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LUSIN AREA FUNCTIONS ON LOCAL FIELDS

JIA-ARNG CHAO

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We show that over a local field, Lusin area functions and nontangential maximal functions of a regular function are equivalent in the L^p "norm" for $0 < p < \infty$. As a consequence, we have that "nice" singular integral transforms preserve H^p -spaces for $0 < p < \infty$.

1. By a local field, we mean a locally compact, nondiscrete, totally disconnected, (complete) field. Various aspects of harmonic analysis on local fields have been studied. A list of references can be found in [4]. We also refer to [4] for notation and preliminaries.

Let K be a fixed local field with the ring of integers \mathcal{O} . $\mathcal{O}/\mathcal{P} \cong GF(q)$ where \mathcal{P} is the maximal ideal in \mathcal{O} and q is a prime power. For $k \in \mathbf{Z}$, let $\mathcal{P}^{-k} = \{x \in K: |x| \leq q^k\}$, ($\mathcal{O} = \mathcal{P}^0$). $\mathcal{P}_y^{-k} = y + \mathcal{P}^{-k}$ are spheres. The Haar measure on K has been normalized so that $|\mathcal{O}| = \int_{\mathcal{O}} dx = 1$ and $|\mathcal{P}_y^{-k}| = q^k$ for all k . The theory of regular functions which are the local field analogue of harmonic functions is studied in [10] and [4]. In particular, distributions on K have been identified with regular functions on $K \times \mathbf{Z}$ and the regularization kernel $R_k(x) = q^{-k}\Phi_{-k}(x)$, where Φ_{-k} is the characteristic function of \mathcal{P}^{-k} , serves as the Poisson kernel.

Write $(\mathcal{P}_y^{-l}, k) = \{(x, k) \in K \times \mathbf{Z}: x \in \mathcal{P}_y^{-l}\}$. For a nonnegative integer l and $z \in K$, let $\Gamma_l(z) = \{(x, k) \in K \times \mathbf{Z}: |x - z| \leq q^{k+l}\} = \bigcup_k (\mathcal{P}_z^{-(k+l)}, k)$. For a distribution f on K or a regular function $f(x, k)$ on $K \times \mathbf{Z}$, denote $d_k f(x) = f(x, k) - f(x, k + 1)$. The Lusin area function of f with respect to Γ_l is given by

$$S^{(l)} f(z) = (\sum |d_k f(x)|^2)^{1/2}$$

where the sum runs over distinct $(\mathcal{P}_x^{-k}, k) \subset \Gamma_l(z)$. Write $Sf(z) = S^{(0)} f(z) = (\sum_k |d_k f(z)|^2)^{1/2}$. The nontangential maximal function of f with respect to Γ_l is given by

$$m^{(l)} f(z) = \sup_{(x,k) \in \Gamma_l(z)} |f(x, k)|.$$

Write $f^*(z) = m^{(0)} f(z) = \sup_k |(z, k)|$.

Let us suppose that $f(x, k) \rightarrow 0$ as $k \rightarrow \infty$ for each $x \in K$. Let $\|f\|_p = \sup_k \|f(\cdot, k)\|_p$ for $0 < p < \infty$. It is shown in [10] that for $1 < p < \infty$,

$$(1) \quad A_p \|f\|_p \leq \|Sf\|_p \leq B_p \|f\|_p \text{ with constants } A_p, B_p > 0.$$

It is easy to see that for $1 < p < \infty$

$$(2) \quad \|f\|_p \leq \|f^*\|_p \leq C_p \|f\|_p \text{ with constant } C_p > 0.$$

In other words,

$$(3) \quad \|Sf\|_p \approx \|f\|_p \approx \|f^*\|_p \text{ for } 1 < p < \infty.$$

From [4], we have that, for all nonnegative l and h ,

$$(4) \quad \begin{aligned} \{x \in K: S^{(l)}f(x) < \infty\} &\cong \{x \in K: \lim_{k \rightarrow \infty} f(x, k) \text{ exists}\} \\ &\cong \{x \in K: m^{(h)}f(x) < \infty\}; \end{aligned}$$

i.e., the above sets are equal except possibly for a set of measure 0. Our main objective is to show that

$$\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p \text{ for } 0 < p < \infty.$$

As a consequence, we show that “nice” singular integral transforms preserve H^p -space ($0 < p < \infty$) which is the space of distributions whose maximal function are in L^p . The last result is the main contribution of [5].

The euclidean version of the main theorem can be found in [2] (see also [7]); its martingale version about Sf and f^* is proved in [1]. Our work has been motivated by these results. In Appendix we shall discuss briefly how our argument can be applied to certain martingales.

