CLASSIFICATION OF SINGULAR INTEGRALS OVER A LOCAL FIELD

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The singular integral operators over a local field $K$ whose kernels are multiplicative characters of the unit sphere of $K$ are shown to be precisely those continuous operators on $\mathcal{L}_2(K)$ which commute with translation and dilation, anti-commute with an appropriately defined rotation, and whose multipliers satisfy a smoothness condition. The characterization is analogous to that of the Hilbert transform over the real numbers.

1. Classically, the Hilbert transform over $\mathbb{R}$ is, up to a constant multiple, the only continuous operator on $\mathcal{L}_2(\mathbb{R})$ which commutes with translation and (positive) dilation and anti-commutes with reflection. See [9], page 55. The Hilbert transform is a singular integral operator with kernel the only (nontrivial) multiplicative character of the unit sphere of $\mathbb{R}$.

Singular integrals over a local field have been developed. (See, for example, Phillips [6], Phillips-Taibleson [7], and Chao [1].) Those with kernel a multiplicative character of the unit sphere satisfy a classification similar to that of the classical Hilbert transform.

The classification theorem is in § 4. The main results are Theorems 4.1 and 4.2. Section 3 contains the necessary results regarding the character group of the unit sphere of a local field; § 2 contains other preliminary results, notation, and definitions.

2. Let $\mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}, \mathbb{R},$ and $\mathbb{C}$ denote the integers, the positive integers, the rational number, the real numbers, and the complex numbers, respectively. $F_p^n$ will denote the (unique) field with $p^n$ elements. The symbols $\mathbb{Q}_p$ and $\mathbb{Z}_p$ will denote the $p$-adic numbers and the $p$-adic integers, respectively. For any set $S$, $\xi_S$ will denote the characteristic function of $S$. The complement of $S$ will be written $S^\complement$.

The necessary analysis on local fields is stated without proof below. Most of it may be found in Chapters I and II of Weil [11].

A local field is a nondiscrete, locally compact, zero-dimensional topological (commutative) field. These have been completely classified. Those of characteristic $p \neq 0$ can be identified as the fields of formal power series over a finite field. Those of characteristic 0 are either the $p$-adic numbers of finite extensions of the $p$-adic numbers. See [11], page 11.
Let $K$ be a local field with $\lambda$ Haar measure for $(K, +)$. The modular function for $K$, $| \cdot |$, is given by $|x| = \lambda(xS)/\lambda(S)$ for $0 < \lambda(S) < \infty$. Haar measure for the multiplicative group $K^\times = K \sim \{0\}$ is $\lambda/| \cdot |$.

Let $R$ be the ring of integers of the local field $K$ and $P$ be the unique maximal ideal of $R$. Then $\text{ord}(R/P) = q$, the module of $K$, a prime power. The ideal $P$ has a generator $\pi$, so that $\pi R = P$. We have $|\pi| = q^{-1}$, and, in fact, any $x \in K$ with $|x| = q^{-1}$ will generate $P$. Those elements of modulus $q^{-1}$ will be called primes in $K$.

For $n \in \mathbb{Z}$ we define

$$P^n = \{x \in K : |x| \leq q^{-n}\}; \quad D^n = \{x \in K : |x| = q^{-n}\}.$$

Then $P^1 = P$, $P^0 = R$, and $R \sim P = D^0$. The set $\{P^n\}_{n=0}^\infty$ is a neighborhood base at 0 of open and closed subgroups of $(K, +)$. The set $\{1 + P^n\}_{n=1}^\infty$ is a neighborhood base at 1 of open and closed subgroups for the topological group $(K^\times, \cdot)$.

We define the operators $\tau_\delta$ for $\delta \neq 0$ on functions by $\tau_\delta f(x) = f(\delta x)$. Regarding the prime $\pi$ as fixed, we single out a set of such operators, the dilation operators, $\mathcal{D}_j$, defined by $\mathcal{D}_j f(x) = f(\pi^j x) j \in \mathbb{Z}$. A function $f$ is homogeneous degree zero if $\mathcal{D}_j f = f$ for all $j \in \mathbb{Z}$. For $x \in K$, translation operators $T_x$ are defined on functions by $T_x f(y) = f(x + y)$.

