

Pacific Journal of Mathematics

**COLLECTIVELY COMPACT SETS AND THE ERGODIC
THEORY OF SEMI-GROUPS**

M. V. DESHPANDE

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Let $\{T(t): t \geq 0\}$ be a uniformly bounded semi-group of linear operators on a Banach space X such that 1 is an eigenvalue of each $T(t)$ and $T(a)$ is compact for some $a > 0$. Then the ergodic limit $A(t) = \lim_{n \rightarrow \infty} (1/n)\{T(t) + T^2(t) + \dots + T^n(t)\}$ exists for each t . In this paper it is proved that if each $T(t)$, $t > 0$, is compact and 1 is, in a certain sense, an isolated eigenvalue of all $T(t)$, then for $t > 0$, the dimension of the null space of $T(t) - I$ is independent of t . Sufficient conditions are also obtained for the $\lim_{t \rightarrow \infty} T(t)$ to exist.

Suppose X is a real or complex Banach space. Let B denote the unit ball in X and $[X]$ the space of bounded linear operators on X into X . A set $\mathcal{K} \subset [X]$ is said to be collectively compact if $\mathcal{K}B = \{Kx: K \in \mathcal{K}, x \in B\}$ is relatively compact. Basic properties of such sets were obtained by Anselone and Palmer [1, 2]. Some of their results will be applied to semi-groups in the following sections.

2. Ergodic family associated with a semi-group. Let $\{T(t): t \geq 0\}$ be a uniformly bounded semi-group of linear operators on a B -space X such that $T(a)$ is compact for some $a > 0$ and 1 is an eigenvalue of each $T(t)$. Then [4, VIII. 8.4] yields that

$$\begin{aligned} \lim_{n \rightarrow \infty} (1/n)\{T(t) + T^2(t) + \dots + T^n(t)\} \\ &= \lim_{n \rightarrow \infty} (1/n)\{T(t) + \dots + T(nt)\} \\ &= A(t) \end{aligned}$$

say, exists in the uniform operator topology and defines a projection operator satisfying $A(t) = T(t)A(t) = A(t)T(t)$. Also, $A(t)$ is the residue operator in the Laurent expansion of $(\lambda - T(t))^{-1}$ in the neighbourhood of 1 and is represented by the Dunford integral

$$A(t) = (1/2\pi i) \int_C (\lambda - T(t))^{-1} d\lambda$$

where C is a sufficiently small circle with centre at 1 and will, in general, depend upon t . For $t = 0$, $T(0) = A(0) = I$ the identity operator. $\{A(t): t \geq 0\}$ will be called the ergodic family associated with $\{T(t): t \geq 0\}$.

When $T(a)$ is compact for some $a > 0$, the family $\{T(t): t \geq a\}$ is collectively compact and totally bounded in the uniform operator

topology [3]. But nothing can be said about $T(t)$ for $t < a$. However, the following proposition shows that the situation is more satisfactory in respect of the family $\{A(t)\}$.

PROPOSITION 2.1. *Let $\{T(t): t \geq 0\}$ be a uniformly bounded semi-group on a B -space X such that 1 is an eigenvalue of each $T(t)$ and $T(a)$ is compact for some $a > 0$. Then for each $\varepsilon > 0$, the ergodic family $\{A(t): t \geq \varepsilon\}$ is collectively compact.*

Proof. Since $A(t) = T(t)A(t)$, by repeated application, we have $A(t) = T(nt)A(t)$ for all positive integers n . For $t > 0$ $T(nt)$ is compact for sufficiently large n . Hence, $A(t)$ will also be compact. Now for $\varepsilon > 0$, choose a positive integer k so that $k\varepsilon \geq a$. Then $kt \geq a$ for $t \geq \varepsilon$. Therefore, $\mathcal{K} = \{T(kt): t \geq \varepsilon\}$ is collectively compact. Now, $\|T(t)\| \leq M$ implies $\|A(t)\| \leq M$. Hence, $\mathcal{M} = \{A(t): t \geq \varepsilon\}$ is uniformly bounded. We can now conclude that

$$\mathcal{KM} = \{T(kt) \circ A(s): t, s \geq \varepsilon\}$$

is collectively compact, because, by [1, Prop. 2.3], if \mathcal{K} is collectively compact and \mathcal{M} is uniformly bounded, then \mathcal{KM} is collectively compact. The result now follows from the fact that $\{A(t): t \geq \varepsilon\}$ is a subset of \mathcal{KM} .

