Pacific Journal of Mathematics

θ-CLOSED SUBSETS OF HAUSDORFF SPACES

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Vol. 59, No. 2 June 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 59, No. 2, 1975

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R. F. DICKMAN, JR. AND JACK R. PORTER

A topological property of subspaces of a Hausdorff space, called θ-closed, is introduced and used to prove and interrelate a number of different results. A compact subspace of a Hausdorff space is θ-closed, and a θ-closed subspace of a Hausdorff space is closed. A Hausdorff space X with property that every continuous function from X into a Hausdorff space is closed is shown to have the property that every θ-continuous function from X into a Hausdorff space is closed. Those Hausdorff spaces in which the Fomin H-closed extension operator commutes with the projective cover (absolute) operator are characterized. An H-closed space is shown not to be the countable union of θ-closed nowhere dense subspaces. Also, an equivalent form of Martin's Axiom in terms of the class of H-closed spaces with the countable chain condition is given.

1. **Preliminaries.** For a space X and $A \subseteq X$, the *θ-closure* of A, denoted as $\mathrm{cl}_\theta A$, is $\{x \in X$: every closed neighborhood of x meets $A\}$. The subset A is θ-closed if $\mathrm{cl}_\theta A = A$. Similarly, the *θ-interior* of A, denoted as $\mathrm{int}_\theta A$, is $\{x \in X$: some closed neighborhood of x is contained in $A\}$. Clearly, $\mathrm{cl}_\theta A$ is closed and $\mathrm{int}_\theta A$ is open. The concept of θ-closure was introduced by Velicko [15] and used by the authors in [3]. Also introduced in [15] is the concept of a H-set: a subset A of a Hausdorff space X is an *H-set* if every cover of A by sets open in X has a finite subfamily whose closures in X cover A; this concept was independently introduced in [11] and called *H-closed relative to X*. An *open filter* is a filter with a filter base consisting of open sets. A maximal open filter is called an *open ultrafilter*. A filter $\mathscr{F}$ on X is said to be *free* if $\mathrm{ad}_X \mathscr{F} \neq \varnothing$, otherwise, $\mathscr{F}$ is said to be *fixed*. A subset A of X is *far from the remainder* (f.f.r.) [1] in X if for every free open ultrafilter $\mathscr{U}$ on X, there is open $U \in \mathscr{U}$ such that $\mathrm{cl}_X U \cap A = \varnothing$; a subset A of X is *rigid in X* [3] if for every filter base $\mathscr{F}$ on X such that $A \cap \bigcap \{\mathrm{cl}_\theta F: F \in \mathscr{F}\} = \varnothing$, there is open set U containing A and $F \in \mathscr{F}$ such that $\mathrm{cl}\, U \cap F = \varnothing$. The following facts are used in the sequel:

(1.1) In $A \subseteq B \subseteq X$ and A is θ-closed in X, then A is θ-closed in B.

(1.2) A compact subset of a Hausdorff space is θ-closed.

(1.3) [15] *A θ-closed subset of an H-closed space is an H-set.*

(1.4) [3] Let A be a subset of a space X. The following are equivalent:

(a) A is rigid in X.

(b) For any filter base $\mathscr{F}$ on X, if $A \cap \bigcap\{\mathrm{cl}_\theta F: F \in \mathscr{F}\} = \varnothing$, then for some $F \in \mathscr{F}$, $A \cap \mathrm{cl}_\theta F = \varnothing$.

(c) For each cover $\mathscr{A}$ of A by open subsets of X, there is a finite subfamily $\mathscr{B} \subseteq \mathscr{A}$ such that $A \subseteq \mathrm{int\,cl}(\bigcup \mathscr{B})$.

(d) For every open filter $\mathscr{G}$ on X such that $A \cap \bigcap\{\mathrm{cl}\, U: U \in \mathscr{G}\} = \varnothing$, there is $U \in \mathscr{G}$ such that $A \cap \mathrm{cl}\, U = \varnothing$,

(1.5) [3] Disjoint rigid subsets in a Hausdorff space can be separated by disjoint open sets.