REMARK 1. The equivalence in L^p “norm” is interpreted in the obvious way, i.e., if one side is finite, so is the other and is bounded by a constant multiple of the former one. The restriction that $f(x, k) \rightarrow 0$ as $k \rightarrow \infty$ is needed only for the first inequality of (1) and $\|m^{(h)}f\|_p \leq A_p \|S^{(l)}f\|_p$.

REMARK 2. A trivial modification gives us the same result for K^n , the n -dimensional vector space over K . The “ Φ -inequalities” of Burkholder-Gundy [1][2] for $S^{(l)}$ and $m^{(h)}$ could also be proved.

2. We first show that $\|f^*\|_p \approx \|m^{(l)}f\|_p$ for $0 < p < \infty$.

LEMMA 1. For $\lambda > 0$,

$$|\{x \in K: f^*(x) > \lambda\}| \leq |\{z \in K: m^{(l)}f(z) > \lambda\}| \leq q^l |\{x \in K: f^*(x) > \lambda\}|.$$

Proof. $|\{f^* > \lambda\}| \leq |\{m^{(l)}f > \lambda\}|$ is obvious since $f^* \leq m^{(l)}f$.

Suppose $m^{(l)}f(z) > \lambda$. Then there exists $(x, k) \in \Gamma_i(z)$ such that $|f(x, k)| > \lambda$. Hence $\mathcal{S}_x^{-k} \subset \{f^* > \lambda\}$ and $z \in \mathcal{S}_x^{-(k+l)}$. Therefore

$$|\{m^{(l)}f > \lambda\}| \leq q^l |\{f^* > \lambda\}| .$$

THEOREM 1. $\|f^*\|_p \leq \|m^{(l)}f\|_p \leq q^{l/p} \|f^*\|_p$ for $0 < p < \infty$.

Proof. This follows from Lemma 1 and the following identity:

$$(5) \quad \|g\|_p^p = p \int_0^\infty \lambda^{p-1} |\{g > \lambda\}| d\lambda, \quad 0 < p < \infty .$$

Now let us break up the proof of $\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p (0 < p < \infty)$ into several lemmas:

LEMMA 2. $\|S^{(l)}f\|_2^2 = q^l \|Sf\|_2^2 = q^l \|f\|_2^2$.

Proof. Easy and known. (See Lemma 2.8(c) of [4].)

LEMMA 3. $\|f^*\|_p \leq A_p \|Sf\|_p$ for $0 < p < 2$.

Proof. By (5), it suffices to show the following estimate:

$$(6) \quad |\{f^* > \lambda\}| \leq A\lambda^{-2} \int_0^\lambda t |\{Sf > t\}| dt \text{ for } \lambda > 0 .$$

For a fixed $\lambda > 0$, let

$$\sigma(x) = \sup \{n: S_n f(z) > \lambda \text{ for some } z \in \mathcal{P}_x^{-(n+1)}\}$$

where $S_n f(z) = (\sum_{k \geq n} |d_k f(z)|^2)^{1/2}$. (Convention: $\sup \emptyset = -\infty$.)
For $x \in K$ with $\sigma(x) = n$, let

$$g(x, k) = \begin{cases} f(x, k) & \text{if } k \geq n + 1, \\ f(x, n + 1) & \text{if } k \leq n. \end{cases}$$

Hence $Sg(x) \leq \lambda$ and $Sg(x) \leq Sf(x)$ for all x . Moreover, for $x \in \{\sigma = -\infty\} \subset \{Sf \leq \lambda\}$, we have $g^*(x) = f^*(x)$ and $Sg(x) = Sf(x)$. On the other hand, suppose $\sigma(x) = n > -\infty$. Then there exists $z \in \mathcal{P}_x^{-(n+1)}$ such that $S_n f(z) > \lambda$. Thus $\mathcal{P}_z^{-n} \subset \{z: Sf(x) > \lambda\}$ with $x \in \mathcal{P}_z^{-(n+1)}$. Therefore we have

$$|\{x: \sigma(x) > -\infty\}| \leq q |\{z: Sf(x) > \lambda\}| .$$

Now

$$\begin{aligned} |\{f^* > \lambda, \sigma > -\infty\}| &\leq q |\{Sf > \lambda\}| \\ &\leq 2q\lambda^{-2} \int_0^\lambda t |\{Sf > t\}| dt \end{aligned}$$

and, by Lemma 2 and (5),

$$\begin{aligned}
 |\{f^* > \lambda, \sigma = -\infty\}| &\leq |\{g^* > \lambda\}| \leq 2\lambda^{-2} \|g\|_2^2 \\
 &= 2\lambda^{-2} \|Sg\|_2^2 = 4\lambda^{-2} \int_0^\infty t |\{Sg > t\}| dt \\
 &= 4\lambda^{-2} \int_0^\lambda t |\{Sg > t\}| dt \\
 &\leq 4\lambda^{-2} \int_0^\lambda t |\{Sf > t\}| dt .
 \end{aligned}$$