COROLLARY 2.2. *The family $\{A(t): t \geq \varepsilon\}$ is totally bounded in the uniform operator topology.*

Proof. The above arguments in respect of $\{T(t)\}$ also hold for the dual semi-group $\{T^*(t)\}$. Hence, $\{A(t): t \geq \varepsilon\}$ and $\{A^*(t): t \geq \varepsilon\}$ are both collectively compact. The desired conclusion follows from the fact that if $\mathcal{K}, \mathcal{K}^*$ are both collectively compact, then \mathcal{K} is totally bounded in the uniform operator topology [6, Thm. 3.1].

3. Spectral properties. In Section 2 strong continuity of $T(t)$ was not necessary. But, for the spectral properties to be discussed now, we shall require strong continuity. Let, as usual, $\sigma(T(t))$, $\tilde{\rho}(T(t))$ denote the spectrum and the extended resolvent set of $T(t)$ and $R(\lambda, T(t))$ the inverse $(\lambda - T(t))^{-1}$. The following proposition proved in [3] leads to the strong continuity of $A(t)$ which is crucial for the main result.

PROPOSITION 3.1. *Let $\{T(t): t \geq 0\}$ be a uniformly bounded, strongly continuous semi-group on a B -space X such that $T(t)$ is compact for each $t > 0$. For $a > 0$, let Ω be a neighbourhood of $\sigma(T(a))$, and $\Lambda = \tilde{\rho}(T(a)) - \Omega$. Then*

- (i) *there exists a $\delta > 0$ such that, for $|t - a| < \delta$, $\sigma(T(t)) \subset \Omega$ and $\Lambda \subset \tilde{\rho}(T(t))$.*
- (ii) *As $t \rightarrow a$, $(\lambda - T(t))^{-1}x \rightarrow (\lambda - T(a))^{-1}x$ uniformly on Λ .*

PROPOSITION 3.2. *Let $\{T(t): t \geq 0\}$ be a uniformly bounded, strongly continuous semi-group on a B -space X such that 1 is an eigenvalue of each $T(t)$ and $T(t)$ is compact for each $t > 0$. Suppose further, that for each $a > 0$, there exists a circle C with centre at 1 in the complex plane and a real number δ such that, for $|t - a| < \delta$, $T(t)$ has no eigenvalue in C except 1. Then for $t > 0$, $A(t)$ is strongly continuous in t .*

Proof. We know that

$$A(t)x = (1/2\pi i) \int_{\Gamma} (\lambda - T(t))^{-1}x d\lambda$$

where, in general, Γ depends upon t . But, under the hypothesis on C , we may assume that

$$A(t)x = (1/2\pi i) \int_C (\lambda - T(t))^{-1}x d\lambda$$

for all t with $|t - a| < \delta$. Again, by proposition 3.1, $(\lambda - T(t))^{-1}x \rightarrow (\lambda - T(a))^{-1}x$ uniformly on C as $t \rightarrow a$. Hence, for any $a > 0$, we have $A(t)x \rightarrow A(a)x$ as $t \rightarrow a$. This completes the proof.

Let $R(T)$, $N(T)$ denote the range and null space of an operator T and $\dim R(T)$, $\dim N(T)$ their dimensions. When $T_n \rightarrow T$ pointwise, $\dim R(T_n) > \dim R(T)$ eventually. But, when we are dealing with projections, the following result [2, 4.2] gives a more precise estimate.

THEOREM 3.3. *Let E and E_n , $n \geq 1$, be projections in $[X]$ such that $E_n \rightarrow E$ pointwise and $\{E_n - E\}$ is collectively compact. Then, eventually $\dim E_n X = \dim EX$.*

THEOREM 3.4. *Let $\{T(t)\}$ be the semi-group of Proposition 3.2. Then for each $a > 0$, there exists $\delta > 0$ such that $\dim N(T(t) - I) = \dim N(T(a) - I)$ whenever $|t - a| < \delta$.*

Proof. Observe first that $R(A(t)) = N(T(t) - I)$ since $A(t)$ is the ergodic limit of $T(t)$. Suppose that there does not exist any δ with the required property. Then, by using Proposition 2.1, it is possible to construct a sequence of projections for which the Theorem 3.3 will not be true.

THEOREM 3.5. *If $\{T(t)\}$ is the semi-group of Proposition 3.4, then for $t > 0$, $\dim N(T(t) - I)$ is independent of t .*

Proof. By the last theorem the function $t \rightarrow \dim N(T(t) - I)$ is continuous and since it takes only positive integral values, it must be a constant function.

4. The nature of $T(t)$ at $t = \infty$. The first result of this section will be a lemma proved in [5] by using collective compactness. However, the proof given here throws some light on the nature of $T(t)$ at $t = \infty$.