(1.6) [3] If A is rigid in X, then A is f.f.r. in X.

Since any closed subset of a regular Hausdorff space is θ-closed and since there are regular Hausdorff spaces with noncompact closed subsets, then the converse of 1.2 is false. In [3], it was shown that every rigid subset of a Hausdorff space is an H-set. Thus, the converse of 1.3 is false since the subset X in the space Y described in Example 1.1 in [3] is rigid in Y but is not θ-closed in Y. On the other hand, by Theorem 4 in [15] a subset of an H-closed, Urysohn space is θ-closed if and only if it is an H-set. Since an H-closed regular space is compact, then a subset of an H-closed, regular space is θ-closed if and only if it is compact. By 1.2 and 1.3, the concept of "θ-closedness" is similar to the concept of "H-closure" in the sense that both are bracketed by the concepts of "compactness" and "H-set".

Also, needed in the sequel is a few definition about semiregularity, θ-continuity, and extensions. For a space X, X_s is used to denote X plus the topology generated by the regular-open subsets (a subset is regular-open if it is the interior of the closure of itself). A space X is *semi-regular* if $X = X_s$; in particular, $(X_s)_s = X_s$.

A function $f: X \rightarrow Y$, where X and Y are spaces, is *θ-continuous* if for each $x \in X$ and open subset U of $f(x)$, there is an open subset V of x such that $f(\mathrm{cl}\, V) \subseteq \mathrm{cl}\, U$. The Katětov extension [9] (resp. Fomin extension [5]) of a Hausdorff space X is denoted as κX (resp. σX); these H-closed extensions are studied in [12, 13]. In [11], it is shown that if Y is an H-closed extension of X, then there is a continuous surjection $f: \kappa X \rightarrow Y$ such that $f(x) = x$ for $x \in X$.

2. θ-closed subsets of H-closed spaces.

For a space X and a subset $A \subseteq X$, we will let X/A denote the set X with A identified to a point and endowed with the quotient topology.

(2.1) Let X be a Hausdorff space and $A \subseteq X$. The following are equivalent:

(a) A is θ-closed in X.

(b) X/A is Hausdorff.

(c) A is the point-inverse of a continuous function from X into a Hausdorff space.

(d) A is the point-inverse of a θ-continuous function from X into a Hausdorff space.

Proof. The proof of the equivalence of (a) and (b) is straightforward to prove. Clearly, (b) implies (c) and (c) implies (d). To show (d) implies (a), let $f: X \rightarrow Y$ be a θ-continuous function into a Hausdorff space Y, $A = f^{-1}(y)$ for some $y \in Y$, and $x \notin A$. There is open set U of $f(x)$ in Y such that $y \notin \mathrm{cl}\, U$. Since there is open set V of x such that $f(\mathrm{cl}\, V) \subseteqq \mathrm{cl}\, U$, then $\mathrm{cl}\, V \cap A = \varnothing$.

(2.2) Let X be a Hausdorff space and $A \subseteqq X$. The following are equivalent:

(a) A is θ-closed in κX.

(b) A is rigid in X.

(c) A is f.f.r. in X and A is θ-closed in X.

Proof. (*a*) *implies* (*b*). Let $\mathscr{A}$ be a cover of A by open subsets of X. For $p \in \kappa X \backslash A$, let U_p be an open subset of κX containing p such that $\mathrm{cl}_{\kappa X} U_p \cap A = \varnothing$. There is a finite subset $\mathscr{B} \subseteqq \mathscr{A}$ and finite subset $B \subseteqq \kappa X \backslash A$ such that

$$\kappa X = \bigcup \{\mathrm{cl}_{\kappa X} U_p : p \in B\} \cup \bigcup \{\mathrm{cl}_{\kappa X} V : V \in \mathscr{B}\} .$$

Thus, $A \subseteqq X \backslash \bigcup \{\mathrm{cl}_X (U_p \cap X) : p \in B\} \subseteqq \bigcup \{\mathrm{cl}_X V : V \in \mathscr{B}\}$, and by 1.4, A is rigid in X.

