

Pacific Journal of Mathematics

METRIZATION OF SPACES WITH COUNTABLE LARGE BASIS DIMENSION

GARY FRED GRUENHAGE AND PHILLIP LEE ZENOR

METRIZATION OF SPACES WITH COUNTABLE LARGE BASIS DIMENSION

GARY GRUENHAGE AND PHILLIP ZENOR

With the following results, we generalize known metrization theorems for spaces with large basis dimension 0 i.e., non-archimedean spaces) to the higher dimensions: *Theorem.* If X is a normal Σ -space with countable large basis dimension, then X is metrizable. *Theorem.* If X is a normal $w\Delta$ -space with countable large basis dimension, then X is metrizable.

I. Introduction. A collection Γ of subsets of a set X is said to have rank 1 if whenever g_1 and g_2 are in Γ with $g_1 \cap g_2 \neq \emptyset$ then $g_1 \subset g_2$ or $g_2 \subset g_1$. According to P. J. Nyikos [13], a topological space X has large basis dimension $\leq n$ (denoted $\text{Bad } X \leq n$) if X has a basis which is the union of $n + 1$ rank 1 collections of open sets. X has countable large basis dimension ($\text{Bad } X \leq \aleph_0$) if X has a basis which is the union of a countable number of rank 1 collections such that each point of X has a basis belonging to one of the collections (a property which is automatically true in the finite case). $\text{Bad } X$ coincides with $\text{Ind } X$ and $\text{dim } X$ for metric spaces.

Spaces having large basis dimension 0 are usually called non-archimedean spaces. Theorems of Nyikos [11] and A. V. Archangelskii [3] show that a non-archimedean space is metrizable if and only if it is a Σ -space or a $w\Delta$ -space. In this paper we show that these results are valid, under mild assumptions, for the higher dimensions. Our results also improve a result of G. Gruenhagen [6], who showed that compact spaces having finite large basis dimension are metrizable.

II. Main results. According to Nyikos [11], a tree of open sets is a collection Γ of open sets such that if $g \in \Gamma$, then the set $\{g' \in \Gamma \mid g' \supset g\}$ is well-ordered by reverse inclusion; that is, $g \leq g'$ if and only if $g \supset g'$. Nyikos shows that the rank 1 collections for spaces with $\text{Bad } X \leq \aleph_0$ can be considered as rank 1 trees of open sets. The following fact will be used in our proofs:

LEMMA 1. Let T be a rank 1 tree of open subsets of a regular space X which contains a basis at each point of a subset X' of X . Then if \mathcal{U} is a cover of X' by open subsets of X , there exists a subset T' of T such that

- (i) T' is a cover of X' ;
- (ii) the elements of T' are pairwise disjoint;

(iii) $t \in T'$ implies that either t is degenerate or \bar{t} is a proper subset of some member of \mathcal{U} .

Proof. Put t in T' if and only if (a) either t is degenerate or there is a member U of \mathcal{U} such that \bar{t} is a proper subset of U and (b) there is no predecessor of t in T whose closure is a proper subset of some element of \mathcal{U} . Since T contains a basis at each point of X' and since the predecessors of a given $t \in T$ are well-ordered, it is easy to see that T' covers X' . Further, since T is a tree, the members of T' are mutually exclusive.

Nyikos calls a space *basically screenable* if it has a basis which is the union of countably many rank 1 trees of open sets. Every space X with $\text{Bad } X \leq \aleph_0$ is basically screenable. Basically screenable spaces are, of course, screenable; that is, every open cover has a σ -pairwise disjoint open refinement. While the following result is known, for the sake of completeness, we include its easy proof:

LEMMA 2. *A screenable countably compact space X is compact [2].*

Proof. Let \mathcal{U} be an open cover of X and let $\mathcal{V} = \cup \{ \mathcal{V}_n \mid n = 1, 2, \dots \}$ be an open refinement of \mathcal{U} covering X such that, for each i , the members of \mathcal{V}_i are mutually exclusive. The set $\{ V_n = \bigcup \mathcal{V}_n \mid n = 1, 2, \dots \}$ is a countable open cover of X ; hence, there exists a finite subcover $\{ V_{n_1}, V_{n_2}, \dots, V_{n_k} \}$. Then $\mathcal{V}_{n_1} \cup \mathcal{V}_{n_2} \cup \dots \cup \mathcal{V}_{n_k}$ is a point-finite refinement of \mathcal{U} . Thus, X is metacompact and it is well-known that a metacompact countably compact space is compact.