Thus

$$\begin{aligned}
 |\{f^* > \lambda\}| &\leq |\{f^* > \lambda, \sigma > -\infty\}| + |\{f^* > \lambda, \sigma = -\infty\}| \\
 &\leq (2q + 4)\lambda^{-2} \int_0^\lambda t |\{Sf > t\}| dt .
 \end{aligned}$$

This establishes (6) and Lemma 3.

LEMMA 4. For $l > 0$ and $0 < p < 2$,

$$\|S^{(l)}f\|_p \leq B_p \|m^{(l)}f\|_p .$$

Proof. Again, it suffices to show that for $l > 0$ and $\lambda > 0$,

$$|\{S^{(l)}f > \lambda\}| \leq B\lambda^{-2} \int_0^\lambda t |\{m^{(l)}f > t\}| dt .$$

Let $\mu(z) = \sup\{n : |f(x, n)| > \lambda \text{ for some } x \in \mathcal{P}_z^{-(n+l)}\}$. For $z \in K$ with $\mu(z) = n$, we have $\mu(x) = n$ for all $x \in \mathcal{P}_z^{-(n+l)}$; and let

$$g(z, k) = \begin{cases} f(x, k) & \text{if } k \geq n + 1 , \\ f(x, n + 1) & \text{if } k \leq n . \end{cases}$$

Hence $\{\mu = -\infty\} = \{m^{(l)}f \leq \lambda\}$ and for $\mu(z) = -\infty$, we have $g(x, k) = f(x, k)$ if $x \in \mathcal{P}_z^{-(k+l)}$ or $(x, k) \in \Gamma_l(z)$. Thus on $\{z : \mu(z) = -\infty\}$, $S^{(l)}g(z) = S^{(l)}f(z)$ and $m^{(l)}g(z) = m^{(l)}f(z) \leq \lambda$. Now

$$\begin{aligned}
 |\{S^{(l)}f > \lambda, \mu > -\infty\}| &\leq |\{m^{(l)}f > \lambda\}| \\
 &\leq 2\lambda^{-2} \int_0^\lambda t |\{m^{(l)}f > t\}| dt ,
 \end{aligned}$$

and by Lemma 2 and (5),

$$\begin{aligned}
 |\{S^{(l)}f > \lambda, \mu = -\infty\}| &\leq |\{S^{(l)}g > \lambda\}| \leq \lambda^{-2} \|S^{(l)}g\|_2^2 \\
 &= q^l \lambda^{-2} \|g\|_2^2 \leq q^l \lambda^{-2} \|m^{(l)}g\|_2^2 \\
 &\leq q^l \lambda^{-2} \cdot 2 \int_0^\infty t |\{m^{(l)}g > t\}| dt \\
 &\leq 2q^l \lambda^{-2} \int_0^\lambda t |\{m^{(l)}f > t\}| dt
 \end{aligned}$$

Hence

$$|\{S^{(l)}f > \lambda\}| \leq 2(q^l + 1)\lambda^{-2} \int_0^\lambda t |\{m^{(l)}f > t\}| dt.$$

Therefore Lemma 4 is proved.

LEMMA 5. For $l \geq 0$ and $2 < p < \infty$,

$$\|S^{(l)}f\|_p \leq C_p \|f\|_p.$$

Proof. Suppose $p > 4$ and let r be the conjugate index of $p/2$. Thus $1 < r < 2$. Consider a fixed $k \in \mathbf{Z}$. For $x \in K$, let $\{x_i\}_{i=1}^{q^l}$ be the distinct coset representatives such that $\mathcal{P}_{x_i}^{-(k-l+1)} \subset \mathcal{P}_x^{-(k+1)}$. For $g \in L^r$ with $\|g\|_r = 1$, we have

$$\begin{aligned} \int_K \sum_{i=1}^{q^l} |d_k f(x_i)|^2 |g(x)| dx &= \sum_i \int_K |d_k f(x_i)|^2 |g(x, k+1)| dx \\ &= \sum_i \int_K |d_k f(x_i)|^2 |g(x_i, k+1)| dx \\ &= q^l \int_K |d_k f(x)|^2 |g(x, k+1)| dx. \end{aligned}$$