(*b*) *implies* (*c*). By 1.6, A is f.f.r. in X. Suppose $p \in X \backslash A$. Then A and p are disjoint rigid subsets and, by 1.5, can be separated by disjoint open sets. Hence, A is θ-closed in X.

(*c*) *implies* (*a*). Let $p \in X \backslash A$. Since A is θ-closed in X, then there is an open set U in X such that $p \in U$ and $\mathrm{cl}_X U \cap A = \varnothing$. Since X is open in κX, then U is open in κX and $\mathrm{cl}_{\kappa X} U = \mathrm{cl}_X U \cup B$ where $B = \{q \in \kappa K \backslash K : U \in q\}$. Thus, $A \cap \mathrm{cl}_{\kappa X} U = \varnothing$. Suppose $p \in \kappa X \backslash X$ (thus, $p \notin A$). Then p is a free open ultrafilter on X and there is open set $U \in p$ such that $\mathrm{cl}_X U \cap A = \varnothing$. Now, $U \cup \{p\}$ is open in κX and contains p and $\mathrm{cl}_{\kappa X} (U \cup \{p\}) = \mathrm{cl}_X U \cup B$ where B is the same as above. Thus, $A \cap \mathrm{cl}_{\kappa X}(U \cup \{p\}) = \varnothing$.

By 2.2 and 1.1, it follows that a rigid subset of a Hausdorff space is θ-closed in the space.

Let X and Y be Hausdorff spaces and $f: X \rightarrow Y$ a continuous function. We say f is *absolutely closed* [17] if f cannot be continuously extended to a proper Hausdorff extension Z of X and is *regular closed* [2] if the image of the closure of an open set is closed. Dickman [2] proved that f is absolutely closed if and only if f is regular closed and point-inverses are f.f.r. in X. By 2.1 and 2.2, this statement converts into the following:

(2.3) Let $f: X \to Y$ be a continuous where X and Y are Hausdorff spaces. The following are equivalent:
(a) f is absolutely closed.
(b) f is regular closed and point-inverses are f.f.r. in X.
(c) f is regular closed and point-inverses are rigid in X.

Another consequence of 2.2, in combination with 1.5, is the following result.

(2.4) Disjoint θ-closed subsets of an H-closed space are contained in disjoint open subsets.

In [9], Katětov shows that if every closed subset of an Hausdorff space X is H-closed, then X is compact. Similarly, by 6.1.1 in [3], if every closed subset of a Hausdorff space X is rigid, then X is compact. A Hausdorff space X in which every closed subset is an H-set is called *C-compact* [16], and there are noncompact, C-compact spaces [17, Example 2]. The next result will help us prove a property possessed by C-compact spaces.

(2.5) If $f: X \to Y$ is θ-continuous where X and Y are Hausdorff and if A is H-subset of X, then $f(A)$ is an H-subset of Y.

Proof. Let $\mathscr{C}$ be cover of $f(A)$ by open subsets of Y. For each $a \in A$, there is open set $U_a \in \mathscr{C}$ such that $f(a) \in U_a$. There is an open set V_a of a such that $f(\mathrm{cl}\, V_a) \subseteq \mathrm{cl}\, U_a$. There is finite subset $B \subseteq A$ such that $A \subseteq \bigcup \{\mathrm{cl}\, V_a : a \in B\}$. It follows that $f(A) \subseteq \bigcup \{\mathrm{cl}\, U_a : a \in B\}$.

A Hausdorff space X is called *functionally compact* [4] if every continuous function from X into a Hausdorff space is closed. A C-compact space is functionally compact [4], and by 2.5, every θ-continuous function from a C-compact space into a Hausdorff space is closed. Clearly, a Hausdorff space X in which every θ-continuous function from X into a Hausdorff space is closed, is functionally compact. Surprisingly, the converse is true. We need the following definition and theorem to prove the converse.