According to C. R. Borges [4], a space X is a $w\Delta$ -space if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X such that whenever $x \in X$ and $x_n \in \text{St}(x, \mathcal{G}_n)$ for each n , then $\{x_1, x_2, \dots\}$ has a cluster point.

THEOREM 1. *If X is a regular $w\Delta$ -space with countable large basis dimension, then X has a point countable basis.*

Proof. Let $\mathcal{G}_1, \mathcal{G}_2, \dots$ be a sequence of open covers of X satisfying the properties given in the definition of a $w\Delta$ -space. Let $\mathcal{B}_1, \mathcal{B}_2, \dots$ and X_1, X_2, \dots be sequences such that $X = \bigcup \{ X_i \mid i = 1, 2, \dots \}$ and, for each i , \mathcal{B}_i is a rank 1 tree of open sets containing a basis at each point of X_i .

For each $i < \omega_0$ and $\alpha < \omega_1$, we construct a collection $\mathcal{B}(i, \alpha)$ as follows: let $\mathcal{B}(i, 1)$ be a collection of mutually exclusive members of \mathcal{B}_i that refines \mathcal{G}_1 and covers X_i .

Suppose $\mathcal{B}(i, \beta)$ has been defined for $\beta < \alpha$. If α is not a limit ordinal, applying Lemma 1, let $\mathcal{B}(i, \alpha)$ be a collection of mutually

exclusive members of \mathcal{B}_i such that

- (i) if $j < \omega_0$, then $\mathcal{B}(i, j)$ refines \mathcal{G}_j ;
- (ii) $\mathcal{B}(i, \alpha)$ covers $(\cup \mathcal{B}(i, \alpha - 1)) \cap X_i$;

and (iii) $g \in \mathcal{B}(i, \alpha)$ implies \bar{g} is a proper subset of some member of $\mathcal{B}(i, \alpha - 1)$, or g is degenerate. If α is a limit ordinal, for each $x \in X_i$, let $B(\alpha, x) = \text{Int}(\bigcap_{\beta < \alpha} \{g \in \mathcal{B}(i, \beta) \mid x \in g\})$. Note that if x and y are in X_i , then either $B(\alpha, x) = B(\alpha, y)$ or $B(\alpha, x) \cap B(\alpha, y) = \emptyset$. Let $\mathcal{B}(i, \alpha) = \{B(\alpha, x) \mid x \in X_i\}$.

Let $\mathcal{B}_i^* = \bigcup_{\alpha < \omega_1} \mathcal{B}(i, \alpha)$. We will show that \mathcal{B}_i^* is a point countable collection forming a basis for X_i in X .

We will say that g is a *chain in \mathcal{B}_i^** if g is a function from an initial segment of ω_1 into \mathcal{B}_i^* so that (1) $g(\alpha) \in \mathcal{B}(i, \alpha)$ and (2) if $\beta < \alpha$, then $g(\beta) \supset \overline{g(\alpha)}$. Note that by our construction, if $\beta < \alpha$, then $g(\beta) \supset \overline{g(\alpha)}$. Furthermore, if $x \in X_i$, then there is exactly one maximal chain, say g , such that $g(\alpha)$ contains x for every α in the domain of g .

Claim 1. The domain of each maximal chain in \mathcal{B}_i^* is countable (and so, \mathcal{B}_i^* is point countable in X).

Proof of Claim 1. Suppose the contrary; i.e., there is a chain, say g , of length ω_1 .