LEMMA 4.1. *Let $T \in [X]$ be such that $\{T^n: n \geq 1\}$ is uniformly bounded and T^k is compact for some $k \geq 1$. Then $\{T^n: n \geq 1\}$ is totally bounded in the uniform operator topology.*

Proof. The conditions on T ensure that $\sigma(T)$ is contained in the closed unit disc with centre 0 in the complex plane and that there are only a finite number of simple poles of $R(\lambda, T)$, say $\lambda_1, \dots, \lambda_p$ on the circumference of the disc [4, VIII. 8.1]. If A_1, \dots, A_p are the residue operators in the Laurent expansion of $R(\lambda, T)$ in the neighbourhoods of $\lambda_1, \dots, \lambda_p$ respectively, then by using the standard technics of operational calculus, it can be easily proved that

$$(I) \quad T^n x = (1/2\pi i) \int_{C_0} \lambda^n R(\lambda, T) x d\lambda + \lambda_1^n A_1(x) + \dots + \lambda_p^n A_p(x)$$

where C_0 is a circle with centre at 0 and radius less than 1. Since $|\lambda| < 1$ on C_0 , the first term on the right tends to zero uniformly on C_0 and on bounded sets of X . Again $|\lambda_k| = 1$ for $k = 1, \dots, p$. Hence $\{\lambda_k^n: n \geq 1, k = 1, \dots, p\}$ is totally bounded. It is easy to extract a sequence $\{n_i\}$ such that $\lambda_k^{n_i}$ converges to μ_k say, for $k = 1, \dots, p$. It then follows that

$$T^{n_i} x \longrightarrow \mu_1 A_1(x) + \dots + \mu_p A_p(x).$$

The convergence is obviously uniform on bounded sets of X .

PROPOSITION 4.2. *Let $\{T(t): t \geq 0\}$ be a uniformly bounded semi-group such that $T(a)$ is compact for some $a > 0$. For any $t > 0$, let $A_1 = A_1(t), \dots, A_p = A_p(t)$ be the residue operators in the Laurent expansion of $R(\lambda, T(t))$ in the neighbourhoods of its poles $\lambda_1, \dots, \lambda_p$ say, lying on the circumference of the unit disc. Then, there exists a sequence $n_i = n_i(t)$ such that, as $n_i \rightarrow \infty$, $T(n_i, t)x$ converges to a linear combination of $A_1 x, \dots, A_p x$ for each x .*

Proof. Follows immediately when the previous lemma is applied to $T(t)$.

PROPOSITION 4.3. *Let $\{T(t): t \geq 0\}$ be a uniformly bounded semi-group on a B -space X such that $T(a)$ is compact for some $a > 0$. Suppose 1 is the only eigenvalue of $T(t)$ on the unit circle in the complex plane for each $t > 0$. Then $\lim_{n \rightarrow \infty} T(nt) = A$ exists in the uniform operator topology for each $t > 0$ and A is independent of t .*

Proof. When the conditions of the proposition are satisfied equation (I) of this section applied to $T(t)$ gives

$$(II) \quad T(nt)x = T^n(t)x = (1/2\pi i) \int_{C_0} \lambda^n (\lambda - T(t))^{-1} x d\lambda + A(t)x$$

where $A = A(t)$ is the residue operator in the Laurent expansion of $(\lambda - T(t))^{-1}$ around 1 and C_0 is a circle with centre at 0 and radius less than 1. Now $|\lambda| < 1$ on C_0 and therefore $\lambda^n \rightarrow 0$ as $n \rightarrow \infty$. Hence taking limits in the above equation as $n \rightarrow \infty$ we get $\lim_{n \rightarrow \infty} T(nt)x = A(t)x$, the limit being uniform on bounded sets of X . Now, to prove that $A(t) = A$ is independent of t note that $A(t)A(s) = \lim_{n \rightarrow \infty} T(nt)T(ns) = \lim_{n \rightarrow \infty} T(n(s+t)) = A(s+t) = A(s)A(t)$. Hence $\{A(t)\}$ is a semi-group. Again, for $t \neq s$, let $t < s$. Then by the semi-group property $R(A(t)) \supset R(A(s))$ where R denotes the range. Also $A(t)$ is a projection. Therefore $A(t) = A^n(t) = A(nt)$ for each positive integer n . Choose n so that $nt > s$. Then we have $R(A(t)) = R(A(nt)) \subset R(A(s))$. Hence $R(A(t)) = R(A(s))$. Now, it is easy to prove that if P, Q are projections such that $PQ = QP$ and $R(P) = R(Q)$ then $P = Q$. Therefore we must have $A(t) = A(s)$. This completes the proof.

COROLLARY 4.4. *$N(T(t) - I)$ is independent of t for $t > 0$.*

Proof. An application of corollary, Theorem 2 in [7, VIII, § 3] leads to $N(T(t) - I) = R(A(t)) = R(A)$.

