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**METRIZATION OF SPACES WITH COUNTABLE LARGE BASIS
DIMENSION**

GARY FRED GRUENHAGE AND PHILLIP LEE ZENOR

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With the following results, we generalize known metrization theorems for spaces with large basis dimension 0 i.e., non-archimedean spaces) to the higher dimensions: *Theorem.* If X is a normal Σ -space with countable large basis dimension, then X is metrizable. *Theorem.* If X is a normal $w\Delta$ -space with countable large basis dimension, then X is metrizable.

I. Introduction. A collection Γ of subsets of a set X is said to have rank 1 if whenever g_1 and g_2 are in Γ with $g_1 \cap g_2 \neq \emptyset$ then $g_1 \subset g_2$ or $g_2 \subset g_1$. According to P. J. Nyikos [13], a topological space X has large basis dimension $\leq n$ (denoted $\text{Bad } X \leq n$) if X has a basis which is the union of $n + 1$ rank 1 collections of open sets. X has countable large basis dimension ($\text{Bad } X \leq \aleph_0$) if X has a basis which is the union of a countable number of rank 1 collections such that each point of X has a basis belonging to one of the collections (a property which is automatically true in the finite case). $\text{Bad } X$ coincides with $\text{Ind } X$ and $\dim X$ for metric spaces.

Spaces having large basis dimension 0 are usually called non-archimedean spaces. Theorems of Nyikos [11] and A. V. Arhangel'skii [3] show that a non-archimedean space is metrizable if and only if it is a Σ -space or a $w\Delta$ -space. In this paper we show that these results are valid, under mild assumptions, for the higher dimensions. Our results also improve a result of G. Gruenhage [6], who showed that compact spaces having finite large basis dimension are metrizable.

II. Main results. According to Nyikos [11], a tree of open sets is a collection Γ of open sets such that if $g \in \Gamma$, then the set $\{g' \in \Gamma \mid g' \supset g\}$ is well-ordered by reverse inclusion; that is, $g \leq g'$ if and only if $g \supset g'$. Nyikos shows that the rank 1 collections for spaces with $\text{Bad } X \leq \aleph_0$ can be considered as rank 1 trees of open sets. The following fact will be used in our proofs:

LEMMA 1. *Let T be a rank 1 tree of open subsets of a regular space X which contains a basis at each point of a subset X' of X . Then if \mathcal{U} is a cover of X' by open subsets of X , there exists a subset T' of T such that*

- (i) T' is a cover of X' ;
- (ii) the elements of T' are pairwise disjoint;

(iii) $t \in T'$ implies that either t is degenerate or \bar{t} is a proper subset of some member of \mathcal{U} .

Proof. Put t in T' if and only if (a) either t is degenerate or there is a member U of \mathcal{U} such that \bar{t} is a proper subset of U and (b) there is no predecessor of t in T whose closure is a proper subset of some element of \mathcal{U} . Since T contains a basis at each point of X' and since the predecessors of a given $t \in T$ are well-ordered, it is easy to see that T' covers X' . Further, since T is a tree, the members of T' are mutually exclusive.

Nyikos calls a space *basically screenable* if it has a basis which is the union of countably many rank 1 trees of open sets. Every space X with $\text{Bad } X \leq \aleph_0$ is basically screenable. Basically screenable spaces are, of course, screenable; that is, every open cover has a σ -pairwise disjoint open refinement. While the following result is known, for the sake of completeness, we include its easy proof:

LEMMA 2. *A screenable countably compact space X is compact [2].*

Proof. Let \mathcal{U} be an open cover of X and let $\mathcal{V} = \cup \{ \mathcal{V}_n \mid n = 1, 2, \dots \}$ be an open refinement of \mathcal{U} covering X such that, for each i , the members of \mathcal{V}_i are mutually exclusive. The set $\{ V_n = \bigcup \mathcal{V}_n \mid n = 1, 2, \dots \}$ is a countable open cover of X ; hence, there exists a finite subcover $\{ V_{n_1}, V_{n_2}, \dots, V_{n_k} \}$. Then $\mathcal{V}_{n_1} \cup \mathcal{V}_{n_2} \cup \dots \cup \mathcal{V}_{n_k}$ is a point-finite refinement of \mathcal{U} . Thus, X is metacompact and it is well-known that a metacompact countably compact space is compact.

According to C. R. Borges [4], a space X is a $w\Delta$ -space if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X such that whenever $x \in X$ and $x_n \in \text{St}(x, \mathcal{G}_n)$ for each n , then $\{x_1, x_2, \dots\}$ has a cluster point.