A Hausdorff space X is called *θ-seminormal* [6] if for every θ-closed subset $A \subseteq X$ and every open set G containing A, there is regular open set R such that $A \subseteq R \subseteq G$.

(2.6) [6] A Hausdorff space is functionally compact if and only if it is H-closed and θ-seminormal.

(2.7) A Hausdorff space X is functionally compact if and only if every θ-continuous function from X into a Hausdorff space is closed.

Proof. The proof of one direction is obvious. To prove the converse, suppose X is functionally compact and $f: X \to Y$ is a θ-continuous function where Y is Hausdorff. To prove f is closed, suppose $B \subseteq X$ is a closed subset and $p \in \mathrm{cl}_Y f(B)$. By Corollary 2.1 in [4], X is H-closed. By 2.5, $f(X)$ is H-subset and, hence, closed in Y. So, $p \in f(X)$. Assume, by way of contradiction, that $p \notin f(B)$. So, $f^{-1}(p) \subseteq X \backslash B$. By 2.1, $f^{-1}(p)$, is θ-closed in X and by 2.6, there is regular open set R such that $f^{-1}(p) \subseteq R \subseteq X \backslash B$. Now, $B \subseteq X \backslash R$, but $X \backslash R$, the closure of an open set, is H-closed by 1.2 in [9]. By 2.5, $f(X/R)$ is an H-set, and hence, closed. This leads to a contradiction as $f(B) \subseteq f(X \backslash R)$ and $p \notin f(X \backslash R)$.

Problem. Characterize those Hausdorff spaces X with this property: every weakly θ-continuous function from X into a Hausdorff space is closed. A function $f: X \to Y$ is weakly θ-continuous [5, 3] if for every $x \in X$ and open set V of $f(x)$, there is open set U of x such that $f(U) \subseteq \mathrm{cl}\, V$. Every compact Hausdorff space has this property; we are unaware of any noncompact Hausdorff space with this property.

3. θ-closure in H-closed extensions.

With the use of the next result, we will derive a new characterization of those subsets of a Hausdorff space X that are θ-closed in κX.

(3.1) If Y is a Hausdorff extension of X and A is a rigid subset of X, then A is rigid in Y.

Proof. By 2.2, it suffices to show that A is θ-closed in κY. By 4.4 in [11], there is a continuous surjection $f: \kappa X \to \kappa Y$ such that that $f(x) = x$ for $x \in X$. Since κX is H-closed, then f is absolutely closed. Let $z \in \kappa Y \backslash A$. Then $f^{-1}(z)$ is rigid in κX by 2.3. Using that $\kappa(\kappa X) = \kappa X$, it follows by 2.2 that A is rigid in κX. By 1.5, there is open set U in κX such that $A \subseteq U$ and $\mathrm{cl}_{\kappa X} U \cap f^{-1}(z) = \varnothing$. Let $W = \kappa Y \backslash f(\mathrm{cl}_{\kappa X} U)$. Since f is regular closed by 2.3, W is open; also, $z \in W$. Now, $f^{-1}(W)$ is open in X and $f^{-1}(W) \cap \mathrm{cl}_{\kappa X} U = \varnothing$. So $\mathrm{cl}_{\kappa X} f^{-1}(W) \cap A = \varnothing$. Since $A = f^{-1} f(A)$ by 1.8 in [13], $f(\mathrm{cl}_{\kappa X} f^{-1}(W)) \cap A = \varnothing$. Again, by 2.3, $f(\mathrm{cl}_{\kappa X} f^{-1}(W))$ is closed implying $\mathrm{cl}_{\kappa Y} W \cap A = \varnothing$. Thus, A is θ-closed in κY.

(3.2) Let X be a Hausdorff space and $A \subseteq X$. The following are equivalent:

(a) A is θ-closed in κX.