There is a character $\chi$ of the additive group of $K$ which is identically one on $R$ and nontrivial on $P^{-1}$. Then for any $y \in K$, $\chi_y (x) = \chi(xy)$ defines a character of $K$. In fact, the mapping $y \rightarrow \chi_y$ is a topological isomorphism of $(K, +)$ onto its dual. We thus identify $K$ with its dual.

The Fourier transform for $K$ is initially defined on $\mathcal{S}_1(K)$ by

$$\mathcal{F} f(x) = \hat{f}(x) = \int_K f(y) \overline{\chi(xy)} dy$$

[The integral is taken with respect to $\lambda$. Here and elsewhere the $\lambda$ will be suppressed.] The transform $\mathcal{F}^{-1}$ is defined by $\mathcal{F}^{-1} f(x) = \hat{f}(x) = \int_K f(y) \overline{\chi(xy)} dy$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ extend uniquely to $\mathcal{S}_1$. It is easy to see that, as $\mathcal{S}_1$ operators, $\tau_\delta \mathcal{F} = |\delta|^{-1} \mathcal{F} \tau_\delta^{-1}$ and $\tau_\delta \mathcal{F}^{-1} = |\delta|^{-1} \mathcal{F}^{-1} \tau_\delta^{-1}$.

The following result will be used extensively in the sequel: Let $L$ be a continuous linear operator from $\mathcal{L}_1(K)$ to $\mathcal{L}_1(K)$. Then a necessary and sufficient condition that $L$ commute with translation is that there exist a function $m$, in $\mathcal{L}_1(K)$, such that $\mathcal{F}(Lf) = m \mathcal{F} f$ for all $f \in \mathcal{L}_1(K)$. See [5], pp. 92-94.

The space $\mathcal{I}$ of test functions on $K$ and its topological dual $\mathcal{I}'$, the space of distributions, are defined as in [8]. Both are
complete linear spaces. The action of a \( \mu \in \mathcal{I}' \) on an \( f \in \mathcal{I} \) will be denoted \((\mu, f)\).

The space \( \mathcal{I} \) is contained densely in \( \mathcal{L}_p, 1 \leq p < \infty \). The Fourier transform is thus well-defined on \( \mathcal{I} \). The Fourier transform on \( \mathcal{I}' \) is given by \((\hat{\mu}, f) = (\mu, \hat{f})\). Thus defined, the Fourier transform is a linear topological isomorphism on both \( \mathcal{I} \) and \( \mathcal{I}' \).

Functions and measures will be identified with the distributions they induce. Convolution of a distribution and a test function is defined by \( \mu * f(x) = (\mu, T_x f) \), where \( f(x) = f(-x) \).

Let \( \mu \in \mathcal{I}' \), and let \( \sigma \) be a (not necessarily unitary) multiplicative character of \( K^\times (= K \sim \{0\}) \). Then, as in [8], we say \( \mu \) is homogeneous of degree \( \sigma \) if for all \( t \in K^\times, \mu_t = \sigma(t)\mu \), where \( \mu_t \) is that distribution defined by \((\mu_t, \phi) = (\mu, |t|^{-1}\tau_{t^{-1}}\phi)\).

We take \( M \) to be \( M^\times \cup \{0\} \), where \( M^\times \) is the group of roots of unity in \( K \) of order prime to \( p \). Then \( M^\times \) is the unique cyclic group of order \( q - 1 \) ([11], p. 16). Let \( g \) be a generator of \( M^\times \). Then each \( 0 \neq x \in K \) may be written uniquely as \( x = \pi^i g^j (1 + p_x) \), where \( k, j \in \mathbb{Z}, 0 \leq k \leq q - 2, p_x \in P \). A multiplicative character of \( K^\times \) is given by its values at \( \pi, g \), and on \( 1 + P \).

