LUSIN AREA FUNCTIONS ON LOCAL FIELDS

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We show that over a local field, Lusin area functions and nontangential maximal functions of a regular function are equivalent in the $L^p$ "norm" for $0 < p < \infty$. As a consequence, we have that "nice" singular integral transforms preserve $H^p$-spaces for $0 < p < \infty$.

1. By a local field, we mean a locally compact, nondiscrete, totally disconnected, (complete) field. Various aspects of harmonic analysis on local fields have been studied. A list of references can be found in [4]. We also refer to [4] for notation and preliminaries.

Let $K$ be a fixed local field with the ring of integers $\mathcal{O}$. $\mathcal{O}/\mathcal{P} \cong GF(q)$ where $\mathcal{P}$ is the maximal ideal in $\mathcal{O}$ and $q$ is a prime power. For $k \in \mathbb{Z}$, let $\mathcal{P}^{-k} = \{x \in K: |x| \leq q^k\}$, $\mathcal{O} = \mathcal{P}^0$. $\mathcal{P}^{-k} = y + \mathcal{P}^{-k}$ are spheres. The Haar measure on $K$ has been normalized so that $|\mathcal{O}| = \int_\mathcal{O} dx = 1$ and $|\mathcal{P}^{-k}_y| = q^k$ for all $k$. The theory of regular functions which are the local field analogue of harmonic functions is studied in [10] and [4]. In particular, distributions on $K$ have been identified with regular functions on $K \times \mathbb{Z}$ and the regularization kernel $R_k(x) = q^{-k} \Phi_{-k}(x)$, where $\Phi_{-k}$ is the characteristic function of $\mathcal{P}^{-k}$, serves as the Poisson kernel.

Write $(\mathcal{P}_y^{-k}, k) = \{(x, k) \in K \times \mathbb{Z}: x \in \mathcal{P}_y^{-k}\}$. For a nonnegative integer $l$ and $z \in K$, let $\Gamma_l(z) = \{(x, k) \in K \times \mathbb{Z}: |x - z| \leq q^{l+1}\} = \bigcup_k (\mathcal{P}_y^{-k+1}, k)$. For a distribution $f$ on $K$ or a regular function $f(x, k)$ on $K \times \mathbb{Z}$, denote $d_k f(x) = f(x, k) - f(x, k + 1)$. The Lusin area function of $f$ with respect to $\Gamma_l$ is given by

$$S^{(1)} f(z) = (\sum |d_k f(x)|^2)^{1/2}$$

where the sum runs over distinct $(\mathcal{P}_y^{-k}, k) \subset \Gamma_l(z)$. Write $S f(z) = S^{(0)} f(z) = (\sum_k |d_k f(z)|^2)^{1/2}$. The nontangential maximal function of $f$ with respect to $\Gamma_1$ is given by

$$m^{(1)} f(z) = \sup_{(x, k) \in \Gamma_1(z)} |f(x, k)|.$$

Write $f^*(z) = m^{(0)} f(z) = \sup_k |(z, k)|$.

Let us suppose that $f(x, k) \to 0$ as $k \to \infty$ for each $x \in K$. Let $\|f\|_p = \sup_k \|f(\cdot, k)\|_p$ for $0 < p < \infty$. It is shown in [10] that for $1 < p < \infty$,

$$A_p \|f\|_p \leq \|S f\|_p \leq B_p \|f\|_p$$

with constants $A_p, B_p > 0$. 

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It is easy to see that for $1 < p < \infty$

\[(2) \|f\|_p \leq \|f^*\|_p \leq C_p\|f\|_p \quad \text{with constant } C_p > 0.\]

In other words,

\[(3) \|Sf\|_p \approx \|f\|_p \approx \|f^*\|_p \quad \text{for } 1 < p < \infty.\]

From [4], we have that, for all nonnegative $l$ and $h$,

\[\{x \in K: S^{(l)}f(x) < \infty\} \approx \{x \in K: \lim_{k \to \infty} f(x, k) \text{ exists}\}\]
\[\approx \{x \in K: m^{(h)}f(x) < \infty\};\]

i.e., the above objective is to show that

\[\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p \quad \text{for } 0 < p < \infty.\]

As a consequence, we show that “nice” singular integral transforms preserve $H^p$-space ($0 < p < \infty$) which is the space of distributions whose maximal function are in $L^p$. The last result is the main contribution of [5].

The euclidean version of the main theorem can be found in [2] (see also [7]); its martingale version about $Sf$ and $f^*$ is proved in [1]. Our work has been motivated by these results. In Appendix we shall discuss briefly how our argument can be applied to certain martingales.