(b) A is θ-closed in every Hausdorff extension of X.

(c) A is θ-closed in σX.

(d) A is θ-closed in some H-closed extension of X.

Proof. By 3.1 and 2.2, (a) implies (b). Clearly, (b) implies (c) and (c) implies (d).

(d) implies (a). Suppose A is θ-closed in an H-closed extension Y of X. By 4.4 in [11], there is a continuous surjection $f: \kappa X \to Y$ such that $f(x) = x$ for $x \in X$. Let $z \in \kappa X \backslash A$. Since $f^{-1}f(A) = A$ by 1.8 in [13], then $f(z) \in Y \backslash A$. So, $\{f(z)\}$ and A are contained in disjoint open sets. By the continuity of f, $\{z\}$ and A are contained in disjoint open sets. So, A is θ-closed in κX.

It is not possible to replace "H-closed" in 3.4(d) by "Hausdorff" as a subset A of X can be θ-closed in some Hausdorff extension Y of X while A is not θ-closed in κX. For example, if X is Hausdorff but not H-closed, then X is θ-closed in the trival Hausdorff extension X *of* X, but X is not θ-closed in κX.

For each Hausdorff space X, we let θX denote $\{q: q$ is open ultrafilter on $X\}$. For each open set U in X, let $G(U)$ denote $\{q \in \theta X: U \in q\}$; $\{G(U): U$ open in $X\}$ forms a basis for an extremally disconnected, compact Hausdorff topology on θX [8]. By 5.2 in [13] there is a θ-continuous, perfect irreducible function $\pi: \theta X \to \sigma X$ defined by $\pi(q) = q$ for each free open ultrafilter q on X and $\pi(q) = x$ where x is the unique convergent point of the fixed open ultrafilter q.

(3.3) Let X be a Hausdorff space and U, V open subsets of X.

(a) $G(U) \cap G(V) = G(U \cap V)$ and $G(U) \cup G(V) = G(U \cup V)$.

(b) If $x \in X$ and $\pi^{-1}(x) \subseteq G(U)$, then $x \in \operatorname{int}_X \operatorname{cl}_X U$.

(3.4) If X is a Hausdorff space and $A \subseteq X$, then $\pi^{-1}(A)$ is compact if and only if A is θ-closed in κX.

Proof. Suppose $\pi^{-1}(A)$ is compact. By 3.2, it suffices to show A is θ-closed in σX. Suppose $y \in \sigma X \backslash A$. By the compactness of $\pi^{-1}(A)$ and $\pi^{-1}(y)$, the Hausdorffness of θX, and 3.3(a), there are open sets U and V in X such that $\pi^{-1}(A) \subseteq G(U)$, $\pi^{-1}(y) \subseteq G(V)$, and $G(U) \cap G(V) = \varnothing$. Now, by 3.3.(b), $A \subseteq \operatorname{int}_X \operatorname{cl}_X U$ and $y \in \operatorname{int}_X \operatorname{cl}_X V$. Since $\varnothing = G(U) \cap G(V) = G(U \cap V)$ and since every nonempty open set is contained in some open ultrafilter, then $U \cap V = \varnothing$. By 2.14 in [11], $\operatorname{int}_X \operatorname{cl}_X U \cap \operatorname{int}_X \operatorname{cl}_X V = \varnothing$. Thus, A and y are contained in disjoint open sets in X and by 4.1(c) in [11], in κX.

Conversely, suppose A is θ-closed in κX and, hence, by 3.2, θ-closed in σX. It suffices to show $\pi^{-1}(A)$ is closed in θX. Let $y \in \theta X \backslash \pi^{-1}(A)$. Then $\pi(y) \notin A$, and there is open neighborhood U of $\pi(y)$ in σX such that $\mathrm{cl}_{\sigma X} U \cap A = \varnothing$. So $\pi^{-1}(A) \cap \pi^{-1}(\mathrm{cl}_{\sigma X} U) = \varnothing$. But $y \in \pi^{-1}(\pi(y)) \subseteq \mathrm{int}_\theta \pi^{-1}(\mathrm{cl}_{\sigma X} U)$. Hence, $\pi^{-1}(A)$ is closed in θX.

