |\{m^{(l)} f > \lambda\}| \]
\[ \leq 2\lambda^{-2}\int_0^1 t|\{m^{(l)} f > t\}| dt , \]
and by Lemma 2 and (5),
\[ |\{S^{(l)} f > \lambda, \mu = -\infty\}| \leq |\{S^{(l)} g > \lambda\}| \leq \lambda^{-2}||S^{(l)} g||^2 \]
\[ = q^l\lambda^{-2}||g||^2 \leq q^l\lambda^{-2}||m^{(l)} g||^2 \]
\[ \leq q^l\lambda^{-2} \cdot 2\int_0^\infty t|\{m^{(l)} g > t\}| dt \]
\[ \leq 2q^l\lambda^{-2}\int_0^1 t|\{m^{(l)} f > t\}| dt \]

Hence
Therefore Lemma 4 is proved.

**Lemma 5.** For \( l \geq 0 \) and \( 2 < p < \infty \),

\[
\| S^{(l)} f \|_p \leq C_p \| f \|_p .
\]

**Proof.** Suppose \( p > 4 \) and let \( r \) be the conjugate index of \( p/2 \). Thus \( 1 < r < 2 \). Consider a fixed \( k \in \mathbb{Z} \). For \( x \in K \), let \( \{x_i\}_{i=1}^q \) be the distinct coset representatives such that \( K^q \sim (k-l+1) \subset \mathcal{S}_x^{-(k-1)} \). For \( g \in L^r \) with \( \|g\|_r = 1 \), we have

\[
\int_K \sum_{i=1}^q d_k f(x_i) \left| g(x) \right| dx = \sum_{i=1}^q \int_K d_k f(x_i) \left| g(x, k+1) \right| dx
\]

\[
= \sum_{i=1}^q \int_K d_k f(x_i) \left| g(x, k+1) \right| dx
\]

\[
= q \int_K \sum_{i=1}^q d_k f(x) \left| g(x, k+1) \right| dx .
\]

Hence it follows from this, Hölder's inequality, (1) and (2) that

\[
\int_K [S_n f(x)]^q \left| g(x) \right| dx = \sum_{k \in \mathbb{Z}} \int_K \sum_{i=1}^q d_k f(x_i) \left| g(x) \right| dx
\]

\[
= \sum_{k \in \mathbb{Z}} q \int_K [S_n f(x)]^q \left| g(x, k+1) \right| dx
\]

\[
\leq q \|S_n f\|_p^q \|g^*\|_r
\]

\[
\leq B_p \|f\|_p^p .
\]

where \( B_p \) depends only on \( p \) and \( q \). Thus

\[
\|S_n f\|_p^p = \|S_n f^q\|_{p/2} = \sup_{g \in L^r, \|g\|_r = 1} \left| \int_K [S_n f(x)]^q g(x) dx \right|
\]

\[
\leq B_p \|f\|_p^p .
\]

Therefore \( \|S f\|_p \leq C_p \|f\|_p \) for \( 4 < p < \infty \).

Apply the Marcinkiewicz interpolation theorem to this and Lemma 2, we have

\[
\|S f\|_p \leq C_p \|f\|_p \text{ for } 2 < p < \infty .
\]

**Theorem 2.** For \( l, h \geq 0 \) and \( 0 < p < \infty \),

\[
\|S^{(l)} f\|_p \approx \|m^{(h)} f\|_p .
\]

**Proof.** The case of \( p = 2 \) is obvious.
If $0 < p < 2$, then, from Lemma 3, Lemma 4 and Theorem 1, we have for $I > 0$,
\[
\|f^*\|_p \leq A_p \|SF\|_p \leq A_p \|S^{(i)}f\|_p \\
\leq A_p B_p \|m^{(i)}f\|_p \approx \|f^*\|_p .
\]

If $2 < p < \infty$, then, by Theorem 1, (3) and Lemma 5,
\[
\|m^{(h)}f\|_p \approx \|f^*\|_p \approx \|F\|_p \approx \|Sf\|_p \\
\leq \|S^{(i)}f\|_p \leq C_p \|f\|_p .
\]

Therefore $\|S^{(i)}f\|_p \approx \|m^{(h)}f\|_p$ for $0 < p < \infty$ and the proof of the theorem is completed.