REMARK 4.5. In the proof of the Proposition 4.3 $A(t)$ is proved to be independent of t by using the fact that $A(t)A(s) = A(t+s) = A(s)A(t)$ and that $A(t)$ is a projection. This means that the only semi-group of projections is the semi-group $T(t) = P$ a projection for all t .

In the next proposition it will be seen that the pointwise convergence of $T(nt)$ in t proved in Proposition 4.3 can be strengthened so as to be locally uniform in t . This essential for the final ergodic theorem.

PROPOSITION 4.6. *Let $\{T(t): t \geq 0\}$ be a uniformly bounded, strongly continuous semi-group on a B -space X such that $T(t)$ is compact for each $t > 0$. Suppose there exists a circle in the complex plane with centre at 0 and radius less than 1 such that 1 is the only eigenvalue of each $T(t)(t > 0)$ outside this circle. Then for each $t > 0$, $\lim_{n \rightarrow \infty} T(nt) = A$ exists in the uniform operator topology, the convergence being locally uniform on t , i.e., for each $a > 0$ there exists a positive δ depending upon a such that the convergence is uniform on the interval $|t - a| < \delta$.*

Proof. The equation (II) of this section is now

$$(III) \quad T(nt)x = (1/2\pi i) \int_{C_0} \lambda^n (\lambda - T(t))^{-1} x d\lambda + Ax$$

where in general C_0 depends upon t . But, under the hypothesis made, C_0 may be assumed to be independent of t . Then by Proposition 3.1 the set $\{(\lambda - T(t))^{-1}: \lambda \in C_0, |t - a| < \delta\}$ is uniformly bounded for some $\delta > 0$, i.e., $\|(\lambda - T(t))^{-1}\| < M$ say, for all $\lambda \in C_0$ and $|t - a| < \delta$. Hence by equation (III) we have

$$\begin{aligned} \|T(nt)x - Ax\| &\leq |(1/2\pi i)| \int_{C_0} \|\lambda^n (\lambda - T(t))^{-1} x\| |d\lambda| \\ &< (M/2\pi) \int_{C_0} |\lambda|^n |d\lambda| \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ since } |\lambda| < 1 \text{ on } C_0. \end{aligned}$$

It is obvious that the convergence is of the required type.

LEMMA 4.7. *Suppose $\{T(t): t \geq 0\}$ is a family of bounded linear operators on a B -space X such that $\lim_{n \rightarrow \infty} T(nt) = A$ exists in the uniform operator topology where A is independent of t and the convergence is uniform on some open interval. Then $\lim_{t \rightarrow \infty} T(t) = A$ in the uniform operator topology.*

Proof. We may assume that $T(nt) \rightarrow A$ uniformly on $|t - a| < \delta$. Then for $\epsilon > 0$, there exists $\delta > 0$ and a positive integer N such that $\|T(nt) - A\| < \epsilon$ for all $n \geq N$ and all t in $|t - a| < \delta$ or equivalently $\|T(t) - A\| < \epsilon$ for $n \geq N$ and $|t - na| < n\delta$. Let I_n be the interval $(na - n\delta, na + n\delta)$. Then I_n has a nonempty intersection with I_{n+1} if and only if $na + n\delta > (n + 1)a - (n + 1)\delta$ or equivalently $(2n + 1)\delta > a$. Choose M so that $n \geq M$ implies $(2n + 1)\delta > a$. Then, for each $n \geq M$, I_n and I_{n+1} have a non-empty intersection. Hence, any $t > Ma - M\delta$ falls in some I_n with $n \geq M$. Therefore we must have $\|T(t) - A\| < \epsilon$ for $t > Ma - M\delta$. This completes the proof.

In the light of this lemma, the Proposition 4.6 and the Corollary 4.4, the following ergodic theorem now becomes obvious.

THEOREM 4.8. *If $\{T(t): t \geq 0\}$ is the semi-group of Proposition 4.6 then $\lim_{t \rightarrow \infty} T(t)$ exists in the uniform operator topology, and $N(T(t) - I)$ is constant for $t > 0$.*

REMARK. In §§ 3 and 4 the continuity of $R(\lambda, T(t))$ in t plays a crucial role. If it is assumed that $T(t)$ is compact for each $t > 0$, then the strong continuity of $T(t)$ in t implies uniform continuity and this in turn implies the uniform continuity of $R(\lambda, T(t))$ for $\lambda \in \rho(T(t))$. But if instead of assuming the compactness of $T(t)$ for each $t > 0$, it is assumed that for each $t > 0$ there exists $\delta > 0$ such that $\{T(t) - T(a): |t - a| < \delta\}$ is collectively compact then the strong continuity of $T(t)$ would not imply uniform continuity but the theory of collectively compact operators can be invoked to ensure the strong continuity of $R(\lambda, T(t))$.

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