Hence it follows from this, Hölder's inequality, (1) and (2) that

$$\begin{aligned} \int_K [S_n^{(l)} f(x)]^2 |g(x)| dx &= \sum_{k \geq n} \int_K \sum_{i=1}^{q^l} |d_k f(x_i)|^2 |g(x)| dx \\ &= \sum_{k \geq n} q^l \int_K |d_k f(x)|^2 |g(x, k+1)| dx \\ &\leq q^l \int_K [S_n f(x)]^2 g^*(x) dx \\ &\leq q^l \|S_n f\|_p^2 \|g^*\|_r \\ &\leq B_p \|f\|_p^2 \end{aligned}$$

where B_p depends only on p and q . Thus

$$\|S_n^{(l)} f\|_p^2 = \|[S_n^{(l)} f]^2\|_{p/2} = \sup_{g \in L^r, \|g\|_r=1} \left| \int_K [S_n^{(l)} f(x)]^2 g(x) dx \right| \leq B_p \|f\|_p^2.$$

Therefore $\|S^{(l)}f\|_p \leq C_p \|f\|_p$ for $4 < p < \infty$.

Apply the Marcinkiewicz interpolation theorem to this and Lemma 2, we have

$$\|S^{(l)}f\|_p \leq C_p \|f\|_p \quad \text{for } 2 < p < \infty.$$

THEOREM 2. For $l, h \geq 0$ and $0 < p < \infty$,

$$\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p.$$

Proof. The case of $p = 2$ is obvious.

If $0 < p < 2$, then, from Lemma 3, Lemma 4 and Theorem 1, we have for $l > 0$,

$$\begin{aligned} \|f^*\|_p &\leq A_p \|Sf\|_p \leq A_p \|S^{(l)}f\|_p \\ &\leq A_p B_p \|m^{(l)}f\|_p \approx \|f^*\|_p. \end{aligned}$$

If $2 < p < \infty$, then, by Theorem 1, (3) and Lemma 5,

$$\begin{aligned} \|m^{(h)}f\|_p &\approx \|f^*\|_p \approx \|f\|_p \approx \|Sf\|_p \\ &\leq \|S^{(l)}f\|_p \leq C_p \|f\|_p. \end{aligned}$$

Therefore $\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p$ for $0 < p < \infty$ and the proof of the theorem is completed.

REMARK 3. The above argument simplifies the extension argument as used in §2 of [4] and is essentially similar to the decomposition argument of [5]. It is also a sort of stopping time argument for martingales relative to a regular stochastic basis. (See Appendix.) The main result (with respect to “truncated cones”) could be used to show (4)—the Fatou-Calderón-Stein theorem, in a similar manner as in [2].

3. Let π be a (multiplicative) unitary character on K^* such that it is homogeneous of degree 0 and is ramified of degree $h \geq 1$. Denote $Q(x) = c\pi(x)|x|^{-1}$ where $c = 1/\Gamma(\pi)$. (See [9] for details about Γ -function.) Let $Q_n = R_n * Q$ and $Q_n^N = Q_n \Phi_{-N}$ for $N \geq n + h$. For a distribution f on K or a regular function $f(x, k)$ on $K \times \mathbf{Z}$, we note that $Q_n^N * f(x, k) = Q_n^N * f(x, k) = Q^N * f(x, k)$ for $n \leq k \leq N - h$. Define

$$(T_\pi f)(x, k) = \lim_{N \rightarrow \infty} Q_n^N * f(x, k) \quad \text{for } (x, k) \in K \times \mathbf{Z}.$$

If $f \in L^p(K)$, $1 \leq p < \infty$, then this is just a sort of singular integral transform as been studied in [8], [11] and [4].

For $0 < p < \infty$, let $H^p(K)$ be the space of all distributions f on K whose maximal function $f^* \in L^p(K)$ with the H^p “norm” $\|f^*\|_p$. From [5], we know that for $f \in H^p$, $(T_\pi f)(x, k)$ is a well-defined regular function. The regularization of the corresponding distribution is just $(T_\pi f)(x, k)$. Moreover, the following is also shown:

THEOREM 3. T_π preserves H^p -spaces for $0 < p < \infty$. That is, $\|(T_\pi f)^*\|_p \approx \|f^*\|_p$ for $0 < p < \infty$.

We show here how this result can be obtained as a consequence of Theorem 2.

LEMMA 6. $S^{(h)}f(z) = S^{(h)}T_\pi f(z)$ for all $z \in K$.