**Remark 3.** The above argument simplifies the extension argument as used in §2 of [4] and is essentially similar to the decomposition argument of [5]. It is also a sort of stopping time argument for martingales relative to a regular stochastic basis. (See Appendix.) The main result (with respect to “truncated cones”) could be used to show (4)—the Fatou-Calderón-Stein theorem, in a similar manner as in [2].

3. Let $\pi$ be a (multiplicative) unitary character on $K^*$ such that it is homogeneous of degree 0 and is ramified of degree $h \geq 1$. Denote $Q(x) = c\pi(x)|x|^{-1}$ where $c = 1/\Gamma(\pi)$. (See [9] for details about $\Gamma$-function.) Let $Q_n = R_n \ast Q$ and $Q_n^g = Q_n \Phi_{-\gamma}$ for $N \geq n + h$. For a distribution $f$ on $K$ or a regular function $f(x, k)$ on $K \times \mathbb{Z}$, we note that $Q_n^g \ast f(x, k) = Q_n^g \ast f(x, k)$ for $n \leq k \leq N - h$. Define
\[
(T_\pi f)(x, k) = \lim_{N \to \infty} Q_n^g \ast f(x, k) \quad \text{for} \quad (x, k) \in K \times \mathbb{Z}.
\]

If $f \in L^p(K)$, $1 \leq p < \infty$, then this is just a sort of singular integral transform as been studied in [8], [11] and [4].

For $0 < p < \infty$, let $H^p(K)$ be the space of all distributions $f$ on $K$ whose maximal function $f^* \in L^p(K)$ with the $H^p$ “norm” $\|f^*\|_p$. From [5], we know that for $f \in H^p$, $(T_\pi f)(x, k)$ is a well-defined regular function. The regularization of the corresponding distribution is just $(T_\pi f)(x, k)$. Moreover, the following is also shown:

**Theorem 3.** $T_\pi$ preserves $H^p$-spaces for $0 < p < \infty$. That is, $\|(T_\pi f)^*\|_p \approx \|f^*\|_p$ for $0 < p < \infty$.

We show here how this result can be obtained as a consequence of Theorem 2.

**Lemma 6.** $S^{(h)}f(z) = S^{(h)}T_\pi f(z)$ for all $z \in K$. 

Proof. For a fixed \( k \in \mathbb{Z} \) and \( x \in K \),
\[
d_{k}T_{x}f(x) = T_{x}f(x, k) - T_{x}f(x, k + 1) = T_{x}d_{k}f(x).
\]

For each \( m \in \mathbb{Z} \), let \( \varepsilon_{m}^{i} \), \( i = 1, 2, \ldots, (q - 1)q^{h-1} \), be coset representatives of \( \mathcal{G}^{-(m-h+1)} \) in \( \{ t : |t| = q^{m+1} \} \). Then
\[
T_{x}f(x, k) = c \int_{|t| > q^{h}} f(x - t) \frac{\pi(t)}{|t|} dt
\]
\[
= c \sum_{m=k}^{\infty} q^{-(m+1)} \int_{|t| = q^{m+1}} f(x - t) \pi(t) dt
\]
\[
= cq^{-h} \sum_{m=k}^{\infty} \sum_{i=1}^{(q-1)q^{h-1}} \pi(\varepsilon_{m}^{i}) f(x - \varepsilon_{m}^{i}, m - h + 1).
\]
Thus
\[
(7) \quad T_{x}d_{k}f(x) = cq^{-h} \sum_{i=1}^{(q-1)q^{h-1}} \pi(\varepsilon_{m}^{i}) f(x - \varepsilon_{m}^{i}, k - h + 1).
\]