**Remark 1.** The equivalence in $L^p$ “norm” is interpreted in the obvious way, i.e., if one side is finite, so is the other and is bounded by a constant multiple of the former one. The restriction that $f(x, k) \to 0$ as $k \to \infty$ is needed only for the first inequality of (1) and $\|m^{(h)}f\|_p \leq A_p\|S^{(l)}f\|_p$.

**Remark 2.** A trivial modification gives us the same result for $K^*$, the $n$-dimensional vector space over $K$. The “$\phi$-inequalities” of Burkholder-Gundy [1][2] for $S^{(l)}$ and $m^{(h)}$ could also be proved.

2. We first show that $\|f^*\|_p \approx \|m^{(l)}f\|_p$ for $0 < p < \infty$.

**Lemma 1.** For $\lambda > 0$,

\[|\{x \in K: f^*(x) > \lambda\}| \leq |\{z \in K: m^{(l)}f(z) > \lambda\}| \leq q^l |\{x \in K: f^*(x) > \lambda\}|.\]

**Proof.** $|\{f^* > \lambda\}| \leq |\{m^{(l)}f > \lambda\}|$ is obvious since $f^* \leq m^{(l)}f$.

Suppose $m^{(l)}f(z) > \lambda$. Then there exists $(x, k) \in \Gamma_1(z)$ such that $|f(x, k)| > \lambda$. Hence $\mathcal{P}_{z}^{-k} \subset \{f^* > \lambda\}$ and $z \in \mathcal{P}_{z}^{-(k+1)}$. Therefore
THEOREM 1. \[ \| f^* \|_p \leq \| m^{(i)}f \|_p \leq q^{1/p} \| f^* \|_p \text{ for } 0 < p < \infty. \]

Proof. This follows from Lemma 1 and the following identity:

\[ \| g \|_p^p = p \int_0^\infty \lambda^{p-1} |\{ g > \lambda \}| d\lambda, \quad 0 < p < \infty. \]

Now let us break up the proof of \( \| S^{(i)}f \|_p \approx \| m^{(h)}f \|_p(0 < p < \infty) \) into several lemmas:

LEMMA 2. \[ \| S^{(i)}f \|_p^2 = q^i \| Sf \|_p^2 = q^i \| f \|_p^2. \]

Proof. Easy and known. (See Lemma 2.8(c) of [4].)

LEMMA 3. \[ \| f^* \|_p \leq A_p \| Sf \|_p \text{ for } 0 < p < 2. \]

Proof. By (5), it suffices to show the following estimate:

\[ \| \{ f^* > \lambda \} \| \leq A\lambda^{-2} \int_0^\lambda t \| Sf > t \| dt \text{ for } \lambda > 0. \]

For a fixed \( \lambda > 0 \), let

\[ \sigma(x) = \sup \{ n: S_nf(z) > \lambda \text{ for some } z \in \mathcal{P}_{\lambda}^{-(n+1)} \} \]

where \( S_nf(z) = (\sum_{k \geq n} |d_kf(z)|^2)^{1/2} \). (Convention: \( \sup \emptyset = -\infty \).)

For \( x \in K \) with \( \sigma(x) = n \), let

\[ g(x, k) = \begin{cases} f(x, k) & \text{if } k \geq n + 1, \\ f(x, n + 1) & \text{if } k \leq n. \end{cases} \]

Hence \( S_g(x) \leq \lambda \) and \( S_g(x) \leq S_f(x) \) for all \( x \). Moreover, for \( x \in \{ \sigma = -\infty \} \subset \{ S_f \leq \lambda \} \), we have \( g^*(x) = f^*(x) \) and \( S_g(x) = S_f(x) \). On the other hand, suppose \( \sigma(x) = n > -\infty \). Then there exists \( z \in \mathcal{P}_{\lambda}^{-(n+1)} \) such that \( S_nf(z) > \lambda \). Thus \( \mathcal{P}_{\lambda}^{-(n)} \subset \{ z: S_f(z) > \lambda \} \) with \( x \in \mathcal{P}_{\lambda}^{-(n+1)} \). Therefore we have

\[ |\{ x: \sigma(x) > -\infty \}| \leq q |\{ z: S_f(z) > \lambda \}|. \]

Now

\[ |\{ f^* > \lambda, \sigma > -\infty \}| \leq q |\{ S_f > \lambda \}| \]

\[ \leq 2q\lambda^{-2} \int_0^\lambda t \| S_f > t \| dt \]

and, by Lemma 2 and (5),
Thus
\[ |\{f^* > \lambda\}| \leq |\{f^* > \lambda, \sigma = -\infty\}| + |\{f^* > \lambda, \sigma = -\infty\}| \]
\[ \leq (2q + 4)\lambda^{-2} \int_0^2 t \{Sf > t\} dt . \]

This establishes (6) and Lemma 3.