Proof. For a fixed $k \in \mathbf{Z}$ and $x \in K$,

$$d_k T_\pi f(x) = T_\pi f(x, k) - T_\pi f(x, k + 1) = T_\pi d_k f(x).$$

For each $m \in \mathbf{Z}$, let $\varepsilon_m^i, i = 1, 2, \dots, (q - 1)q^{h-1}$, be coset representatives of $\mathcal{P}^{-(m-h+1)}$ in $\{t: |t| = q^{m+1}\}$. Then

$$\begin{aligned} T_\pi f(x, k) &= c \int_{|t| > q^k} f(x - t) \frac{\pi(t)}{|t|} dt \\ &= c \sum_{m=k}^{\infty} q^{-(m+1)} \int_{|t|=q^{m+1}} f(x - t) \pi(t) dt \\ &= cq^{-h} \sum_{m=k}^{\infty} \sum_{i=1}^{(q-1)q^{h-1}} \pi(\varepsilon_m^i) f(x - \varepsilon_m^i, m - h + 1). \end{aligned}$$

Thus

$$(7) \quad T_\pi d_k f(x) = cq^{-h} \sum_{i=1}^{(q-1)q^{h-1}} \pi(\varepsilon_k^i) f(x - \varepsilon_k^i, k - h + 1).$$

Now let $g(x)$ be the restriction of $d_k f(x)$ on $z + \mathcal{P}^{-(k+1)}$ for any fixed z . Hence from (7) we see that $T_\pi g(x)$ is also supported on $z + \mathcal{P}^{-(k+1)}$. By Plancherel's theorem, since $|\pi| = 1$, we have

$$\|T_\pi g\|_2 = \|(T_\pi g)^\wedge\|_2 = \|\pi^{-1} \hat{g}\|_p = \|\hat{g}\|_2 = \|g\|_2.$$

That is,

$$\sum_{i=1}^{q^h} |d_k f(x_i)|^2 = \sum_{i=1}^{q^h} |d_k T_\pi f_\pi(x_i)|^2$$

where $x_i, i = 1, 2, \dots, q^h$, are coset representatives of $\mathcal{P}^{-(k-h+1)}$ in $\mathcal{P}_z^{-(k+1)}$. Thus summing this up with respect to k , we have

$$S^{(h)} f(z) = S^{(h)} T_\pi f(z).$$

Proof of Theorem 3. It follows immediately from Theorem 2 and Lemma 6 that for $0 < p < \infty$,

$$\|f^*\|_p \approx \|S^{(h)} f\|_p = \|S^{(h)} T_\pi f\|_p \approx \|(T_\pi f)^*\|_p.$$

Appendix. Let (Ω, \mathcal{A}, P) be a probability space and $\{\mathcal{A}_n\}_{n=1}$ a nondecreasing sequence of sub- σ -fields of \mathcal{A} . Let $f = \{f_n\}_{n \geq 1}$ be a real-valued martingale relative to $\{\mathcal{A}_n\}_{n \geq 1}$ and $\{d_k\}_{k \geq 1}$ be the difference sequence of f . For a nonnegative integer l , write

$$m^{(l)} f = \sup_n E(|f_{n+l}| | \mathcal{A}_n)$$

and $S^{(l)} f = [\sum_{k>l} E(d_k^2 | \mathcal{A}_{k-l})]^{1/2}$. $f^* = m^{(0)} f = \sup_n |f_n|$ is the maximal function of f and $Sf = S^{(0)} f = [\sum_{k>0} d_k^2]^{1/2}$ is the square function of f . Burkholder and Gundy [1] proved that for a large class of

martingales,

$$(8) \quad \|Sf\|_p \approx \|f^*\|_p \quad \text{for } 0 < p < \infty.$$

However examples (in [1]) show that

$$(9) \quad \|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p \quad \text{for } 0 < p < \infty$$

fails to hold. Nevertheless by a slight modification of the previous argument, we can show that this is true for martingales relative to a *regular* stochastic basis (after Chow [6]).

Indeed, the crucial part of the proof is to consider the following stopping time:

$$\mu(x) = \inf \{n: E(|f_{n+l}| | \mathcal{A}_n) < \lambda\} \quad (\lambda > 0).$$

Together with the regularity of the stochastic basis and (8), we get (9) by a similar argument as before.

We remark that our argument gives a simplified proof of (8) for martingales relative to a regular stochastic basis. Also the argument used in Lemma 5 similar to the one in [3] provides a new proof of that

$$\|sf\|_p \leq C_p \|f\|_p \quad \text{for } p > 2$$

where $sf = S^{(1)}f = [\sum_{k>1} E(d_k^2 | \mathcal{A}_{k-1})]^{1/2}$ is the *conditioned* square function of the martingale f (relative to any stochastic basis).

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