Let \( \omega \) be a multiplicative character of \( K^\times \). There is some \( n \in \mathbb{Z} \) such that \( \omega \) is trivial on \( 1 + P^n \). If \( \omega \) is trivial on \( 1 + P^n \) but not on \( 1 + P^{n-1}, n \geq 1 \), we say \( \omega \) is ramified of degree \( n \). If \( \omega \) is trivial on \( D^\times \), we say \( \omega \) is unramified. Given a character \( \omega \) of \( 1 + P, \omega \) is the restriction of a character of \( K^\times \), say \( \omega' \). The ramification degree of \( \omega' \) depends only on \( \omega \), and we define the ramification degree of \( \omega \) to be that of \( \omega' \).

We define the local field gamma function on ramified characters of \( K^\times \) by

\[
\Gamma(\omega) = \text{p.v.} \int_K \frac{\chi(x)\omega(x)dx}{|x|},
\]

where

\[
\text{p.v.} \int_K f(x)dx = \lim_{n \to \infty} \int_{P^{-1}(P^+)^n} f(x)dx.
\]

See [8] for details and further definition of \( \Gamma \).

3. \textbf{Lemma 3.1.} Let \( K \) be a local field of characteristic \( p \neq 0 \) with module \( q = p' \). Let \{\( \alpha_i, \ldots, \alpha_f \)\} be a basis for \( F_q \) over \( F_p \). Then given \( x \in P \) and \( N \in \mathbb{Z}^+ \),

\( a \) there are unique integers \( a_{i,j}, n_i, \nu_j, \) with \( 0 \leq a_{i,j} < p \), \( (n_j, p) = 1 \) for \( 1 \leq i \leq f, 1 \leq j \leq N \), such that \( 1 + x = \prod_{j=1}^f \prod_{i=1}^N (1 + \alpha_i \pi^j a_{i,j} p^{j+1}(p^{N+1}) \), and
(b) $1 + x \in (1 + P^N) \sim (1 + P^{N+1})$ if and only if $a_{i,j} = 0$ for $1 \leq i \leq f, 1 \leq j \leq N$ and at least one of the $a_{i,N} \neq 0, 1 \leq i \leq f$.

Proof. The proof is similar to that of Proposition 10, page 34 of [11], and is omitted.

Given $N \in \mathbb{Z}^+$ we establish the following notation to be used in the following lemma and theorem. For each $j, 1 \leq j \leq N$, write $j = n_j P^r$, where $(n_j, p) = 1$; define $m_j$ as the smallest integer such that $m_j \geq \log_p ((N+1)/n_j)$ then define $\beta_j$ as a primitive $p^{m_j}$th root of 1 in $C$.

**Lemma 3.2.** With the above notation, $m_N = v_N + 1$.

Proof. The proof is a direct computation and is omitted.

**Theorem 3.1.** Let $K$ be a local field of characteristic $p \neq 0$ and $\omega$ a character of $1 + P \subset K$ ramified field degree $N+1$. Then for $x \in 1 + P$, $\omega$ is given by

$$\omega(x) = \prod_{j=1}^{N} \prod_{i=1}^{f} \beta_{j,i}^{k_{i,j} n_j} (P^{N+1}),$$

where

$$x \equiv \prod_{j=1}^{N} \prod_{i=1}^{f} (1 + \alpha_i P^{n_j})^{k_{i,j} n_j} (P^{N+1})$$

for some unique $k_{i,j}, 0 \leq k_{i,j} < p^{m_j}$ with at least one of $k_{i,N}, 1 \leq i \leq f$, relatively prime to $p$.

Proof. Since $\omega$ is constant on cosets of $P^{N+1}$ it suffices to consider $x$ mod $P^{N+1}$. For any $x \in 1 + P$, the numbers $a_{i,j}, n_j, \nu_j$ are determined as in Lemma 3.1 so that (*) holds. Clearly $\omega$ will be completely determined by its values on $\{1 + \alpha_i P^{n_j}\}, 1 \leq i \leq f, 1 \leq j \leq N$, and the range of $\omega$ is contained in the $p^N$th power roots of unity.