A liability of the concept "θ-continuity" is that the restriction of a θ-continuous function is not necessarily θ-continuous; this fact is emphasized by 3.4. In particular, if A is a θ-closed, but not H-closed, subspace in an H-closed space Y (e.g., the set of nonisolated points of the space Y of Example 1.1 in [3]), then by 3.4, $\pi^{-1}(A)$ is compact; however, $\pi | \pi^{-1}(A)\colon \pi^{-1}(A) \rightarrow Y$ is not θ-continuous.

For a Hausdorff space X, let EX denote $\{q \in \theta X\colon q \text{ is fixed}\}$. Now, $\pi^{-1}(X) = EX$ and $\pi | EX\colon EX \rightarrow X$ is a θ-continuous, perfect, irreducible function (see [8, Th. 10]). Porter and Votaw [13] proved that $\sigma(EX) = E(\sigma X)$ if and only if the set of nonisolated points of EX is compact. We now characterize when σ and E commute in terms of X.

COROLLARY (3.5). *Let X be a Hausdorff space $\sigma(EX) = E(\sigma X)$ if and only if the set of nonisolated points of X is θ-closed in κX.*

Proof. Let A be the set of nonisolated points of X. By Theorem 5.8 in [13], $\pi^{-1}(A)$ is the set of nonisolated points of EX. The stated result now follows immediately by 3.4.

It is known that [10] no H-closed space is the countable union of compact nowhere dense subspaces and that [10] there exists an H-closed space that is the countable union of closed nowhere dense subspaces. An unsolved problem by Mioduszewski [10] is whether some H-closed space is the countable union of H-closed nowhere dense subspaces. We now show that no H-closed space is the countable union of θ-closed nowhere dense subspaces.

(3.6) An H-closed space is not the countable union of θ-closed nowhere dense subspaces.

Proof. Assume, by way of contradiction, that X is an H-closed space and $X = \bigcup \{A_n\colon n \in N\}$ where each A_n is nowhere dense and θ-closed in X. Since X is H-closed, then $X = \kappa X = \sigma X$ and $\theta X = EX$. By 3.4, $\pi^{-1}(A_n)$ is compact for each $n \in N$. If $\pi^{-1}(A_n)$ contains a nonempty open set, then by the irreducibility and closedness of π [8, Lemma 17], $\pi(\pi^{-1}(A_n)) = A_n$ contains a nonempty open set. So, each $\pi^{-1}(A_n)$ is nowhere dense. Hence, the compact Hausdorff space θX is the countable union of nowhere dense closed subsets, a contradiction.

A space has the *countable chain condition* (c.c.c.) if every family of pairwise disjoint nonempty open sets is countable. One of the equivalent forms (see [14]) of Martin's axiom is the following: Every compact Hausdorff space with ccc is not the union of less than $c(=2^{\aleph_0})$ closed nowhere dense subsets.

(3.7) Martin's axiom is equivalent to

(*) every H-closed space with c.c.c. is not the union of less than c θ-closed nowhere dense subsets.

Proof. Clearly, (*) implies the "compact Hausdorff" form of Martin's axiom. Conversely, suppose Martin's axiom is true and X is an H-closed space with c.c.c. Since X is H-closed, then $\theta X = EX$. Using the fact $\operatorname{int}_X \pi(U) \neq \emptyset$ for every nonempty open set U of EX, it follows that EX has c.c.c. If X is the union of α, a cardinal number, θ-closed nowhere dense subsets, then, as in the proof of 3.6, the compact Hausdorff space EX with c.c.c. is also the union of α closed nowhere dense subsets. Thus, (*) is true.

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Received September 18, 1974 and in revised form May 28, 1975. The research of the second author was partially supported by a University of Kansas research grant.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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