Note that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ is compact. To prove this, we will only show that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ is countably compact; that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ is compact will then follow from Lemma 2. To this end, let N denote a countable subset of $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$. There is an α so that $g(\alpha)$ does not meet N . In particular then, no point of $\overline{g(\alpha + 1)}$ is a limit point of N . Because of property (i), it must be the case that N has a limit point in $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$; and so, $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ is compact. But, $\{\overline{g(\omega_0 + 1)} - \overline{g(\alpha)} \mid \alpha < \omega_1\}$ is an open cover of $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$ with no finite subcover, which is a contradiction from which Claim 1 follows.

Claim 2: \mathcal{B}_i^* is a basis for X_i in X ; in particular, if $x \in X_i$ and g is the maximal chain in \mathcal{B}_i^* centered at x , then $\{g(\alpha) \mid \alpha \text{ is in the domain of } g\}$ is a local basis for x in X .

Proof of Claim 2. Suppose otherwise. Then there is a point x of X_i so that the maximal chain, g , centered at x does not yield a basis at x in X ; i.e., $\{g(\alpha) \mid \alpha \in \text{domain of } g\}$ is not a local basis for x in X . Since the domain of g is countable, there is a first $\alpha_0 < \omega_1$ not in the domain g . There is a member B of \mathcal{B}_i so that if $\alpha < \alpha_0$, then $g(\alpha)$ is not a subset of B but this means that B is a subset of each $g(\alpha)$. Then x is in the interior of $\bigcap_{\alpha < \alpha_0} g(\alpha)$. Thus, by our

construction of $\mathcal{B}(i, \alpha_0)$, there is a member of $\mathcal{B}(i, \alpha_0)$ that contains x . This contradicts the maximality of g and it follows that $\{g(\alpha) \mid \alpha \text{ is in the domain of } g\}$ is a local basis for x in X .

We now have that $\bigcup_{i < \omega_0} \mathcal{B}_i^*$ is a point countable basis for X .

If \mathcal{H} is a cover of the space X and if $x \in X$, then $C(x, H)$ will denote the set $\bigcap \{H \in \mathcal{H} \mid x \in H\}$. According to K. Nagami [9], the space X is a Σ -space if there is a sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ of locally finite closed covers of such that if x_0, x_1, x_2, \dots is a sequence with $x_i \in C(x_0, \mathcal{F}_i)$ for each $0 < i < \omega_0$, then $\{x_i\}$ has a cluster point. The sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ is called a spectral Σ -sequence for X .

We will, without loss of generality, assume that each \mathcal{F}_i is closed under intersections and, for each i , \mathcal{F}_{i+1} refines \mathcal{F}_i .

LEMMA 3. *If X is a space with countable large basis dimension such that each uncountable subset of X has a limit point, then X is Lindelof.*

Proof. Since X has countable large basis dimension, X is screenable. G. Aquaro [1] has proved that every meta-Lindelof (and thus every screenable) space in which every uncountable set has a limit point is Lindelof.

THEOREM 2. *If X is a regular Σ -space with countable large basis dimension then X has a point countable basis.*

Proof. Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be a sequence of locally finite closed coverings of X given in the definition of a Σ -space. For each n , let \mathcal{G}_n be an open cover of X such that each member of \mathcal{G}_n intersects only finitely many members of \mathcal{F}_n . Let $\mathcal{B}_1, \mathcal{B}_2, \dots$ and X_1, X_2, \dots be sequences such that $X = \bigcup_{i < \omega_0} X_i$ and \mathcal{B}_i is a rank 1 tree of open sets which contains a basis for each point of X_i .

Define $\mathcal{B}(i, \alpha)$, $i < \omega_0$, $\alpha < \omega_1$, exactly as in the proof of Theorem 1. Let $\mathcal{B}_i^* = \bigcup_{\alpha < \omega_1} \mathcal{B}(i, \alpha)$ and define chain in \mathcal{B}_i^* as in the proof to Theorem 1.