The definition of $m_j$ as the smallest integer greater than or equal to $\log_p (N+1)/n_j$ makes $m_j$ the smallest integer such that

$$(1 + \alpha_i P^{n_j})^{m_i} \in 1 + P^{N+1}.$$ 

Thus $(\omega(1 + \alpha_i P^{n_j}))^{m_i} = 1$, and $\omega(1 + \alpha_i P^{n_j}) = \beta_j^{k_{i,j}}$ for some unique $k_{i,j}, 0 \leq k_{i,j} < p^{m_i}$. Thus $\omega$ has the form required. The remainder of the theorem follows easily from the fact that $\beta_N$ is a $p^{m_N}$th root of unity and $\omega$ must be nontrivial on $P^N$.

From Proposition 9 of Chapter II, § 3 of [11], we have:
PROPOSITION. Let $K$ be a $d$-dimensional extension of $Q_p$. Then there is an integer $m \geq 0$ such that $1 + P$, as a multiplicative group is isomorphic to the additive group $Z_p^d \times F_{p^m}$, where $m$ is the largest integer such that $K$ contains a primitive $p^m$th root of unity. For proof see [11].

Let $\{u_i\}_{i=1}^d$ be those elements of $1 + P$ which map to the vectors with 1 in the $i$th coordinate and zeros elsewhere by the isomorphism in the proposition. Let $u_{d+1}$ be a primitive $p$th power root of unity in $K$ of maximal order, say $p^m$. Then any $x \in 1 + P$ is given uniquely by $x = \prod_{i=1}^{d+1} u_i^{a_i}$, where $a_i \in Z_p$, $1 \leq i \leq d$ and $a_{d+1} \in Z$, $0 \leq a_{d+1} < p^m$.

LEMMA 3.3. Let $K$ be a $d$-dimensional extension of $Q_p$. Then given nonnegative integers $k_i$, $1 \leq i \leq d$, each $x \in 1 + P \subset K$ has a representation as

$$x = u_{d+1}^{a_{d+1}} \prod_{i=1}^{d} u_i^{a_i},$$

where $b_i \in Z_p$ with $|b_i|_{Z_p} < p^{-k_i}$ and $n_i$ is a nonnegative integer. If $n_i$ is picked to be as small as possible, this representation is unique.

Proof. The proof is direct from the above proposition and the density of $Z^+$ in $Z_p$.

Given $N \in Z^+$, define, for $1 \leq i \leq d + 1$, $\mathcal{L}_i$ to be the smallest integer such that $u_i^{b_i} \in 1 + P^{N+1}$ and $\beta_i$ to be a fixed primitive $p^{i_{th}}$ root of unity. With this notation we have the following:

THEOREM 3.2. Let $K$ be a local field of characteristic 0 and $\omega$ a character of $1 + P \subset K$ ramified of degree $N + 1$. Then for $x \in 1 + P$, $\omega$ is given for some unique $k_i$, $0 \leq k_i < p^i$, $1 \leq i \leq d + 1$, by

$$\omega(x) = \prod_{i=1}^{d} \beta_i^{k_i}, \quad \text{for } x = \prod_{i=1}^{d} u_i^{a_i} u_{d+1}^{a_{d+1}},$$

where for $1 \leq i \leq d$, $b_i \in Z_p$ with $|b_i|_{Z_p} < p^{-k_i}$ and $n_i \in Z^+$.

Proof. The density of $Z$ in $Z_p$ shows that an (additive) character of $Z_p$ is determined by its value at 1. Thus a (multiplicative) character $1 + P$ will be determined by its values at the $u_i$, $1 \leq i \leq d + 1$. Here $\omega(u_i)^{p_i} = 1$ since $u_i^{p_i} \subset 1 + P^{N+1}$ and $\omega$ is ramified of degree $N + 1$. Thus $\omega(u_i) = \beta_i^{k_i}$ for some (unique) $k_i$, $0 \leq k_i < p^i$.

This characterization of the character group of $K$ depends on
the \( p \)th roots of unity in \( K \). Since \( K \) is a finite dimensional extension of \( Q_p \), we look for a relationship between the degree \( d \) of \( K \) over \( Q_p \) and the existence of \( p \)th roots of unity in \( K \).