THEOREM 1. *If X is a regular $w\Delta$ -space with countable large basis dimension, then X has a point countable basis.*

Proof. Let $\mathcal{G}_1, \mathcal{G}_2, \dots$ be a sequence of open covers of X satisfying the properties given in the definition of a $w\Delta$ -space. Let $\mathcal{B}_1, \mathcal{B}_2, \dots$ and X_1, X_2, \dots be sequences such that $X = \bigcup \{ X_i \mid i = 1, 2, \dots \}$ and, for each i , \mathcal{B}_i is a rank 1 tree of open sets containing a basis at each point of X_i .

For each $i < \omega_0$ and $\alpha < \omega_1$, we construct a collection $\mathcal{B}(i, \alpha)$ as follows: let $\mathcal{B}(i, 1)$ be a collection of mutually exclusive members of \mathcal{B}_i that refines \mathcal{G}_1 and covers X_i .

Suppose $\mathcal{B}(i, \beta)$ has been defined for $\beta < \alpha$. If α is not a limit ordinal, applying Lemma 1, let $\mathcal{B}(i, \alpha)$ be a collection of mutually

exclusive members of \mathcal{B}_i such that

- (i) if $j < \omega_0$, then $\mathcal{B}(i, j)$ refines \mathcal{G}_j ;
- (ii) $\mathcal{B}(i, \alpha)$ covers $(\cup \mathcal{B}(i, \alpha - 1)) \cap X_i$;

and (iii) $g \in \mathcal{B}(i, \alpha)$ implies \bar{g} is a proper subset of some member of $\mathcal{B}(i, \alpha - 1)$, or g is degenerate. If α is a limit ordinal, for each $x \in X_i$, let $B(\alpha, x) = \text{Int}(\bigcap_{\beta < \alpha} \{g \in \mathcal{B}(i, \beta) \mid x \in g\})$. Note that if x and y are in X_i , then either $B(\alpha, x) = B(\alpha, y)$ or $B(\alpha, x) \cap B(\alpha, y) = \emptyset$. Let $\mathcal{B}(i, \alpha) = \{B(\alpha, x) \mid x \in X_i\}$.

Let $\mathcal{B}_i^* = \bigcup_{\alpha < \omega_1} \mathcal{B}(i, \alpha)$. We will show that \mathcal{B}_i^* is a point countable collection forming a basis for X_i in X .

We will say that g is a *chain in \mathcal{B}_i^** if g is a function from an initial segment of ω_1 into \mathcal{B}_i^* so that (1) $g(\alpha) \in \mathcal{B}(i, \alpha)$ and (2) if $\beta < \alpha$, then $g(\beta) \supset g(\alpha)$. Note that by our construction, if $\beta < \alpha$, then $g(\beta) \supset \overline{g(\alpha)}$. Furthermore, if $x \in X_i$, then there is exactly one maximal chain, say g , such that $g(\alpha)$ contains x for every α in the domain of g .

Claim 1. The domain of each maximal chain in \mathcal{B}_i^* is countable (and so, \mathcal{B}_i^* is point countable in X).

Proof of Claim 1. Suppose the contrary; i.e., there is a chain, say g , of length ω_1 .

Note that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ is compact. To prove this, we will only show that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ is countably compact; that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ is compact will then follow from Lemma 2. To this end, let N denote a countable subset of $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$. There is an α so that $g(\alpha)$ does not meet N . In particular then, no point of $\overline{g(\alpha + 1)}$ is a limit point of N . Because of property (i), it must be the case that N has a limit point in $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$; and so, $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ is compact. But, $\{g(\omega_0 + 1) - g(\alpha) \mid \alpha < \omega_1\}$ is an open cover of $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ with no finite subcover, which is a contradiction from which Claim 1 follows.

Claim 2: \mathcal{B}_i^* is a basis for X_i in X ; in particular, if $x \in X_i$ and g is the maximal chain in \mathcal{B}_i^* centered at x , then $\{g(\alpha) \mid \alpha \text{ is in the domain of } g\}$ is a local basis for x in X .

Proof of Claim 2. Suppose otherwise. Then there is a point x of X_i so that the maximal chain, g , centered at x does not yield a basis at x in X ; i.e., $\{g(\alpha) \mid \alpha \in \text{domain of } g\}$ is not a local basis for x in X . Since the domain of g is countable, there is a first $\alpha_0 < \omega_1$ not in the domain g . There is a member B of \mathcal{B}_i so that if $\alpha < \alpha_0$, then $g(\alpha)$ is not a subset of B but this means that B is a subset of each $g(\alpha)$. Then x is in the interior of $\bigcap_{\alpha < \alpha_0} g(\alpha)$. Thus, by our

construction of $\mathcal{B}(i, \alpha)$, there is a member of $\mathcal{B}(i, \alpha)$ that contains x . This contradicts the maximality of g and it follows that $\{g(\alpha) \mid \alpha \text{ is in the domain of } g\}$ is a local basis for x in X .

We now have that $\bigcup_{i < \omega_0} \mathcal{B}_i^*$ is a point countable basis for X .