Now let \( g(x) \) be the restriction of \( d_{k}f(x) \) on \( z + \mathcal{G}^{-(h+1)} \) for any fixed \( z \). Hence from (7) we see that \( T_{x}g(x) \) is also supported on \( z + \mathcal{G}^{-(h+1)} \). By Plancherel’s theorem, since \( |\pi| = 1 \), we have
\[
\| T_{x}g \|_{2} = \| (T_{x}g)^{\pi} \|_{2} = \| \pi^{-1}g \|_{p} = \| g \|_{p} = \| g \|_{2}.
\]
That is,
\[
\sum_{i=1}^{q^{h}} |d_{k}f(x_{i})|^{2} = \sum_{i=1}^{q^{h}} |d_{k}T_{x}f(x_{i})|^{2}
\]
where \( x_{i}, i = 1, 2, \ldots, q^{h} \), are coset representatives of \( \mathcal{G}^{-(h-h+1)} \) in \( \mathcal{G}_{z}^{-(h+1)} \). Thus summing this up with respect to \( k \), we have
\[
S^{(k)}f(x) = S^{(k)}T_{x}f(x).
\]

Proof of Theorem 3. It follows immediately from Theorem 2 and Lemma 6 that for \( 0 < p < \infty \),
\[
\| f^{*} \|_{p} \approx \| S^{(k)}f \|_{p} = \| S^{(k)}T_{x}f \|_{p} \approx \| (T_{x}f)^{*} \|_{p}.
\]

Appendix. Let \( (\Omega, \mathcal{A}, P) \) be a probability space and \( \{ \mathcal{A}_{n} \}_{n=1} \) a nondecreasing sequence of sub-\( \sigma \)-fields of \( \mathcal{A} \). Let \( f = \{ f_{n} \}_{n \geq 1} \) be a real-valued martingale relative to \( \{ \mathcal{A}_{n} \}_{n \geq 1} \) and \( \{ d_{k} \}_{k \geq 1} \) be the difference sequence of \( f \). For a nonnegative integer \( l \), write
\[
m^{(l)}f = \sup_{n} E(|f_{n+l}| \mid \mathcal{A}_{n})
\]
and \( S^{(l)}f = \left[ \sum_{k \geq 1} E(d_{k}^{l} \mid \mathcal{A}_{k-1}) \right]^{1/2} \). \( f^{*} = m^{(0)}f = \sup_{n} |f_{n}| \) is the maximal function of \( f \) and \( Sf = S^{(0)}f = \left[ \sum_{k \geq 1} d_{k}^{2} \right]^{1/2} \) is the square function of \( f \). Burkholder and Gundy [1] proved that for a large class of
martingales, 
\begin{equation}
\| S^f \|_p \approx \| f^* \|_p \quad \text{for} \quad 0 < p < \infty.
\end{equation}

However examples (in [1]) show that
\begin{equation}
\| S^{(1)}f \|_p \approx \| m^{(1)}f \|_p \quad \text{for} \quad 0 < p < \infty
\end{equation}

fails to hold. Nevertheless by a slight modification of the previous argument, we can show that this is true for martingales relative to a regular stochastic basis (after Chow [6]).

Indeed, the crucial part of the proof is to consider the following stopping time:
\[ \mu(x) = \inf \{ n : E(|f_{n+1}|^\lambda) < \lambda \} \quad (\lambda > 0). \]

Together with the regularity of the stochastic basis and (8), we get (9) by a similar argument as before.

We remark that our argument gives a simplified proof of (8) for martingales relative to a regular stochastic basis. Also the argument used in Lemma 5 similar to the one in [3] provides a new proof of that
\[ \| sf \|_p \leq C_p \| f \|_p \quad \text{for} \quad p > 2 \]

where \( sf = S^{(1)}f = [\sum_{k>1} E(d^2_k | \mathcal{F}_{k-1})]^{1/2} \) is the conditioned square function of the martingale \( f \) (relative to any stochastic basis).

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