**Theorem 3.3.** Let \( K \) be a local field of characteristic 0. If \( K \) is the \( p \)-adic field \( Q_p \) for some prime \( p \neq 2 \), then \( K \) has no nontrivial \( p \)th roots of unity. If \( K \) is an extension of \( Q_p \), \( p \neq 2 \), let the degree of ramification (see [11]) of \( K \) over \( Q_p \) be \( e \); then,

(a) \( K \) has no \( p \)th roots of 1 if \( (p - 1) \) does not divide \( e \),

(b) \( K \) may or may not have \( p \)th roots of 1 if \( p - 1 \) divides \( e \).

**Proof.** For the proof of (a) see [2]. Part (b) follows from [2] and the fact that the extension of \( Q_p \) by a root of \( x^{p - 1} - p \) is fully ramified of degree \( p - 1 \) and has no \( p \)th roots of unity.

**Lemma 4.1.** Let \( \omega \) be a homogeneous degree zero multiplicative character of \( K^* \), ramified of degree \( k > 0 \). Then \( \omega \) is a kernel for a singular integral operator. The multiplier \( m \) for the singular integral operator \( T \) with kernel \( \omega \) satisfies

\[
m(x) = \omega(-1)T(\omega)\omega^{-1}(x).
\]

**Proof.** The operator \( T \) is defined for \( f \in S_p \), \( 1 \leq p < \infty \) by

\[
Tf(x) = \lim_{k \to \infty} \int_{(p^k)\mathbb{Z}} \frac{\omega(y)}{|y|} (f(x - y)dy.
\]

Theorem 3.1 of [7] gives sufficient conditions on the kernel \( \omega \) for the limit to exist (in \( S_p \)). That \( \omega \) satisfies those conditions is easily verified. Then from [7] we know \( T \) is bounded on \( S_p \), \( 1 < p < \infty \) and weak type \((1, 1)\).

The remainder of the lemma is done by Chao [1] for the case \( \omega \) ramified of degree 1. The same proof establishes the result stated here.

**Note.** Chao [1] uses Theorem 4 of [8] to establish the conclusion of Lemma 4.1 for the case \( \omega \) ramified of degree 1. However, he fails to compensate for the fact that he defines the Fourier transform as herein, i.e., \( \mathcal{F}f(y) = \int f(x)\chi(xy)dx \), while in [8] it is defined as \( \int f(x)\chi(xy)dx \). Thus the result of [1] which corresponds to the conclusion of Lemma 4.4 above does not contain the necessary factor of \( \omega(-1) \).

With notation as in Theorem 3.1, we define rotation operators \( S_{i,j} \) for functions on a \( p \)-series field as follows:
S_{i,0}f(x) = f(gx),

where \( g \) is a fixed primitive \((q - 1)^{st}\) root of unity in \( K \); and

\[
S_{i,j}f(x) = f((1 + \alpha_i\pi^o)x)
\]

for \( 1 \leq i \leq f, j \geq 1 \).

Given \( N \) we determine \( \beta_j, 1 \leq j \leq n \) as in Theorem 3.1, and let \( \beta_0 \) be a \((q - 1)^{st}\) root of unity in \( \mathbb{C} \). Also as in that theorem, note that given \( N \) the choice of integers \( k_{i,j}, 0 \leq k_{i,j} < p^m, 1 \leq i \leq f, 1 \leq j \leq N \) determines a character of \( 1 + P \). If we also pick a \( k_{i,0}, 0 \leq k_{i,0} < q - 1 \), and set \( \omega(g) = \beta^{k_{i,0}} \), then the set \( \{k_{i,j}\} \) determines character of \( D^o \). That character will be called the character determined by \( \{k_{i,j}\} \). As it may be used as a kernel for a singular integral operator, that operator will be identified as the one determined by \( \{k_{i,j}\} \).

We can how state

**Theorem 4.1.** Let \( K \) be a p-series field and \( L \) a continuous linear operator from \( \mathcal{L}_2(K) \) to \( \mathcal{L}_2(K) \) which satisfies

1. \( L \) commutes with translation and dilation,
2. there is some \( N \geq 0 \) such that the multiplier corresponding to \( L \) is constant on cosets of \( P^{N+1} \),
3. \( L \) anti-commutes with the rotations \( S_{i,j}, 1 \leq i \leq f, 1 \leq j \leq N \), and \( S_{i,0} \) in the sense that

\[
LS_{i,j} = \beta_{j}^{-k_{i,j}}S_{i,j}L,
\]

for some \( k_{i,j} \). Then \( L \) is a constant multiple of the singular integral operator determined by \( \{k_{i,j}\} \).