**Lemma 4.** For \( l > 0 \) and \( 0 < p < 2 \),
\[ \| S^{(l)} f \|_p \leq B \| m^{(l)} f \|_p . \]

**Proof.** Again, it suffices to show that for \( l > 0 \) and \( \lambda > 0 \),
\[ |\{S^{(l)} f > \lambda\}| \leq B\lambda^{-2} \int_0^2 t \{m^{(l)} f > t\} dt . \]

Let \( \mu(z) = \sup \{ n: |f(x, n)| > \lambda \text{ for some } x \in \mathcal{G}_z^{-(n+1)} \} \). For \( z \in K \) with \( \mu(z) = n \), we have \( \mu(x) = n \) for all \( x \in \mathcal{G}_z^{-(n+1)} \); and let
\[ g(z, k) = \begin{cases} f(x, k) & \text{if } k \geq n + 1 , \\ f(x, n + 1) & \text{if } k \leq n . \end{cases} \]

Hence \( \{\mu = -\infty\} = \{m^{(l)} f \leq \lambda\} \) and for \( \mu(z) = -\infty \), we have \( g(x, k) = f(x, k) \) if \( x \in \mathcal{G}_z^{-(k+1)} \) or \( (x, k) \in \Gamma(z) \). Thus on \( \{z: \mu(z) = -\infty\}, S^{(l)} g(z) = S^{(l)} f(z) \) and \( m^{(l)} g(z) = m^{(l)} f(z) \leq \lambda \). Now
\[ |\{S^{(l)} f > \lambda, \mu > -\infty\}| \leq |\{m^{(l)} f > \lambda\}| \]
\[ \leq 2\lambda^{-2} \int_0^2 t \{m^{(l)} f > t\} dt , \]

and by Lemma 2 and (5),
\[ |\{S^{(l)} f > \lambda, \mu = -\infty\}| \leq |\{S^{(l)} g > \lambda\}| \leq \lambda^{-2} \| S^{(l)} g \|_p^2 \]
\[ = q^l \lambda^{-2} \| g \|_p^2 \leq q^l \lambda^{-2} \| m^{(l)} g \|_p^2 \]
\[ \leq q^l \lambda^{-2} \int_0^2 t \{m^{(l)} g > t\} dt \]
\[ \leq 2q^l \lambda^{-2} \int_0^2 t \{m^{(l)} f > t\} dt . \]

Hence
\[ \left| \{ S^{(l)}f > \lambda \} \right| \leq 2(q^l + 1)\lambda^{-2} \int_0^t \left| \{ m^{(l)}f > t \} \right| dt. \]

Therefore Lemma 4 is proved.

**Lemma 5.** For \( l \geq 0 \) and \( 2 < p < \infty \),
\[ \| S^{(l)}f \|_p \leq C_p \| f \|_p. \]

**Proof.** Suppose \( p > 4 \) and let \( r \) be the conjugate index of \( p/2 \). Thus \( 1 < r < 2 \). Consider a fixed \( k \in \mathbb{Z} \). For \( x \in K \), let \( \{ x_i \}_{i=1}^q \) be the distinct coset representatives such that \( \mathcal{P}_{x_i}^{(k-1+1)} \subset \mathcal{P}_{x}^{-(k+1)} \). For \( g \in L^r \) with \( \| g \|_r = 1 \), we have
\[
\int_K \sum_{i=1}^q |d_kf(x_i)|^r g(x) |dx = \sum_{i=1}^q \int_K |d_kf(x_i)|^r g(x, k + 1) |dx
\]
\[
= q^l \int_K |d_kf(x)|^r g(x, k + 1) |dx.
\]
Hence it follows from this, Hölder's inequality, (1) and (2) that
\[
\int_K [S^{(l)}f(x)]^r g(x) |dx = \sum_{k \in \mathbb{N}} \int_K \sum_{i=1}^q |d_kf(x_i)|^r g(x) |dx
\]
\[
= \sum_{k \in \mathbb{N}} q^l \int_K [S_k f(x)]^r g^*(x) |dx
\]
\[
\leq q^l \| S_k f \|_r^p \| g^* \|_r
\]
\[
\leq B_p \| f \|_p^2
\]
where \( B_p \) depends only on \( p \) and \( q \). Thus
\[
\| S^{(l)}f \|_p^p = \| [S^{(l)}f]^r \|_{p/2} = \sup_{g \in L^r} \left| \int_K [S^{(l)}f(x)]^r g(x) |dx \right|
\]
\[
\leq B_p \| f \|_p^2.
\]
Therefore \( \| S^{(l)}f \|_p \leq C_p \| f \|_p \) for \( 4 < p < \infty \).