If \mathcal{H} is a cover of the space X and if $x \in X$, then $C(x, H)$ will denote the set $\bigcap \{H \in \mathcal{H} \mid x \in H\}$. According to K. Nagami [9], the space X is a Σ -space if there is a sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ of locally finite closed covers of such that if x_0, x_1, x_2, \dots is a sequence with $x_i \in C(x_0, \mathcal{F}_i)$ for each $0 < i < \omega_0$, then $\{x_i\}$ has a cluster point. The sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ is called a spectral Σ -sequence for X .

We will, without loss of generality, assume that each \mathcal{F}_i is closed under intersections and, for each i , \mathcal{F}_{i+1} refines \mathcal{F}_i .

LEMMA 3. *If X is a space with countable large basis dimension such that each uncountable subset of X has a limit point, then X is Lindelof.*

Proof. Since X has countable large basis dimension, X is screenable. G. Aquaro [1] has proved that every meta-Lindelof (and thus every screenable) space in which every uncountable set has a limit point is Lindelof.

THEOREM 2. *If X is a regular Σ -space with countable large basis dimension then X has a point countable basis.*

Proof. Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be a sequence of locally finite closed coverings of X given in the definition of a Σ -space. For each n , let \mathcal{G}_n be an open cover of X such that each member of \mathcal{G}_n intersects only finitely many members of \mathcal{F}_n . Let $\mathcal{B}_1, \mathcal{B}_2, \dots$ and X_1, X_2, \dots be sequences such that $X = \bigcup_{i < \omega_0} X_i$ and \mathcal{B}_i is a rank 1 tree of open sets which contains a basis for each point of X_i .

Define $\mathcal{B}(i, \alpha)$, $i < \omega_0$, $\alpha < \omega_1$, exactly as in the proof of Theorem 1. Let $\mathcal{B}_i^* = \bigcup_{\alpha < \omega_1} \mathcal{B}(i, \alpha)$ and define chain in \mathcal{B}_i^* as in the proof to Theorem 1.

Claim 1. Every chain in \mathcal{B}_i^* is countable.

Proof of Claim 1. Suppose otherwise; i.e., suppose that g is a chain in \mathcal{B}_i^* with length ω_1 . Let $K = \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$. Every uncountable of $\overline{g(\omega_0)} - K$ has a limit point in $\overline{g(\omega_0)} - K$ for suppose otherwise; that is, suppose that H is an uncountable subset of $\overline{g(\omega_0)} - K$ with no limit point in $\overline{g(\omega_0)} - K$.

Suppose that there is a point, h , of H such that, for each n ,

$C(h, \mathcal{F}_n)$ intersects infinitely many points of H . Then there is a countable subset N of H with a limit point. Since N is countable, there is an $\alpha < \omega_1$ so that $g(\alpha)$ does not intersect N . It follows that no point of K is a limit point of N . Hence, no point of K is a limit point of N ; and so, H has a limit point in $\overline{g(\omega_0)} - K$. This is a contradiction from which it follows that, for each h in H , there is an integer $n(h)$ such that $C(h, n(h))$ intersects only finitely many members of H . Thus, there is an N and an uncountable subset H^* of H so that if $h \in H^*$, then $n(h) = N$ and $\{C(h, F_N) | h \in H^*\}$ is an infinite subcollection of \mathcal{F}_N , each member of which intersects $g(N)$. But, $g(N)$ is in $\mathcal{B}(i, N)$ which contradicts the fact that $\mathcal{B}(i, N)$ refines \mathcal{E}_N . It follows that each uncountable subset of $\overline{g(\omega_0)} - K$ has a limit point in $\overline{g(\omega_0)} - K$; and so, by Lemma 3, $\overline{g(\omega_0)} - K$ is Lindelof. But $\{\overline{g(\omega_0)} - \overline{g(\alpha)} | \alpha < \omega_1\}$ is an open cover of $\overline{g(\omega_0)} - K$ with no countable subcover which is a contradiction from which Claim 1 follows.

That \mathcal{B}_i^* contains a basis at each point of X_i follows exactly as in the proof of Theorem 1. Thus Theorem 2 is proved.

THEOREM 3. *If X is a normal Σ -space with countable large basis dimension, then X is metrizable.*

Proof. R. E. Hodel has proved that every Σ -space is a β -space [8], and that every β -space is countably metacompact [7]. A screenable countably metacompact space is metacompact. Nagami [10] has shown that a normal screenable metacompact space is paracompact. But a paracompact Σ -space with a point-countable base is metrizable [9].

THEOREM 4. *If X is a normal $w\Delta$ -space with countable large basis dimension, then X is metrizable.*

Proof. As above, X is normal, screenable, and metacompact (since every $w\Delta$ -space is a β -space), hence paracompact. But a papacompact $w\Delta$ -space is an M -space, hence a Σ -space. Thus X is metrizable.

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