Claim 1. Every chain in \mathcal{B}_i^* is countable.

Proof of Claim 1. Suppose otherwise; i.e., suppose that g is a chain in \mathcal{B}_i^* with length ω_1 . Let $K = \bigcap_{\alpha < \omega_1} \overline{g(\alpha)}$. Every uncountable of $\overline{g(\omega_0)} - K$ has a limit point in $\overline{g(\omega_0)} - K$ for suppose otherwise; that is, suppose that H is an uncountable subset of $\overline{g(\omega_0)} - K$ with no limit point in $\overline{g(\omega_0)} - K$.

Suppose that there is a point, h , of H such that, for each n ,

$C(h, \mathcal{F}_n)$ intersects infinitely many points of H . Then there is a countable subset N of H with a limit point. Since N is countable, there is an $\alpha < \omega_1$ so that $g(\alpha)$ does not intersect N . It follows that no point of K is a limit point of N . Hence, no point of K is a limit point of N ; and so, H has a limit point in $\overline{g(\omega_0)} - K$. This is a contradiction from which it follows that, for each h in H , there is an integer $n(h)$ such that $C(h, n(h))$ intersects only finitely many members of H . Thus, there is an N and an uncountable subset H^* of H so that if $h \in H^*$, then $n(h) = N$ and $\{C(h, F_N) \mid h \in H^*\}$ is an infinite subcollection of \mathcal{F}_N , each member of which intersects $g(N)$. But, $g(N)$ is in $\mathcal{B}(i, N)$ which contradicts the fact that $\mathcal{B}(i, N)$ refines \mathcal{G}_N . It follows that each uncountable subset of $\overline{g(\omega_0)} - K$ has a limit point in $\overline{g(\omega_0)} - K$; and so, by Lemma 3, $\overline{g(\omega_0)} - K$ is Lindelof. But $\{\overline{g(\omega_0)} - \overline{g(\alpha)} \mid \alpha < \omega_1\}$ is an open cover of $\overline{g(\omega_0)} - K$ with no countable subcover which is a contradiction from which Claim 1 follows.

That \mathcal{B}_i^* contains a basis at each point of X_i follows exactly as in the proof of Theorem 1. Thus Theorem 2 is proved.

THEOREM 3. *If X is a normal Σ -space with countable large basis dimension, then X is metrizable.*

Proof. R. E. Hodel has proved that every Σ -space is a β -space [8], and that every β -space is countably metacompact [7]. A screenable countably metacompact space is metacompact. Nagami [10] has shown that a normal screenable metacompact space is paracompact. But a paracompact Σ -space with a point-countable base is metrizable [9].

THEOREM 4. *If X is a normal $w\Delta$ -space with countable large basis dimension, then X is metrizable.*

Proof. As above, X is normal, screenable, and metacompact (since every $w\Delta$ -space is a β -space), hence paracompact. But a papacompact $w\Delta$ -space is an M -space, hence a Σ -space. Thus X is metrizable.

REFERENCES

1. G. Aquaro, *Point-countable open coverings in countably compact spaces*, General Topology and Its Relations to Modern Analysis and Algebra II, Academia, Prague (1966), 39-41.
2. R. Arens and J. Dugundji, *Remark on the concept of compactness*, Portugaliae Math., **9** (1950), 141-143.
3. A. V. Arhangel'skii and V. V. Filippov, *Spaces with bases of finite rank*, Math. USSR Sb., **16** (1972), 147-158.