Apply the Marcinkiewicz interpolation theorem to this and Lemma 2, we have
\[
\| S^{(l)}f \|_p \leq C_p \| f \|_p \quad \text{for} \quad 2 < p < \infty.
\]

**Theorem 2.** For \( l, h \geq 0 \) and \( 0 < p < \infty \),
\[
\| S^{(l)}f \|_p \approx \| m^{(h)}f \|_p.
\]

**Proof.** The case of \( p = 2 \) is obvious.
If $0 < p < 2$, then, from Lemma 3, Lemma 4 and Theorem 1, we have for $l > 0$,
\[ \| f^* \|_p \leq A_p \| Sf \|_p \leq A_p \| S(f) \|_p \]
\[ \leq A_p B_p \| m(f) \|_p \sim \| f^* \|_p. \]

If $2 < p < \infty$, then, by Theorem 1, (3) and Lemma 5,
\[ \| m(f) \|_p \sim \| f^* \|_p \sim \| f \|_p \sim \| Sf \|_p \]
\[ \leq \| S(f) \|_p \leq C_p \| f \|_p. \]

Therefore $\| S(f) \|_p \sim \| m(f) \|_p$ for $0 < p < \infty$ and the proof of the theorem is completed.

REMARK 3. The above argument simplifies the extension argument as used in §2 of [4] and is essentially similar to the decomposition argument of [5]. It is also a sort of stopping time argument for martingales relative to a regular stochastic basis. (See Appendix.) The main result (with respect to "truncated cones") could be used to show (4)—the Fatou-Calderón-Stein theorem, in a similar manner as in [2].

3. Let $\pi$ be a (multiplicative) unitary character on $K^*$ such that it is homogeneous of degree 0 and is ramified of degree $h \geq 1$. Denote $Q(x) = c\pi(x)|x|^{-1}$ where $c = 1/\Gamma(\pi)$. (See [9] for details about $\Gamma$-function.) Let $Q_n = R_n*Q$ and $Q_n = Q_n\Phi_{-N}$ for $N \geq n + h$. For a distribution $f$ on $K$ or a regular function $f(x, k)$ on $K \times Z$, we note that $Q_n*Q_n f(x, k) = Q_n*Q_n f(x, k) = Q_n*Q_n f(x, k)$ for $n \leq k \leq N - h$. Define
\[ (T_{\pi} f)(x, k) = \lim_{N \to \infty} Q_n f(x, k) \text{ for } (x, k) \in K \times Z. \]

If $f \in L^p(K)$, $1 \leq p < \infty$, then this is just a sort of singular integral transform as been studied in [8], [11] and [4].

For $0 < p < \infty$, let $H^p(K)$ be the space of all distributions $f$ on $K$ whose maximal function $f^* \in L^p(K)$ with the $H^p$ "norm" $\| f^* \|_p$. From [5], we know that for $f \in H^p$, $(T_{\pi} f)(x, k)$ is a well-defined regular function. The regularization of the corresponding distribution is just $(T_{\pi} f)(x, k)$. Moreover, the following is also shown:

THEOREM 3. $T_{\pi}$ preserves $H^p$-spaces for $0 < p < \infty$. That is, $\| (T_{\pi} f)^* \|_p \approx \| f^* \|_p$ for $0 < p < \infty$.

We show here how this result can be obtained as a consequence of Theorem 2.

LEMMA 6. $S^{(h)} f(z) = S^{(h)} T_{\pi} f(z)$ for all $z \in K$. 