Before proving Theorem 4.1, we consider the p-adic case. Let \( u_i, 1 \leq i \leq d + 1 \) be as in Theorem 3.2, and let \( u_0 = g \), the fixed \((q - 1)^{st}\) root of 1 in \( K \). In this case we define rotation operators as:

\[
S_i f(x) = f(u_i x), 0 \leq i \leq d + 1.
\]

Given \( N \), we determine \( \beta_i, 0 \leq i \leq d + 1 \) by: \( \beta_0 \) is a primitive \((q - 1)^{st}\) root of 1 in \( \mathbb{C} \); \( \beta_i, 1 \leq i \leq d + 1 \), is a primitive \( p^{l_i}\)th root of 1 in \( \mathbb{C} \), where \( l_i \) is the smallest integer such that \( u_i^{l_i} \in 1 + P^{N+1} \). Also, for each \( i \) we consider integers \( k_i \) such that \( 0 \leq k_0 < q - 1, 0 \leq k_i < p^{l_i}, 1 \leq i \leq d + 1 \).

By Theorem 3.2 and the fact that \( D^o = M^{x}(1 + P) \), the set \( \{k_i\}_{i=0}^{d+1} \) determines a unique character of \( D^o \) by \( \omega(u_i) = \beta_{i}^{k_{i}} \). The character \( \omega \) will be called the character determined by the \( \{k_{i}\} \). It is clearly constant on \( 1 + P^{N+1} \).

**Theorem 4.2.** Let \( K \) be a local field of characteristic 0, and let \( L \) be a continuous linear operator from \( \mathcal{L}_2(K) \) to \( \mathcal{L}_2(K) \) which
satisfies

(a) $L$ commutes with translation and dilation,
(b) there is some $N$ such that the multiplier for $L$ is constant on cosets of $P^{N+1}$,
(c) $L$ anti-commutes with the rotations $S_i$, $0 \leq i \leq d + 1$ in the sense that

$$LS_i = \beta_i S_i L.$$ 

Then $L$ is a constant multiple of the singular integral transform determined by the $\{k_i\}$. The proof of Theorems 4.1 and 4.2 will utilize the following Lemma.

**Lemma 4.2.** Let $K$ be a local field. If characteristic $K = 0$, let $L$ satisfy the hypothesis of Theorem 4.2. If characteristic $K = p \neq 0$, let $L$ satisfy the hypothesis of Theorem 4.1. Then for $f \in \mathcal{S}$, $L$ is given by convolution with a unique distribution $\mu$, homogeneous of degree $\omega \| \cdot \|$, where $\omega$ is the character of $D^0$ determined the $\{k_i\}$ or $\{k_i\}$ in the characteristic $p \neq 0$ and characteristic 0 case, respectively.

**Proof.** Since $L$ is a bounded linear operator from $L_2^1$ to $L_2$ which commutes with translation, by Theorem 9 of [10], it is given, on $\mathcal{S}$, by convolution with a unique distribution $\mu$. We need only to show $\mu$ homogeneous of degree $\nu$, where $\nu(x) = \omega(x)/\|x\|$, $x \neq 0$.

There is a function $m$ in $L_\infty(K)$ so that for $f \in L_2(K)$, $(Lf)^\wedge = mf$. For $f \in \mathcal{S}$, $\hat{f} \in \mathcal{S}$, thus $m\hat{f} \in L_2(K)$ since $m \in L_\infty(K)$. Then $Lf = (m\hat{f})^\vee$ is continuous since it is the inverse Fourier transform of an $L_1$ function.