4. C. J. R. Borges, *On metrizable topological spaces*, *Canad. J. Math.*, **20** (1968), 795-804.
5. J. Dugundji, *Topology*, Allyn and Bacon, 1966.
6. G. Gruenhagen, *Large basis dimension and metrizable spaces*, to appear in *Proc. AMS*.
7. R. E. Hodel, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, *Duke Math. J.*, **39** (1972), 253-263.
8. ———, *Spaces characterized by sequences of covers which guarantee that certain sequences have cluster points*, *Proceedings of the U. of Houston Point Set Topology Conference* (1971), 105-114.
9. K. Nagami, *Σ -spaces*, *Fund. Math.*, **65** (1969), 169-192.
10. ———, *Paracompactness and strong screenability*, *Nagoya Math. J.*, **8** (1955), 83-88.
11. P. J. Nyikos, *Some surprising base properties in topology*, *Studies in topology*, New York, Academic Press, 1975.
12. T. Shiraki, *M -spaces, their generalization, and metrization theorems*, *Sci. Rep. Tokyo Kyoiku Daigaku Ser. A*, **11** (1971), 57-67.

Received January 22, 1975 and in revised form June 3, 1975.

AUBURN UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.),
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Aharon Atzmon, <i>A moment problem for positive measures on the unit disc</i>	317
Peter W. Bates and Grant Bernard Gustafson, <i>Green's function inequalities for two-point boundary value problems</i>	327
Howard Edwin Bell, <i>Infinite subrings of infinite rings and near-rings</i>	345
Grahame Bennett, Victor Wayne Goodman and Charles Michael Newman, <i>Norms of random matrices</i>	359
Beverly L. Brechner, <i>Almost periodic homeomorphisms of E^2 are periodic</i>	367
Beverly L. Brechner and R. Daniel Mauldin, <i>Homeomorphisms of the plane</i>	375
Jia-Arng Chao, <i>Lusin area functions on local fields</i>	383
Frank Rimi DeMeyer, <i>The Brauer group of polynomial rings</i>	391
M. V. Deshpande, <i>Collectively compact sets and the ergodic theory of semi-groups</i>	399
Raymond Frank Dickman and Jack Ray Porter, <i>θ-closed subsets of Hausdorff spaces</i>	407
Charles P. Downey, <i>Classification of singular integrals over a local field</i>	417
Daniel Reuven Farkas, <i>Miscellany on Bieberbach group algebras</i>	427
Peter A. Fowler, <i>Infimum and domination principles in vector lattices</i>	437
Barry J. Gardner, <i>Some aspects of T-nilpotence. II: Lifting properties over T-nilpotent ideals</i>	445
Gary Fred Gruenhagen and Phillip Lee Zenor, <i>Metritzation of spaces with countable large basis dimension</i>	455
J. L. Hickman, <i>Reducing series of ordinals</i>	461
Hugh M. Hilden, <i>Generators for two groups related to the braid group</i>	475
Tom (Roy Thomas Jr.) Jacob, <i>Some matrix transformations on analytic sequence spaces</i>	487
Elyahu Katz, <i>Free products in the category of k_w-groups</i>	493
Tsang Hai Kuo, <i>On conjugate Banach spaces with the Radon-Nikodým property</i>	497
Norman Eugene Liden, <i>K-spaces, their antispace and related mappings</i>	505
Clinton M. Petty, <i>Radon partitions in real linear spaces</i>	515
Alan Saleski, <i>A conditional entropy for the space of pseudo-Menger maps</i>	525
Michael Singer, <i>Elementary solutions of differential equations</i>	535
Eugene Spiegel and Allan Trojan, <i>On semi-simple group algebras. I</i>	549
Charles Madison Stanton, <i>Bounded analytic functions on a class of open Riemann surfaces</i>	557
Sherman K. Stein, <i>Transversals of Latin squares and their generalizations</i>	567
Ivan Ernest Stux, <i>Distribution of squarefree integers in non-linear sequences</i>	577
Lowell G. Sweet, <i>On homogeneous algebras</i>	585
Lowell G. Sweet, <i>On doubly homogeneous algebras</i>	595
Florian Vasilescu, <i>The closed range modulus of operators</i>	599
Arthur Anthony Yanushka, <i>A characterization of the symplectic groups $P\text{Sp}(2m, q)$ as rank 3 permutation groups</i>	611
James Juei-Chin Yeh, <i>Inversion of conditional Wiener integrals</i>	623