Proof. For a fixed \( k \in \mathbb{Z} \) and \( x \in K \),

\[
d_k T_x f(x) = T_x f(x, k) - T_x f(x, k + 1) = T_x d_k f(x).
\]

For each \( m \in \mathbb{Z} \), let \( \varepsilon_m, i = 1, 2, \ldots, (q - 1)q^{h-1} \), be coset representatives of \( \mathcal{G}^{-(m-h+1)} \) in \( \{ t : |t| = q^{m+1} \} \). Then

\[
T_x f(x, k) = c \int_{|t| > q^k} f(x - t) \pi(t) \cdot d \cdot dt
= c \sum_{m=k}^{\infty} q^{- (m+1)} \int_{|t| = q^{m+1}} f(x - t) \pi(t) \cdot dt
= cq^{-h} \sum_{m=k}^{\infty} \sum_{i=1}^{(q-1)q^{h-1}} \pi(\varepsilon_m^i) f(x - \varepsilon_m^i, m - h + 1).
\]

Thus

\[
(7) \quad T_x d_k f(x) = cq^{-h} \sum_{i=1}^{(q-1)q^{h-1}} \pi(\varepsilon_m^i) f(x - \varepsilon_m^i, k - h + 1).
\]

Now let \( g(x) \) be the restriction of \( d_k f(x) \) on \( z + \mathcal{G}^{-(h+1)} \) for any fixed \( z \). Hence from (7) we see that \( T_x g(x) \) is also supported on \( z + \mathcal{G}^{-(h+1)} \). By Plancherel’s theorem, since \( |\pi| = 1 \), we have

\[
\| T_x g \|_2 = \| (T_x g)^* \|_2 = \| \pi^{-1} \hat{g} \|_p = \| \hat{g} \|_2 = \| g \|_2.
\]

That is,

\[
\sum_{i=1}^{h} |d_k f(x_i)|^2 = \sum_{i=1}^{h} |d_k T_x f(x_i)|^2
\]

where \( x_i, i = 1, 2, \ldots, q^h \), are coset representatives of \( \mathcal{G}^{-(h+1)} \) in \( \mathcal{G}^{-(h+1)} \). Thus summing this up with respect to \( k \), we have

\[
S^{(h)} f(z) = S^{(h)} T_x f(z).
\]

Proof of Theorem 3. It follows immediately from Theorem 2 and Lemma 6 that for \( 0 < p < \infty \),

\[
\| f^* \|_p \approx \| S^{(h)} f \|_p = \| S^{(h)} T_x f \|_p \approx \| (T_x f)^* \|_p.
\]

Appendix. Let \((\Omega, \mathcal{A}, P)\) be a probability space and \( \{ \mathcal{A}_n \}_{n=1} \) a nondecreasing sequence of sub-\( \sigma \)-fields of \( \mathcal{A} \). Let \( f = \{ f_n \}_{n \geq 1} \) be a real-valued) martingale relative to \( \{ \mathcal{A}_n \}_{n \geq 1} \) and \( \{ d_k \}_{k \geq 1} \) be the difference sequence of \( f \). For a nonnegative integer \( l \), write

\[
m^{(l)} f = \sup_n E(\| f_n+1 \|_{\mathcal{A}_n})
\]

and \( S^{(l)} f = [\sum_{k \geq 1} E(\| d_k^l \|_{\mathcal{A}_{k-1}})]^{1/2} \). \( f^* = m^{(0)} f = \sup_n \| f_n \| \) is the maximal function of \( f \) and \( S f = S^{(0)} f = [\sum_{k \geq 0} d_k^l]^{1/2} \) is the square function of \( f \). Burkholder and Gundy [1] proved that for a large class of
martingales,
\begin{equation}
\|Sf\|_p \approx \|f^*\|_p \quad \text{for } 0 < p < \infty .
\end{equation}
However examples (in [1]) show that
\begin{equation}
\|S^{(1)}f\|_p \approx \|m^{(k)}f\|_p \quad \text{for } 0 < p < \infty
\end{equation}
fails to hold. Nevertheless by a slight modification of the previous argument, we can show that this is true for martingales relative to a regular stochastic basis (after Chow [6]).

Indeed, the crucial part of the proof is to consider the following stopping time:
\[ \mu(x) = \inf \{ n : E(\|f_{n+1}\|_{\mathcal{F}_n}) < \lambda \} \quad (\lambda > 0) . \]
Together with the regularity of the stochastic basis and (8), we get (9) by a similar argument as before.

We remark that our argument gives a simplified proof of (8) for martingales relative to a regular stochastic basis. Also the argument used in Lemma 5 similar to the one in [3] provides a new proof of that
\[ \|sf\|_p \leq C_p \|f\|_p \quad \text{for } p > 2 \]
where \( sf = S^{(1)}f = [\sum_{k>1} E(d_k|\mathcal{F}_{k-1})]^{1/2} \) is the conditioned square function of the martingale \( f \) (relative to any stochastic basis).

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