Let $\gamma \in 1 + P^{N+1}$. Then $\gamma^{-1} \in 1 + P^{N+1}$, and, since $m$ is constant on cosets of $P^{N+1}$, we have:

$$(L\tau_{t} f)^\wedge(x) = m(\tau_t f)^\wedge(x) = m(x)\hat{f}(\gamma^{-1} x)$$

$$= m(\gamma^{-1} x)\hat{f}(\gamma^{-1} x) = \tau_{\gamma^{-1}} m\hat{f}(x).$$

Thus

$$L\tau_t f = \tau_t Lf \quad \text{in} \quad L_2.$$

Fix $t \in K^*$. By (a) and (c) of Theorems 4.1 and 4.2 and the above equality, we have:

$$L\tau_t f = \omega^{-1}(t)\tau_t Lf \quad \text{in} \quad L_2$$

and

$$L\tau_t f(x) = \omega^{-1}(t)\tau_t Lf(x) \quad \text{a.e.}$$
But since both \( L\tau_t f \) and \( \tau_t Lf \) are continuous, we have the above equality everywhere.

For \( f \in \mathcal{F} \),
\[
\mu \ast \tau_t f(0) = \omega^{-1}(t)(\mu \ast f)(t \cdot 0),
\]
and
\[
(\mu, \tau_t f) = \omega^{-1}(t)(\mu, f).
\]

Thus
\[
(\mu, \mu_\tau f) = \omega^{-1}(t)(\mu, f).
\]

Since this holds for all \( f \in \mathcal{F}, \mu_\tau = \nu(t)\mu \).

Now we are ready for the

**Proof of Theorems 4.1 and 4.2.** By Lemma 4.2 for \( f \in \mathcal{F}, \)
\( Lf = \mu \ast f \), where \( \mu \) is homogeneous of degree \( \omega // | \cdot |. \) But by Lemma 5 of [8], the only distributions which are homogeneous of degree \( \sigma \), \( \sigma \) multiplicative character of \( K^\times \) such that \( \sigma(x) \) is not identically \( |x|^{-1} \), are constant multiples of \( \sigma \). Thus \( \mu = \omega |/\cdot| \), and \( Lf = (\omega)(/\cdot)(f), f \in \mathcal{F} \). Thus, on the test functions, a dense subset of \( \mathcal{L}_2 \), \( L \) agrees with \( L' \), the singular integral operator defined by
\[
L'f(x) = c \int (\omega(y))/(y)f(x - y)dy.
\]
But since \( L \) and \( L' \) are continuous, \( L = L' \) on \( \mathcal{L}_2 \).

5. Example. The conclusions of Theorems 4.1 and 4.2 may be obtained by direct calculation. We indicate the method in the case \( q = 3 \) and \( \omega \) ramified of degree 1. Here \( M^\times = \{1, -1\} \) and \( \omega \) will assume only the values \( \pm 1 \). [This is the “exact” analog of the Hilbert transform for the reals.]

Let \( H \) be the singular integral operator with \( \omega \) as kernel. Both theorems then have the form: Theorem: Let \( K \) be local field with module \( q = 3 \) and \( L \) be a continuous operator on \( \mathcal{L}_2(K) \) which satisfies:

(a) \( L \) commutes with translation and dilation;
(b) the multiplier, \( m \), for \( L \) is constant on \( 1 + P \);
(c) \( L \) anti-commutes with the rotation \( \tau_{-1} \) by \( L\tau_{-1} = -\tau_{-1}L \).

Then \( L \) is a constant multiple of \( H \).

**Proof.** From the relation \((Lf)^\wedge = m\hat{f}\) it follows as in the real
case (see [9]) that \( m(-x) = -m(x) \). Since any \( x \in K^\times \) may be written
\[ x = \pm \pi^i(1 + \rho_x), \rho_x \in P, \ m(x) = \pm m(1) = \omega^{-i}(x)m(1). \]
The theorem then follows from Lemma 4.1.

Lemma 4.1 may also be shown directly. For the case above we may even evaluate the multiplier \( m_H \) explicitly. Taking the fundamental character \( \chi \) to be that given in [1] (a variation of that given in [6]), and the form of \( m_H \) from [7], we obtain \( m_H(x) = (i)/(\sqrt{3}) \omega(x) \). As in [1], a similar easy calculation gives \( \Gamma(\omega) = -i/\sqrt{3} \), exemplifying Lemma 4.1. In further analogy with the real case, it is apparent from the multiplier that \( H^2 = -(1/3)I \).

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